## ON

# LINEAR DIFFERENTIAL EQUATIONS 

AND<br>\section*{ALGEBRAIC GROUPS OF TRANSFORMATIONS}

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1.     - The analogies between linear differential equations and algebraic equation have been pointed out for a long time now and pursued in different directions. Meanwhile, I believe that no one has sought to explore how Galois's celebrated theory of algebraic equations can be extended to linear differential equations. By employing a method that presents a strong analogy with the one that the illustrious geometer used, one will arrive at a proposition that corresponds, in some way, to Galois's fundamental theorem, and one will then be led to the notion of what I call the group of linear transformations of the differential equation. I employ the expression group of transformations that was used already in a general manner by Sophus Lie in a series of extremely remarkable articles that I have had occasion to cite numerous times in order to distinguish that group from the one that one generally calls the group of the linear equation $\left(^{(1)}\right.$.

Therefore, consider a linear differential equation:

$$
\begin{equation*}
\frac{d^{m} y}{d x^{m}}+p_{1} \frac{d^{m-1} y}{d x^{m-1}}+\cdots+p_{m} y=0 \tag{1}
\end{equation*}
$$

in which we suppose that the coefficients are rational functions of the variable $x$, and let $y_{1}, y_{2}, \ldots$, $y_{m}$ be a fundamental system of integrals.

I envision the following expression:

$$
V=A_{1,1} y_{1}+A_{1,2} y_{2}+\ldots+A_{1, m} y_{m}+A_{2,1} \frac{d y_{1}}{d x}+\cdots+A_{2, m} \frac{d y_{m}}{d x}+\cdots+A_{m, m} \frac{d^{m-1} y_{m}}{d x^{m-1}}
$$

[^0]which is, as one sees, a linear and homogeneous expression in $y_{1}, y_{2}, \ldots, y_{m}$, and their derivatives up to order $m-1$.

The $m^{2}$ coefficients $A$ represent arbitrary rational fractions of $x$. The function $V$ satisfies a linear equation of order $m^{2}$ that is easy to form. Let us denote that equation by:

$$
\begin{equation*}
\frac{d^{m^{2}} V}{d x^{m^{2}}}+P_{1} \frac{d^{m^{2}-1} V}{d x^{m^{2}-1}}+\cdots+P_{m^{2}} V=0 \tag{3}
\end{equation*}
$$

in which $P$ are obviously rational functions.
Moreover, upon differentiating $V$ a number of times equal to $m^{2}-1$, one will have $m^{2}$ equations of first degree in $V$ and its derivatives, which will give:

$$
\begin{aligned}
& y_{1}=\alpha_{1} V+\alpha_{2} \frac{d V}{d x}+\cdots+\alpha_{m^{2}-1} \frac{d^{m^{2}-1} V}{d x^{m^{2}-1}} \\
& y_{2}=\beta_{1} V+\beta_{2} \frac{d V}{d x}+\cdots+\beta_{m^{2}-1} \frac{d^{m^{2}-1} V}{d x^{m^{2}-1}} \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
& y_{m}=\lambda_{1} V+\lambda_{2} \frac{d V}{d x}+\cdots+\lambda_{m^{2}-1} \frac{d^{m^{2}-1} V}{d x^{m^{2}-1}},
\end{aligned}
$$

in which the $\alpha, \beta, \ldots, \lambda$ are rational in $x$.
Any integral of equation (2) corresponds to a system of integrals $y_{1}, y_{2}, \ldots, y_{m}$ of equation (1). That system cannot be fundamental. That will happen if the determinant of $y_{1}, y_{2}, \ldots, y_{m}$, and their derivatives up to order $m-1$ is zero. Upon writing that out, one will obtain a certain equation in $V$ :

$$
\begin{equation*}
\varphi\left(V, \frac{d V}{d x}, \ldots, \frac{d^{k} V}{d x^{k}}\right)=0 \tag{3}
\end{equation*}
$$

in which $k$ is equal to at most $m^{2}-1$.
One will then have a fundamental system $y_{1}, y_{2}, \ldots, y_{m}$ if one takes $V$ to be an integral of equation (2) that does not satisfy equation (3).

Having said that, it will happen, in general [i.e., if equation (1) is taken arbitrarily], that equation (2) has no common solution with a (linear or nonlinear) differential equation with rational coefficients of order less than $m^{2}$, if one ignores solutions that satisfy equation (3).

However, things can be otherwise in certain cases. Therefore, suppose that the differential equation of order $p$ :

$$
\begin{equation*}
f\left(x, V, \frac{d V}{d x}, \ldots, \frac{d^{p} V}{d x^{p}}\right)=0 \tag{4}
\end{equation*}
$$

( $f$ represents a rational function) fulfills that condition. Furthermore, I shall suppose that the preceding equation is irreducible, in the sense that was employed by Kœnigsberger (Theorie der Differentialgleichungen, 1882), i.e., not having any common solution with an equation of the same form and lower order. Under those conditions, all functions $V$ that satisfy equation (4) will satisfy equation (2), and moreover, equation (4) will have no common solution with equation (3). As a result, each solution to the equation:

$$
f=0
$$

will correspond to a fundamental system of integrals for the proposed linear equation.
Therefore, let $y_{1}, y_{2}, \ldots, y_{m}$ be the fundamental system that corresponds to a certain solution $V$ of the equation $f=0$, and let $Y_{1}, Y_{2}, \ldots, Y_{m}$ be the system that corresponds to an arbitrary solution $V$ of the same equation. One will have:

$$
\begin{aligned}
& Y_{1}=a_{1,1} y_{1}+a_{1,2} y_{2}+\ldots+a_{1, m} y_{m}, \\
& Y_{2}=a_{2,1} y_{1}+a_{2,2} y_{2}+\ldots+a_{2, m} y_{m}, \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& Y_{m}=a_{m, 1} y_{1}+a_{m, 2} y_{2}+\ldots+a_{m, m} y_{m} .
\end{aligned}
$$

The coefficients $a$ depends upon only $p$ arbitrary parameters, and we shall easily see that one can consider them to be algebraic functions of $p$ arbitrary parameters.

Indeed, consider the general integral of the equation:

$$
f\left(x, V, \frac{d V}{d x}, \ldots, \frac{d^{p} V}{d x^{p}}\right)=0 .
$$

That general integral will necessarily have the form:

$$
V=\alpha_{1} v_{1}+\alpha_{2} v_{2}+\ldots+\alpha_{m^{2}} v_{m^{2}},
$$

in which the $v$ are solutions to equation (2), and the $a$ are suitable algebraic functions of $p$ arbitrary parameters. Therefore, $p$ arbitrary constants will enter algebraically in:

$$
Y_{1}, Y_{2}, \ldots, Y_{m},
$$

and as a result, if $y_{1}, y_{2}, \ldots, y_{m}$ denote a fundamental system that corresponds to a particular solution $v$ of the equation $f=0$ then the coefficients $a$ in the substitution:

$$
\left\{\begin{array}{c}
Y_{1}=a_{1,1} y_{1}+a_{1,2} y_{2}+\cdots+a_{1, m} y_{m},  \tag{8}\\
Y_{2}=a_{2,1} y_{1}+a_{2,2} y_{2}+\cdots+a_{2, m} y_{m}, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
Y_{m}=a_{m, 1} y_{1}+a_{m, 2} y_{2}+\cdots+a_{m, m} y_{m}
\end{array}\right.
$$

will depend upon $p$ arbitrary parameters algebraically. We shall denote those parameters by $\lambda_{1}, \lambda_{2}$, $\ldots, \lambda_{p}$.

It is obvious that if one considers two substitutions such that (S) corresponds to two distinct systems of values of the parameters $\lambda$ then the product of those substitutions will be a substitution of the same form with the parameters $\lambda$ being replaced with a third system of values. The substitutions ( S ) then form a continuous group of transformations, to use the expression that Sophus Lie adopted in his profound and important research [see Theorie der Transformationsgruppen (Math. Ann. Bd. XVI) and a series of papers during 1876-1878-1879 in the Norwegian archives for Mathematikk og naturvitenskap].

I let $G$ denote that continuous and algebraic group of linear transformations and call it the group of transformations relative to the linear equation (1). Obviously, that group of transformations must note be confused with the one that is generally called the group of the linear equation.
2. - One can establish the following proposition in regard to that group, which is called Galois's fundamental theorem in the theory of algebraic equations:

Any rational function of $x$ and $y_{1}, y_{2}, \ldots, y_{m}$, as well as their derivatives, which are expressed rationally as functions of $x$, will remain invariable when one performs the substitutions of the group $G$ on $y_{1}, y_{2}, \ldots, y_{m}$.

Indeed, consider one such function. Upon replacing $y_{1}, y_{2}, \ldots, y_{m}$ in it with their values as functions of $V$ and equating it to a rational function, one will have:

$$
F\left(x, V, \frac{d V}{d x}, \ldots, \frac{d^{p} V}{d x^{p}}\right)=R(x),
$$

in which $F$ and $R$. Now, that equation is found to be verified for a certain solution $V$ of the equation $f=0$, and as a result, it will be verified for all solutions, from the irreducibility of the latter equation. That amounts to saying that the rational function considered does not change when one performs the substitution $S$ on $y_{1}, y_{2}, \ldots, y_{m}$.

One can add the following proposition to that theorem:

Any rational function of $x, y_{1}, y_{2}, \ldots, y_{m}$, and their derivatives that remains invariable under the substitutions of the group $G$ is a uniform function of $x$.

Let:

$$
f\left(x, y_{1}, y_{2}, \ldots, y_{m}\right)
$$

be such a function. The group of the equation (with the usual terminology), i.e., the group of substitutions that are performed on $y_{1}, y_{2}, \ldots, y_{m}$ when one makes the variables describe all possible
paths, will obviously belong to the group of transformations, i.e., all of the substitutions of the group of the equation are included in the substitutions $(S)$ for suitable values of the parameter $\lambda$. Therefore, the function $f$ will remain invariable when one performs all of the substitutions of the group of the equation on the $y$. As a result, that function $f$ will be a uniform function of $x$. One can add that it will be a rational function of $x$ if the differential equation has no singular points, for which the integrals will be regular.
3. - Those are the results to which one is quite naturally led when one seeks to develop a theory of linear differential equations that is analogous to Galois's theory of algebraic equations. We thus find that we are led to consider groups of transformations that are linear and algebraic.

Let us first make a general remark in regard to that subject.
Imagine one such group of transformations:

$$
\begin{aligned}
& Y_{1}=a_{1,1} y_{1}+a_{1,2} y_{2}+\ldots+a_{1, n} y_{n}, \\
& Y_{2}=a_{2,1} y_{1}+a_{2,2} y_{2}+\ldots+a_{2, n} y_{n}, \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\
& Y_{n}=a_{n, 1} y_{1}+a_{n, 2} y_{2}+\ldots+a_{n, n} y_{n},
\end{aligned}
$$

in which the $a$ are algebraic functions of $r$ arbitrary parameters $\left(r<n^{2}\right)$. Differentiate those equations once, twice, $\ldots$, up to order $(n-1)$ times. One will first have:

$$
\begin{aligned}
& \frac{d Y_{1}}{d x}=a_{1,1} \frac{d y_{1}}{d x}+\cdots+a_{1, n} \frac{d y_{n}}{d x} \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
& \frac{d Y_{n}}{d x}=a_{n, 1} \frac{d y_{1}}{d x}+\cdots+a_{n, n} \frac{d y_{n}}{d x}
\end{aligned}
$$

and some other similar equations. Eliminate the parameters from $r+1$ of those equations, which are $n^{2}$ in number. We will have a relation between the $Y$ and their derivatives, and between the $y$ and their derivatives, and it will be an entire algebraic relation.

Let:

$$
\begin{equation*}
F\left(Y_{1}, \frac{d Y_{1}}{d x}, \ldots, y_{1}, \frac{d y_{1}}{d x}, \ldots\right)=0 \tag{1}
\end{equation*}
$$

Now imagine that one performs an arbitrary substitution from the group of transformations on $y_{1}$, $y_{2}, \ldots, y_{n}$. Upon denoting the transformed values of $y_{1}, y_{2}, \ldots, y_{n}$ by $y_{1}^{\prime}, y_{2}^{\prime}, \ldots, y_{n}^{\prime}$, one will have:

$$
\begin{aligned}
& Y_{1}=A_{1,1} y_{1}^{\prime}+A_{1,2} y_{2}^{\prime}+\cdots+A_{1, n} y_{n}^{\prime} \\
& Y_{2}=A_{2,1} y_{1}^{\prime}+A_{2,2} y_{2}^{\prime}+\cdots+A_{2, n} y_{n}^{\prime}
\end{aligned}
$$

$$
Y_{n}=A_{n, 1} y_{1}^{\prime}+A_{n, 2} y_{2}^{\prime}+\cdots+A_{n, n} y_{n}^{\prime} .
$$

As a result, upon making the same elimination as before, one will have:

$$
\begin{equation*}
f\left(Y_{1}, \frac{d Y_{1}}{d x}, \ldots, y_{1}^{\prime}, \frac{d y_{1}^{\prime}}{d x}, \ldots\right)=0 . \tag{2}
\end{equation*}
$$

The relations (1) and (2) must be true no matter what the $Y$ and their derivatives might be. The coefficients of the various terms in $T$ and $d Y / d x, \ldots$ will then be the same, up to a factor. One will then obtain polynomials:

$$
\varphi\left(y_{1}, \frac{d y_{1}}{d x}, \ldots\right)
$$

in $y_{1}, y_{2}, \ldots, y_{n}$ and their derivatives [up to order at most $(n-1)$ ]. Those polynomials will be invariants of the group, i.e., upon performing a substitution of the group on the $y$, and simultaneously on their derivatives, one will have:

$$
\varphi\left(y_{1}^{\prime}, \frac{d y_{1}^{\prime}}{d x}, \ldots\right)=\mu \varphi\left(y_{1}, \frac{d y_{1}}{d x}, \ldots\right)
$$

in which $\mu$ is a constant.
The study of groups of transformations is then intimately linked with the search for polynomials or systems of polynomials in:

$$
\begin{array}{cccc}
y_{1}, & y_{2}, & \cdots & y_{n} \\
\frac{d y_{1}}{d x}, & \frac{d y_{2}}{d x}, & \cdots & \frac{d y_{n}}{d x} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{d^{n-1} y_{1}}{d x^{n-1}}, & \frac{d^{n-1} y_{2}}{d x^{n-1}}, & \cdots & \frac{d^{n-1} y_{n}}{d x^{n-1}}
\end{array}
$$

that are reproduced, up to a factor, when one simultaneously performs a set of linear substitutions on the terms in the same row.
4. - We shall now seek to find out what the direct study of algebraic groups of linear transformations will amount to. It is necessary to recall Lie's general theorems on groups of transformations (Math. Ann., Bd. XVI). With that eminent geometer, consider $n$ equations:

$$
x_{i}^{\prime}=f_{i}\left(x_{1}, x_{2}, \ldots, x_{n} ; a_{1}, a_{2}, \ldots, a_{r}\right) \quad(i=1,2, \ldots, n)
$$

that define a family of transformations between the $x$ and the $x^{\prime}$ in which $r$ arbitrary parameters $a_{1}, \ldots, a_{r}$ appear. Those transformations form a group if the succession of two transformations from that family is again a transformation of the same family.

Lie called a transformation infinitesimal if it has the form:

$$
x_{i}^{\prime}=x_{i}+X_{i}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \delta t
$$

or

$$
\delta x_{i}=X_{i}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \delta t
$$

in which $\delta t$ denotes an infinitely-small quantity.
Lie showed that every group of transformations with $r$ parameters contains $r$ independent infinitesimal transformation. Those infinitesimal transformations determine the group of transformations completely.

Let:

$$
\delta_{q} x_{1}=X_{q, 1} \delta t, \quad \delta_{q} x_{2}=X_{q, 2} \delta t, \quad \ldots, \quad \delta_{q} x_{n}=X_{q, n} \delta t \quad(q=1,2, \ldots, r)
$$

be the $r$ infinitesimal transformations of the group. One will have $r(r-1) / 2$ relations of the form:

$$
\begin{equation*}
A_{j}\left(X_{q i}\right)-A_{q}\left(X_{j i}\right)=\sum_{s} C_{j q s} X_{s i} \tag{E}
\end{equation*}
$$

for each value of $i$, upon setting:

$$
A_{q}(F)=X_{q, i} \frac{\partial F}{\partial x_{1}}+\cdots+X_{q, n} \frac{\partial F}{\partial x_{n}} .
$$

The $C_{j q s}$ are independent constants that are $i$ in number. One will then have $r(r-1) / 2$ identities between the $X$ that express the necessary and sufficient condition for the $r$ infinitesimal transformations considered to be the infinitesimal transformations of an $r$-parameter group. Those conditions can be further put into the form:

$$
\left(A_{j} A_{q}\right),
$$

i.e.:

$$
A_{j}\left[A_{q}(F)\right]-A_{q}\left[A_{j}(F)\right]=\sum_{s} C_{j q s} A_{s}(F) .
$$

When those conditions are fulfilled, one will have the group itself by proceeding as follows: Consider the differential equations:

$$
\frac{d x_{1}}{d x}=\sum_{k=1}^{r} \lambda_{k} X_{k, 1}, \quad \frac{d x_{2}}{d x}=\sum \lambda_{k} X_{k, 2}, \quad \ldots, \quad \frac{d x_{n}}{d x}=\sum \lambda_{k} X_{k, n}
$$

in which the $\lambda$ are considered to be constants. Let:

$$
W_{i}\left(x_{1}, x_{2}, \ldots, x_{n}, \lambda_{1} t, \lambda_{2} t, \ldots, \lambda_{r} t\right)=\alpha_{i} \quad(i=1,2, \ldots, n)
$$

be $n$ first integrals of that system, in which the $\lambda$ have been emphasized, and set:

$$
W_{i}\left(x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{n}^{\prime}, \lambda_{1} t, \ldots, \lambda_{r} t\right)=W_{i}\left(x_{1}, x_{2}, \ldots, x_{n}, \lambda_{1} t_{0}, \ldots, \lambda_{r} t_{0}\right) .
$$

One infers from those $n$ equations that:

$$
x_{i}^{\prime}=f_{i}\left[x_{1}, x_{2}, \ldots, x_{n}, \lambda_{1}\left(t-t_{0}\right), \ldots, \lambda_{r}\left(t-t_{0}\right)\right] .
$$

That will be the desired group when one replaces:

$$
\lambda_{1}\left(t-t_{0}\right), \lambda_{2}\left(t-t_{0}\right), \ldots, \lambda_{r}\left(t-t_{0}\right)
$$

with the $r$ parameters $a_{1}, a_{2}, \ldots, a_{r}$.
Having recalled those general theorems, we now return to linear groups of transformations. The infinitesimal substitutions will obviously be linear substitutions here. We must then start with $r$ linear substitutions, between which one supposes that the identities $(E)$ that we spoke of above are verified. The system:

$$
\frac{d x_{1}}{d x}=\sum \lambda_{k} X_{k, 1}, \quad \ldots, \quad \frac{d x_{n}}{d x}=\sum \lambda_{k} X_{k, n}
$$

is a system of linear equations with constant coefficients here. From the foregoing, we must look for the integrals of that system that become equal to $x_{1}, x_{2}, \ldots, x_{n}$, respectively, for $t=t_{0}$. Now, those integrals will necessarily have the form:

$$
\left\{\begin{array}{l}
x_{1}^{\prime}=A_{11} x_{1}+A_{12} x_{2}+\cdots+A_{1 n} x_{n}  \tag{S}\\
x_{2}^{\prime}=A_{21} x_{1}+A_{22} x_{2}+\cdots+A_{2 n} x_{n} \\
\cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
x_{n}^{\prime}=A_{n 1} x_{1}+A_{n 2} x_{2}+\cdots+A_{n n} x_{n}
\end{array}\right.
$$

in which the $A$ are linear and homogeneous expressions in $e^{\mu_{1}\left(t-t_{0}\right)}, e^{\mu_{2}\left(t-t_{0}\right)}, \ldots, e^{\mu_{n}\left(t-t_{0}\right)}$, while $\mu_{1}$, $\mu_{2}, \ldots, \mu_{n}$ are the roots of the fundamental equation that relates to the system of linear equations. That equation will obviously be of the form:

$$
\mu^{n}+\varphi_{1}\left(\lambda_{1}, \ldots, \lambda_{r}\right) \mu^{r-1}+\cdots+\varphi_{n}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}\right)=0,
$$

in which $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{n}$ are homogeneous polynomials in $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}$ whose degrees are equal to their respective indices. The $\mu$ will then be algebraic functions that are homogeneous of degree one in $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}$. As for the coefficients of $e^{\mu_{1}\left(t-t_{0}\right)}, e^{\mu_{2}\left(t-t_{0}\right)}, \ldots, e^{\mu_{n}\left(t-t_{0}\right)}$ in the $A$, they will be homogeneous functions of degree zero in $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}$.

If one then sets:

$$
\lambda_{1}\left(t-t_{0}\right)=a_{1}, \quad \lambda_{2}\left(t-t_{0}\right)=a_{2}, \quad \ldots, \quad \lambda_{r}\left(t-t_{0}\right)=a_{r}
$$

and:

$$
\mu\left(t-t_{0}\right)=\rho
$$

in addition, then one will have the equation for $\rho$ :

$$
\rho^{n}+\varphi_{1}\left(a_{1}, \ldots, a_{r}\right) \rho^{n-1}+\cdots+\varphi_{n}\left(a_{1}, \ldots, a_{r}\right)=0
$$

whose roots will be denoted by $\rho_{1}, \rho_{2}, \ldots, \rho_{n}$.
The $A$ will then be homogeneous linear functions in $e^{\rho_{1}}, e^{\rho_{2}},, \ldots, e^{\rho_{n}}$, whose coefficients will be rational functions in $a_{1}, a_{2}, \ldots, a_{r}, \rho_{1}, \rho_{2}, \ldots, \rho_{n}$.

The equations ( $\mathbf{S}$ ) will then give the group of transformations that has the $r$ initial substitutions for its infinitesimal substitutions. That group will not be algebraic, in general, and the question to ask will be how one must choose the $r$ initial substitutions in order that the group will be algebraic: I reserve the right to return to that general question later. At the moment, we shall examine only certain special cases of it that are still quite extensive in scope.
5. - In one of his papers [Norwegian Archives (1878), pp. 110], Sophus Lie incidentally considered a particular group of linear transformations. As above, denote the $r$ infinitesimal substitutions by $A_{1}(F), A_{2}(F), \ldots, A_{r}(F)$, and suppose that the bracket $\left(A_{i}, A_{i+k}\right)$ is a linear function of $A_{1}, A_{2}, \ldots, A_{i+k-1}$.

Lie showed that under those conditions, one can give the expressions $A(F)$ the common form:

$$
\alpha x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1} x_{1}+\beta_{2} x_{2}\right) \frac{\partial F}{\partial x_{2}}+\left(\gamma_{1} x_{1}+\gamma_{2} x_{2}+\gamma_{3} x_{3}\right) \frac{\partial F}{\partial x_{3}}+\cdots+\left(\rho_{1} x_{1}+\rho_{2} x_{2}+\cdots+\rho_{n} x_{n}\right) \frac{\partial F}{\partial x_{n}}
$$

by a convenient choice of independent variables.
It is the groups whose $r$ infinitesimal substitutions have that form that we shall consider.
We might remark that any group with two parameters will necessarily be included in the preceding hypothesis, and that is the case that we shall address first. Therefore, let:

$$
\begin{aligned}
& A_{1}(F)=\alpha x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1} x_{1}+\beta_{2} x_{2}\right) \frac{\partial F}{\partial x_{2}}+\cdots+\left(\rho_{1} x_{1}+\rho_{2} x_{2}+\cdots+\rho_{n} x_{n}\right) \frac{\partial F}{\partial x_{n}} \\
& A_{2}(F)=\alpha^{\prime} x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1}^{\prime} x_{1}+\beta_{2}^{\prime} x_{2}\right) \frac{\partial F}{\partial x_{2}}+\cdots+\left(\rho_{1}^{\prime} x_{1}+\rho_{2}^{\prime} x_{2}+\cdots+\rho_{n}^{\prime} x_{n}\right) \frac{\partial F}{\partial x_{n}}
\end{aligned}
$$

be the two expressions that correspond to the two fundamental substitutions. We must write out the idea that $\left(A_{1} A_{2}\right)$ is a linear combination of $A_{1}(F)$ and $A_{2}(F)$. We see, as a result, that we must set:

$$
\left(A_{1} A_{2}\right)=k\left(\alpha^{\prime} A_{1}-\alpha A_{2}\right)
$$

because the term in $\partial F / \partial x_{1}$ is missing from $\left(A_{1} A_{2}\right)$.
The coefficients of $\frac{\partial F}{\partial x_{2}}, \frac{\partial F}{\partial x_{3}}, \ldots, \frac{\partial F}{\partial x_{n}}$ in $\left(A_{1} A_{2}\right)$ do not contain terms in $x_{2}, x_{3}, \ldots, x_{n}$. It then results that:

$$
\frac{\alpha}{\alpha^{\prime}}=\frac{\beta_{2}}{\beta_{2}^{\prime}}=\frac{\gamma_{3}}{\gamma_{3}^{\prime}}=\ldots=\frac{\rho_{n}}{\rho_{n}^{\prime}},
$$

and we shall see that those relations will give us the solution to the problem. The roots $\mu_{1}, \mu_{2}, \ldots$, $\mu_{r}$ of the characteristic equation that corresponds to the system of linear equations that was formed in the preceding paragraph are:

$$
\mu_{1}=\alpha \lambda_{1}+\alpha^{\prime} \lambda_{2}, \quad \mu_{2}=\beta_{1} \lambda_{1}+\beta^{\prime} \lambda_{2}, \quad \ldots, \quad \mu_{r}=\rho_{n} \lambda_{1}+\rho_{n}^{\prime} \lambda_{2}
$$

here, so:

$$
\frac{\mu}{\alpha}=\frac{\mu_{2}}{\beta_{2}}=\frac{\mu_{3}}{\gamma_{3}}=\ldots=\frac{\mu_{n}}{\rho_{n}} .
$$

The exponentials $e^{\mu_{1}\left(t-t_{0}\right)}, e^{\mu_{2}\left(t-t_{0}\right)}, \ldots, e^{\mu_{n}\left(t-t_{0}\right)}$ will then be functions of one of them, and algebraic functions if $\alpha, \beta_{2}, \gamma_{3}, \ldots, \rho_{n}$ have commensurable ratios. As for the coefficients of $e^{\mu_{i}\left(t-t_{0}\right)}$ in the $A$, they will be functions of $\lambda_{2} / \lambda_{1}$. We can then take the parameters to be:

$$
\frac{\lambda_{2}}{\lambda_{1}}=a_{1}, \quad e^{\mu_{1}\left(t-t_{0}\right)}=a_{2} .
$$

The necessary and sufficient condition for $A_{1}(F)$ and $A_{2}(F)$, which are assumed to be capable of generating a group of transformations, to generate an algebraic group is then that $\alpha, \beta_{2}, \gamma_{3}, \ldots$, $\rho_{n}$ should have commensurable ratios, and as a result, that theorem will permit us to find all twoparameter algebraic groups of linear transformations.

The preceding statement answers the question that was posed, but it is easy to terminate the calculation completely. One can simplify those calculations by supposing that:

$$
A_{2}(F)=a_{1} x_{1} \frac{\partial F}{\partial x_{1}}+a_{2} x_{2} \frac{\partial F}{\partial x_{2}}+\cdots+a_{n} x_{n} \frac{\partial F}{\partial x_{n}}
$$

which is permissible. The identity will then give relations whose general type is:

$$
\begin{aligned}
& v_{1}\left(a_{1}-a_{i}\right)=k a_{1} v_{1}, \\
& v_{2}\left(a_{2}-a_{i}\right)=k a_{1} v_{2}, \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& v_{i-1}\left(a_{i-1}-a_{i}\right)=k a_{1} v_{i-1},
\end{aligned}
$$

and $\alpha a_{i}=a_{1} v_{i}$, when one supposes that the coefficient of $\partial F / \partial x_{i}$ in $A_{1}(F)$ is:

$$
v_{1} x_{1}+v_{2} x_{2}+\ldots+v_{i} x_{i}
$$

One can make different hypotheses. Take the general case in which $a_{1}, a_{2}, \ldots, a_{n}$ are distinct. If one then considers all of the differences:

$$
a_{1}-a_{i} \quad(i=2,3, \ldots, n)
$$

then they cannot be equal. Let $\beta_{1} \neq 0$ in $A_{1}(F)$.
One will then have:

$$
a_{1}-a_{2}=k a_{1}
$$

None of the other differences:

$$
a_{1}-a_{i} \quad(i=3, \ldots, n)
$$

can be equal to $k a_{1}$. As a result:

$$
\gamma_{1}=\delta_{1}=\ldots=\rho_{1}=0 .
$$

Similarly, let $\gamma_{2} \neq 0$, so one will have:

$$
a_{2}-a_{3}=k a_{1}
$$

and one must write:

$$
\delta_{2}=\ldots=\rho_{2}=0 .
$$

Upon continuing in that way, one will see that under the hypothesis that $\beta_{1} \gamma_{2} \delta_{3} \ldots \rho_{n-1} \neq 0$, one will have:

$$
a_{1}-a_{2}=a_{2}-a_{3}=\ldots=a_{n-1}-a_{n}=k a_{1},
$$

and

$$
\begin{gathered}
\gamma_{1}=\delta_{1}=\ldots=\rho_{1}=0, \\
\gamma_{2}=\delta_{2}=\ldots=\rho_{2}=0, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\
\rho_{n-2}=0 .
\end{gathered}
$$

One can look for algebraic groups of linear transformations with an arbitrary number of parameters in the same way, provided that the infinitesimal substitutions fulfill the conditions that were indicated at the beginning of the preceding section. It will suffice to examine the case of three parameters. Therefore, let:

$$
\begin{aligned}
& A_{1}(F)=\alpha x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1} x_{1}+\beta_{2} x_{2}\right) \frac{\partial F}{\partial x_{2}}+\left(\gamma_{1} x_{1}+\gamma_{2} x_{2}+\gamma_{3} x_{3}\right) \frac{\partial F}{\partial x_{3}}+\cdots+\left(\rho_{1} x_{1}+\cdots+\rho_{n} x_{n}\right) \frac{\partial F}{\partial x_{n}}, \\
& A_{2}(F)=\alpha^{\prime} x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1}^{\prime} x_{1}+\beta_{2}^{\prime} x_{2}\right) \frac{\partial F}{\partial x_{2}}+\left(\gamma_{1}^{\prime} x_{1}+\gamma_{2}^{\prime} x_{2}+\gamma_{3}^{\prime} x_{3}\right) \frac{\partial F}{\partial x_{3}}+\cdots+\left(\rho_{1}^{\prime} x_{1}+\cdots+\rho_{n}^{\prime} x_{n}\right) \frac{\partial F}{\partial x_{n}} \\
& A_{3}(F)=\alpha^{\prime \prime} x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1}^{\prime \prime} x_{1}+\beta_{2}^{\prime \prime} x_{2}\right) \frac{\partial F}{\partial x_{2}}+\left(\gamma_{1}^{\prime \prime} x_{1}+\gamma_{2}^{\prime \prime} x_{2}+\gamma_{3}^{\prime \prime} x_{3}\right) \frac{\partial F}{\partial x_{3}}+\cdots+\left(\rho_{1}^{\prime \prime} x_{1}+\cdots+\rho_{n}^{\prime \prime} x_{n}\right) \frac{\partial F}{\partial x_{n}} .
\end{aligned}
$$

One must have:

$$
\begin{aligned}
& \left(A_{2} A_{3}\right)=\lambda A_{1}+\mu A_{2}+v A_{3}, \\
& \left(A_{3} A_{1}\right)=\lambda^{\prime} A_{1}+\mu^{\prime} A_{2}+v^{\prime} A_{3}, \\
& \left(A_{1} A_{2}\right)=\lambda^{\prime \prime} A_{1}+\mu^{\prime \prime} A_{2}+v^{\prime \prime} A_{3},
\end{aligned}
$$

identically, in which the $\lambda, \mu, v$ are constants.
There is no term in $\partial F / \partial x_{1}$ in $\left(A_{2} A_{3}\right)$, and the coefficients of $\frac{\partial F}{\partial x_{2}}, \frac{\partial F}{\partial x_{3}}, \ldots, \frac{\partial F}{\partial x_{n}}$ contain no terms in $x_{2}, x_{3}, \ldots, x_{n}$, respectively. It will then result that:

$$
\lambda \alpha+\lambda^{\prime} \alpha^{\prime}+\lambda^{\prime \prime} \alpha^{\prime \prime}=0, \quad \lambda \beta_{2}+\lambda^{\prime} \beta_{2}^{\prime}+\lambda^{\prime \prime} \beta_{2}^{\prime \prime}=0, \quad \lambda \gamma_{3}+\lambda^{\prime} \beta_{2}^{\prime}+\lambda^{\prime \prime} \beta_{2}^{\prime \prime}=0, \ldots, \quad \lambda \rho_{n}+\lambda^{\prime} \rho_{n}^{\prime}+\lambda^{\prime \prime} \rho_{n}^{\prime \prime}=0,
$$

Similarly, the second and third identity give us:
$\lambda^{\prime} \alpha+\mu^{\prime} \alpha^{\prime}+v^{\prime} \alpha^{\prime \prime}=0, \quad \lambda^{\prime} \beta_{2}+\mu^{\prime} \beta_{2}^{\prime}+v^{\prime} \beta_{2}^{\prime \prime}=0, \quad \lambda^{\prime} \gamma_{3}+\mu^{\prime} \gamma_{3}^{\prime}+v^{\prime} \gamma_{3}^{\prime \prime}=0, \ldots, \lambda^{\prime} \rho_{n}+\mu^{\prime} \rho_{n}^{\prime}+v^{\prime} \rho_{n}^{\prime \prime}=0$, $\lambda^{\prime \prime} \alpha+\mu^{\prime \prime} \alpha^{\prime}+v^{\prime \prime} \alpha^{\prime \prime}=0, \lambda^{\prime \prime} \beta_{2}+\mu^{\prime \prime} \beta_{2}^{\prime}+v^{\prime \prime} \beta_{2}^{\prime \prime}=0, \lambda^{\prime \prime} \gamma_{3}+\mu^{\prime \prime} \beta_{2}^{\prime}+v^{\prime \prime} \beta_{2}^{\prime \prime}=0, \ldots, \lambda^{\prime \prime} \rho_{n}+\mu^{\prime \prime} \rho_{n}^{\prime}+v^{\prime \prime} \rho_{n}^{\prime \prime}=0$.

One concludes from this that if one leaves aside some exceptional cases whose discussion, although quite simple, would present little interest then:

$$
\begin{aligned}
& \frac{\beta_{2}}{\alpha}=\frac{\beta_{2}^{\prime}}{\alpha^{\prime}}=\frac{\beta_{2}^{\prime \prime}}{\alpha^{\prime \prime}}, \\
& \frac{\gamma_{2}}{\alpha}=\frac{\gamma_{2}^{\prime}}{\alpha^{\prime}}=\frac{\gamma_{2}^{\prime \prime}}{\alpha^{\prime \prime}}, \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& \frac{\rho_{n}}{\alpha}=\frac{\rho_{n}^{\prime}}{\alpha^{\prime}}=\frac{\rho_{2}^{\prime \prime}}{\alpha^{\prime \prime}} .
\end{aligned}
$$

Those relations will then give us the solution to the problem, in turn. The roots $\mu_{1}, \mu_{2}, \ldots, \mu_{n}$ of the characteristic equation (see no. 4) are:

$$
\mu_{1}=\lambda_{1} \alpha+\lambda_{2} \alpha^{\prime}+\lambda_{3} \alpha^{\prime \prime}, \quad \mu_{2}=\lambda_{1} \beta_{2}+\lambda_{2} \beta_{2}^{\prime}+\lambda_{3} \beta_{2}^{\prime \prime}, \quad \ldots, \quad \mu_{n}=\lambda_{1} \rho_{n}+\lambda_{2} \rho_{n}^{\prime}+\lambda_{3} \rho_{n}^{\prime \prime}
$$

here, so:

$$
\frac{\mu_{1}}{\alpha}=\frac{\mu_{2}}{\beta_{2}}=\frac{\mu_{3}}{\beta_{3}}=\ldots=\frac{\mu_{n}}{\beta_{n}} .
$$

The exponentials $e^{\mu_{1}\left(t-t_{0}\right)}, e^{\mu_{2}\left(t-t_{0}\right)}, \ldots, e^{\mu_{n}\left(t-t_{0}\right)}$ will be functions of one of them then, and algebraic functions if $\alpha, \beta_{2}, \gamma_{3}, \ldots, \rho_{n}$ have commensurable ratios. As for the coefficients of the $e^{\mu_{i}\left(t-t_{0}\right)}$ in
the $A$ (see no. 4), they will be functions of $\lambda_{2} / \lambda_{1}$ and $\lambda_{3} / \lambda_{1}$. We can then take the parameters to be:

$$
\frac{\lambda_{2}}{\lambda_{1}}=a_{1}, \quad \frac{\lambda_{3}}{\lambda_{1}}=a_{2}, \quad e^{\mu_{1}\left(t t_{0}\right)}=a_{3},
$$

and the group will be algebraic under the indicated condition.
7. - In conclusion, take the case of two variables and two parameters as an example.

Let:

$$
\begin{aligned}
& A_{1}(F)=\alpha x_{1} \frac{\partial F}{\partial x_{1}}+\left(\beta_{1} x_{1}+\beta_{2} x_{2}\right) \frac{\partial F}{\partial x_{2}} \\
& A_{2}(F)=a_{1} x_{1} \frac{\partial F}{\partial x_{1}}+a_{2} x_{2} \frac{\partial F}{\partial x_{2}}
\end{aligned}
$$

One will have:

$$
a_{1}-a_{2}=k a_{1}, \quad \alpha a_{2}=a_{1} \beta_{2}
$$

so one sets:

$$
a_{2}=s a_{1}, \quad \beta_{2}=s \alpha .
$$

One has the linear equations:

$$
\begin{aligned}
& \frac{d x_{1}}{d t}=\left(\lambda_{1} a_{1}+\lambda_{2} \alpha\right) x_{1}, \\
& \frac{d x_{2}}{d t}=\lambda_{2} \beta_{1} x_{1}+\left(\lambda_{1} a_{1}+\lambda_{2} \alpha\right) x_{2}
\end{aligned}
$$

The corresponding group will be:

$$
\begin{aligned}
& x_{1}^{\prime}=\sigma_{1} x_{1}, \\
& x_{2}^{\prime}=\sigma_{2} x_{1}+\sigma_{1}^{s} x_{2},
\end{aligned}
$$

in which $p$ and $q$ are two integers.
$x_{1}$ and the determinant $x_{1} \frac{d x_{2}}{d t}-x_{2} \frac{d x_{1}}{d t}$ are the only two distinct invariants of that elementary group.


[^0]:    ( ${ }^{1}$ ) The principal points of this work were announced in a note that was inserted in the Comptes rendus des séances de l'Académie des Sciences in April 1883.

