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On certain topological properties of trajectories of dynamical systems

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CHAPTER I

INTRODUCTION

1.1 – Generalities on dynamical systems. – Since the time of the classic work of H. Poincaré and G. D. Birkhoff [**37**, **37a**, **37b**, **5**], the sense of the locution "dynamical system" has been expanded considerably, to the point that it presently refers to some extremely general structures. A dynamical system is always attached to a partition of a topological space E (viz., the phase space) into subsets (viz., trajectories) that verify certain conditions that relate to topology, the theory of ordered sets, or the theory of groups of transformations. One thus distinguishes between topological dynamical systems, ordered dynamical systems, and general dynamical systems; cf., [**36**]. In a theory of such generality, one proposes to study the topological properties of trajectories and the things that pertain to them: e.g., regularity, recurrence, compactness, almost-periodicity, transitivity, etc… We remark that the theory of fiber bundles [**11**, **16**, **41**] and the theory of foliated manifolds [**39**] are particular theories of topological dynamical systems.

We return to the more restricted notion of dynamical system (D.S.) that was envisioned by H. Poincaré and defined by G. D. Birkhoff and studied by numerous authors and scholars since then. A D.S. is the pair of an n-dimensional differentiable manifold ([1]) V_n and a continuously-differentiable vector field ([1]) E on V_n . Moreover, we propose to study certain global properties of the trajectories of E. We remark that following certain authors [**24a**, **26**], the properties that we have in mind must be "stable": i.e., they must be verified by all fields (or at least by all of the fields in some class) that are sufficiently close to E. Indeed, it is only with that condition that those properties will present any physical interest. (See the last paragraph of **1.3.**)

The general results that are concerned with dynamical systems thus-defined are much less numerous [**51**, page 19]. In order to obtain more extensive results, one will then be led to make some restrictive hypotheses on the D.S. considered: Those hypotheses can be, for example, of an analytical nature (e.g., linear systems, perturbed linear systems, …) or of a topological nature (e.g., the trajectories are required to penetrate into a certain region [**44**]) or of a mechanical nature (e.g., conservative systems, …). The latter hypotheses of a mechanical nature seem to be the most interesting. In **1.2**, we shall attempt to group certain important theories around some principal notions of a mechanical nature that come into play.

1.2. – Review of certain classical theories.

a. Conservative dynamical systems. – Those D.S.'s play an important role, since one comes back to them in analytical mechanics (e.g., celestial mechanics) and differential

([1]) In order to not pointlessly encumber the presentations, we agree once and for all (unless stated to the contrary) to consider only manifolds, maps, homeomorphisms, etc., … that are indefinitely differentiable.

geometry (Finsler spaces, calculus of variations, …). Those systems enjoy the following properties:

1. The existence of Liouville's invariant integral.

2. The trajectories are the extremals of a problem in the calculus of variations.

3. The trajectories are the geodesics of a Riemann space.

4. The existence of canonical variables (p_i , q_i).

These various properties correspond to the following classical theories:

1. Ergodic theory [**22**] and the statistical theory [**32**].

2. The theory of the minimum and the min-max [**41b**, **5a**], as it is perfected and generalized in the global theory of the calculus of variations [**42a**, **34**].

3. The theory of geodesics in a space of negative curvature [**5a**, **21**, **7**] and the theory of symbolic dynamics; see also [**38**].

4. The study of neighboring trajectories to a closed trajectory [**5a**, **37a**], the theory of perturbations [**37**].

b. Periodically-excited dynamical systems. – That class of D.S.'s can be schematized thus: The D.S. is defined by the equation:

$$dx = \mathrm{E}\,(x, t)\, dt,$$

in which x describes a manifold V_n and the vector E (x, t) is a periodic function of time t, and the period T is a constant that is independent of x [**39c**, **12**, **30**, **30a**]. The study of the topological properties of the trajectories such a D.S. comes down to the study of the topological properties of the transformation of V_n onto V_n that associates the point x with its transform at the instant T along the trajectory that issues from x at the instant $t = 0$. A particularly important class of those systems is composed of the D.S.'s that are energy dissipators at large velocities. The latter systems can also be classified according to the rubric:

c. Relaxation oscillations. – These amount to systems that are subject to constraints and forces that are independent of time and have the following peculiarity: The energy of the system diminishes at large velocities and increases at small velocities [**19**, **19a**, **40**, **31**, **29**, **29a**, **33**]. In particular, one studies the systems with one or two degrees of freedom [**46**, **46a**].

Without continuing that classification (which runs the risk of rapidly becoming arbitrary and artificial), we further indicate some classical methods that will present some interest in what follows.

We first point out the use that Birkhoff [**5a**, Chap. IV] made of sectional surfaces (with boundary) in the study of the trajectories of conservative D.S.'s with two degrees of freedom. We mention, in turn, the paper by H. Poincaré [**37**] on the geodesics of convex surfaces and the paper by H. Seifert [**41a**] on the existence of closed trajectories of certain vector fields that are defined on the three-dimensional sphere (cf., **6.1**). Those two papers play a leading role in the D.S.'s whose trajectories are all closed (and whose period is a continuous function). We propose to study precisely those "fibered" systems (F.D.S.) in the present article.

1.3. Objective and plan of the present paper. – The objective of this work is to exhibit the role of fibered dynamical systems (F.D.S.) and the dynamical systems that admit Élie Cartan's integral invariant (I.D.S.), as well as the systems that simultaneously enjoy both of those properties (I.F.D.S.). (The last two chapters VII and VII are only weakly coupled with the previous chapters and have no relationship to the F.D.S.'s and I.D.S.'s; meanwhile, some other considerations will permit one to associate them with the previous chapters. Chapters VII and VIII are collectively the counterpart to chapter III.)

In **5.1**, one will find a list of the principal F.D.S.'s that present themselves in dynamics and differential geometry [**39a** and **b**]. Here are some problems that relate to I.F.D.S.:

a. – The study of the distribution and the nature of periodic solutions of the D.S. that are obtained by perturbing an F.D.S. That study is carried out by using the method of H. Poincaré [**37**, **39**], which has the advantage of giving results that are particularly useful in the case of I.F.D.S., but which present the inconvenience that they are applicable to only weak perturbations that do not present certain accidental peculiarities. By contrast, the Seifert method permits one to study finite perturbations (cf., Chapter VI).

b. – The fibration that is defined by the trajectories of an I.F.D.S. enjoys some remarkable structural properties. In particular, that fibration is never trivial. It then results that the perturbation problem (that was mentioned in *a*) cannot be reduced to the study of the fixed points of a transformation (as in **1.2b**).

c. – One can demand a list of the I.F.D.S., or at least one can look for the topological conditions that are imposed on the phase manifold or the configuration manifold of such a system.

To that effect, we have adopted the following plan:

Some properties of I.D.S.'s are presented in Chapter III. They essentially amount to the non-existence of compact transversal manifolds and certain applications of that property. Some other problems concerned with the existence of I.D.S.'s are mentioned. The special properties of I.F.D.S.'s are studied in Chapter IV. Chapter V is dedicated to H. Poincaré's method of small parameters, while the finite perturbations are examined in VI. The last Chapters VII and VIII are only weakly related to the preceding ones. Chapter VII is concerned with various results that relate to D.S.'s in the plane $\mathbb{R}^2$: e.g., the Liénard equations… The main idea of Chapter VII can be summarized thus: The calculation of the characteristic exponent of a solution to an ordinary differential equation can be done quite simply with the aid of the operations of a differential algebra [**39**].

The questions that were treated in Chapter VIII have very little relation to the theory of F.D.S.'s and I.D.S.'s. At most, one can say that the notion of a P.D.S. that is introduced in Chapter VIII is the exact opposite of the notion of an I.D.S. (A P.D.S. is, in the large, a D.S. that admits only a finite number of recurrent motions, and they are assumed to be periodic.) The distribution of periodic motions of a P.D.S. can be studied by some methods that are analogous in every respect to the proof of the Morse inequalities that Thom proposed [**49**]. The results that will be found recall the statements of the theorems of M. Morse.

The articles [**39a**, **39b**] were already devoted to the questions that were enumerated above. It would seem pointless to recall the results of [**39a**, **39b**] in detail. By contrast, we believe that we have ameliorated certain proofs in those papers and we shall reproduce those proofs here.

The material in Chapters III and IV has been partly treated in [**39a** and **b**]. However, the systematic use of the formulas of differential algebra [**11**] will permit one to make a better grouping of the properties of I.D.S.'s and I.F.D.S.'s. The results that are stated in that paper are more general than the ones in [**39b**], which are concerned solely with I.F.D.S.'s. Chapters III and IV use only the formulas of differential algebra in [**11**] and the usual notions from the homology of manifolds (e.g., the de Rham theorems [**4**]). We have tried to give complete proofs of the other properties that are utilized.

Chapters V and VI are concerned with the study of perturbations of F.D.S.'s. We briefly recall the results that were stated in [**39b**]. However, in **5.3**, we shall show how the formulas of differential algebra permit one to simplify the very laborious proofs in [**39b**, § **4**]. Chapter VI includes some remarks that relate to Seifert's theorem [**41a**] and we shall indicate an extension of that theorem that is important for certain applications. The explicit editing of the proofs of the generalized Seifert theorem will lead us to almost reproduce the paper by that author [**41a**]. We therefore believed it to be good to do that while indicating only Seifert's essential idea and pointing out the modifications that were used in the proof of the generalized theorem.

The last Chapters VII and VIII are concerned with some complete proofs. One can refer to the introductions **7.1** and **8.1** of those chapters for their contents.

It was pointed out above that the theory of foliated manifolds is a theory of "dynamical systems" (cf., **1.1**). It seems to me that the questions that were treated in [**39b**] can be considered similarly to be ones that refer to the theory of dynamical systems.

Finally, it remains for us to say a few words about the stable character of the formulas that were stated in the course of this work (cf., **1.1**). It is clear that the results of Chapters VII and VIII are stable, or that they can at least be considered to be that way after some suitable modifications of the corresponding statements. One easily sees the possible relations (which we do not have the time to specify in this paper) between Chapter VIII and [**24a**]. The stability properties of the statements that relate to periodically-excited D.S.'s (cf., **1**, **2b**) are specified in [**39e**]. As for the stability of the statements in Chapters V and VI, we shall content ourselves with the following remark: If those statements do not present the desirable stability character then they will nonetheless still have something that justifies the interest in them. The statements in Chapters III and IV have a "geometric" nature, and there is therefore no reason to discuss their stability.

1.4. Bibliography. – It is not possible to give a complete bibliography that relates to the topological properties of dynamical systems. The list that follows includes the publications that have been more or less directly useful in the preparation of this work. For more extensive bibliographic references, one can refer to [**25**, **33**, **26**, **28**, **7**, **34a**, **8**, **30**, **40a**, **47**, **23**].

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CHAPTER II

REVIEW OF SOME DEFINITIONS AND PROPERTIES OF DIFFERENTIAL GEOMETRY AND TOPOLOGY

2.1. Differential forms and vector fields on a numerical manifold ([1]). – For a complete and precise presentation of these questions, one can refer to [**11**]. For the notations of exterior algebra, cf. [**6**, **9**, **9a**]. See also [**13**, **14**].

On a numerical manifold V_n , one can define:

Exterior differential forms, which are denoted by Greek letters α, ω, … that are possibly affected with an upper integer index that indicates the degree of the form. The numerical functions on V_n are the forms of degree 0. One defines the exterior product $\alpha \wedge \beta$ of two exterior forms α and β, as well as the exterior differential $d\alpha$ of an exterior differential form α. The operator d increases the degree by one unit, and one recalls the following properties:

$$d\,(\alpha + \beta) = d\alpha + d\beta\,, \tag{1}$$

$$d\,d\alpha = 0, \tag{2}$$

$$d\,(\omega^q \wedge \alpha^p) = d\omega^q \wedge \alpha^p + (-1)^q\,\omega^q \wedge d\alpha^p\,. \tag{3}$$

Along with the exterior differential forms on V_n , one also defines vector fields on V_n , which are denoted by uppercase Latin letters X, Y, … A vector field X on V_n defines the additive group $\mathbb{R}$ of real numbers as a group (or more precisely, a “pseudo-group”) of transformations Γ (X) of V_n .

If $t \cdot x$ denotes the transform of $x \in V_n$ by $t \in \Gamma$ (X) then:

$$\left[\frac{d}{dt}(t \cdot x)\right]_{t=0} = \mathrm{X}\,(x)\,. \tag{4}$$

Let Y be a second vector field. One can associate the group Γ (X) with the transform of Y by X:

$$\theta\,(\mathrm{X}) \cdot \mathrm{Y} = \left[\frac{d}{dt}(t \cdot Y)\right]_{t=0}\,, \tag{5}$$

in which $t \cdot$ Y denotes the transform of Y by $t \in \Gamma$ (X) . The “infinitesimal transformation” operator θ (X) enjoys the following property:

([1]) **Remark**: Unless stated explicitly to the contrary, the manifolds, functions, homeomorphisms, differential forms, and vector fields that we will consider or that we will be led to introduce in our presentation will be assumed to be indefinitely differentiable.

(6) $$\theta\,(\mathrm{X}) \cdot \mathrm{Y} = -\,\theta\,(\mathrm{Y}) \cdot \mathrm{X} = [\mathrm{X}\ \mathrm{Y}]\,.$$

If ω^q is an exterior form of degree q then one can set:

(7) $$\theta\,(\mathrm{X}) \cdot \omega^q = \left[\frac{d}{dt}(t^* \cdot \omega^q)\right]_{t=0},$$

in which $t^* \cdot \omega^q$ denotes the transform of ω^q by the transpose of the map $t \in \Gamma\,(\mathrm{X})$.

Furthermore, one can prove the following formulas:

(8) $$\theta\,(\mathrm{X}) \cdot (\omega^q \wedge \bar{\varpi}^p) = (\theta\,(\mathrm{X})\ \omega^q) \wedge \bar{\varpi}^p + \omega^q \wedge \theta\,(\mathrm{X})\ \bar{\varpi}^p,$$

and in particular, if λ is a numerical function then:

(9) $$\theta\,(\mathrm{X}) \cdot \lambda\omega^q = (\theta\,(\mathrm{X}) \cdot \lambda)\ \omega^q + \lambda\,\theta\,(\mathrm{X}) \cdot \omega^q.$$

The operators d and θ (X) commute:

(10) $$d\,\theta\,(\mathrm{X})\ \omega^q = \theta\,(\mathrm{X}) \cdot d\omega^q.$$

One can associate the field X with the interior product operator i (X), which acts on exterior differential forms.

By definition:

(11) $$< i\,(\mathrm{X}) \cdot \omega^q, u^{q-1} > = < \omega^q, \mathrm{X} \wedge u^{q-1} >,$$

in which $< \alpha^r, u^r >$ denotes the scalar product of the exterior form α^r of degree r with the field of exterior r-vectors u^r. The operator i (X) diminishes the degree by one, and it verifies the following property:

(12) $$i\,(\mathrm{X}) \cdot (\omega^q \wedge \bar{\varpi}^p) = (i\,(\mathrm{X}) \cdot \omega^q) \wedge \bar{\varpi}^p + \varepsilon\,\omega^q \wedge (i\,(\mathrm{X}) \cdot \bar{\varpi}^p),$$

in which:

$$\varepsilon = \pm\,1.$$

Finally, the operators i (X) · θ (X) and d are coupled by the following very important relations:

(13) $$d\,(i\,(\mathrm{X}) \cdot \omega^q) + i\,(\mathrm{X}) \cdot d\omega^q = \theta\,(\mathrm{X}) \cdot \omega^q,$$

(14) $$\theta\,(\mathrm{X}) \cdot i\,(\mathrm{Y}) - i\,(\mathrm{Y}) \cdot \theta\,(\mathrm{X}) = i\,([\mathrm{X}\ \mathrm{Y}]).$$

2.2. Spaces with groups of operators isomorphic to T or $\mathbb{R}$. Manifolds fibered into circles. –

Definition 1. – A *dynamical system* (D. S.) is the pair (X, V_n) of an n-dimensional numerical manifold V_n and a vector field X. The pseudo-group of transformations that is generated by the infinitesimal transformation X is denoted by Γ(X). If $x \in V_n$ and $t \in \Gamma$(X) then $t \cdot x$ will denote the transform of x by t .

Definition 2. – A *general fibered* D.S. (F′.D.S) is a dynamical system that verifies the following conditions:

1. X $(x) \neq 0$ at every point $x \in V_n$.

2. The transformation X defines $\mathbb{R}$ as a group (and not just a pseudo-group) of transformations Γ(X) on V_n .

3. There exists a non-zero element $T \in \Gamma$(X) that fixes the points of V_n .

(Indeed, **2** is a consequence of **3**.)

Remark: It seems natural to replace condition **3** of Definition **2** with a weaker one; for example: There exists a continuous map $T(x)$ of V_n in Γ(X) [in which $T(x) \neq 0$] such that $T(x) \cdot x = x$.

Indeed, one will see in 4.2 that those two conditions are equivalent, at least for the I.D.S.'s.

Definition 3. – A *fibered dynamical system* (F.D.S.) is an F′.D.S that verifies the following condition, in addition to conditions **1**, **2**, and **3** of Definition **2**:

4. If $t_1 \in \Gamma$(X) and $x \in V_n$ then the relation $x = t_1 \cdot x$ will imply that $t_1 = n\,T$, where n is a rational integer.

The quotient group of Γ(X) by the subgroup that is generated by the element T is a group of transformations Γ'(X) of V_n whose abstract group is isomorphic to the torus T_1 (viz., the additive group of real numbers, modulo 1). Conversely, one can define an F′.D.S. by being given V_n and a group of transformations Γ'(X) (which is isomorphic to T_1) without fixed points. The group Γ'(X) is imagined to be the *structure group* of the F′.D.S. The trajectories of the field X that is attached to an F′.D.S. are obviously compact and homeomorphic to T_1 . They are the "fibers" of the F′.D.S. In the particular case of an F.D.S., those trajectories will define the structure of a fiber bundle in V_n that admits T_1 for its structure group.

Definition 4. – The *base space* of an F.D.S. is an $(n-1)$-dimensional manifold that is denoted by V_{n-1} . The canonical projection of V_n onto V_{n-1} is denoted by γ.

Recall that any point x of V_n admits an open neighborhood U_x and a homeomorphism φ of $U_x \times T_1$ onto $\gamma^{-1}(U_x)$ such that $\varphi(y \times T_1) = \gamma^{-1}(y)$ (in which $y \in U_x$).

Definition 5. – An F′.D.S. (X, V_n) is associated with the equivalence relation ρ that identifies the points that belong to the same fiber of V_n . The quotient space of V_n by ρ is once more denoted by V_{n-1} , and the canonical projection of V_n onto V_{n-1} is denoted by γ.

Conforming to the convention that was made at the beginning of this chapter, we point out that the topological space V_{n-1} is not necessarily a manifold and that γ is not necessarily indefinitely differentiable. However, one can verify that any point x of V_{n-1} admits an open neighborhood U_x , and that there exists a finite covering W of $\gamma^{-1}(U_x)$ such that the field p^{-1} (X) define an F.D.S. on W. [p denotes the canonical projection of W onto $\gamma^{-1}(U_x)$.] A particularly important case of this is when V_{n-1} can be endowed with the structure of an indefinitely-differentiable manifold such that the map p is indefinitely differentiable.

Definition 6. – An F′.D.S. that enjoys the property in the preceding statement is an F″.D.S.

CHAPTER III

GLOBAL PROPERTIES THAT ARE DUE TO THE EXISTENCE OF É. CARTAN'S INTEGRAL INVARIANT

3.1. Definitions and notations. –

Definition 1. – An I.D.S. is the pair of a D.S. (E, V_n) (cf., **2.2**, Definition **2**) and a Pfaff form π on V_n that verifies the following conditions:

1. n is odd: $n = 2q + 1$ ($q \geq 1$).

2. $\pi \wedge [d\pi]^q \neq 0$ at any point of V_n .

3. i (E) $\cdot$ $d\pi = 0$.

(cf., **2.1**)

At the end of this paragraph, we shall recall the main questions of dynamics and differential geometry that lead to I.D.S.'s. The stated global properties will be studied in **3.2**, **3.3**, and **3.4**.

Let u be a vector field on V_n and let i (u) be the operator of the interior product with u. The equation:

$$(1) \qquad i\,(u) \cdot d\pi = 0$$

is linear with respect to the vector u. The solutions of (1) at a given point define a one-dimensional vector space. The last statement is a classical consequence of property **2** in Definition **1**. One can then prove that as follows: The solutions to:

$$(2) \qquad i\,(u) \cdot [d\pi]^q = 0$$

form a one-dimensional vector space, since $[d\pi]^q$ is a non-zero, completely-decomposable form of degree $n - 1$. It then results that the solutions to (1) form a vector space whose dimension is at most one, because:

$$i\,(u) \cdot [d\pi]^q = (i\,(u) \cdot d\pi) \wedge [d\pi]^{q-1} \qquad \text{[cf., \textbf{2.1}, (12)].}$$

It remains to show that (1) admits a non-zero solution. That will result from the fact that:

$$i\,(u) \cdot [i\,(v) \cdot d\pi] = -\, i\,(v)\,[i\,(u) \cdot d\pi] = i\,(u \wedge v) \cdot d\pi,$$

which is a relation that shows that the rank of the map $u \to i\,(u)\,d\pi$ is less than n.

We append the equation:

$$(3) \qquad i\,(u) \cdot \pi = \langle u, \pi \rangle = 1$$

to equation (1).

The system of equations (1) and (3) defines a non-zero vector at every point of V_n . It is convenient to suppose that E verifies (3) precisely.

Proposition 1:

Let π be a Pfaff form on V_n that verifies conditions **1** *and* **2** *of Definition* **1**. *There exists a well-defined vector field that verifies condition* **3** *of Definition* **1**. *In other words, being given π is sufficient for one to define an* I.D.S.

Definition 2. – One says that π and $d\pi$ are a *relative integral invariant* and an *absolute integral invariant*, resp., of the unoriented direction field that E defines. The field E is called the *associated field* to π.

The following proposition is also classical:

Proposition 2:

Let λ be a numerical function on V_n , and let $\theta(u)$ be the infinitesimal transformation that is attached to u; the following relations are verified:

(4) $$\theta(\mathrm{E}) \cdot \pi = 0,$$

(5) $$\theta(\lambda\,\mathrm{E}) \cdot d\pi = 0 .$$

In other words, the form π is invariant under the group of transformations that is generated by E, and $d\pi$ is invariant under all one-parameter groups that are generated by the infinitesimal transformations λ E.

The equalities (4) and (5) result from [**2.1**, (13)] and Definition **1**, when one takes into account the fact that $i(\mathrm{E}) \cdot \pi = 1$.

Remarks:

a. Let $W_{2(q+1)}$ be a $2(q + 1)$-dimensional manifold (q is an integer and $q > 0$) on which a Pfaff form ω is defined that verifies the following relation:

(6) $$[d\omega]^{q+1} = 0.$$

Let V_n ($n = 2q + 1$) be a manifold that is embedded in $W_{2(q+1)}$ by way of a map φ of rank n. The image $\pi = \varphi^*(\omega)$ of ω under the transposed map φ^* to φ verifies: $[d\pi]^q \neq 0$.

b. Let V_{q+1} be a $(q + 1)$-dimensional manifold. The covariant vectors (x, p) (in which $x \in V_{q+1}$, and p is an element of the dual vector space that is tangent to V_{q+1} at x) define a $2(q + 1)$-dimensional manifold W_{n+1} . Let ω be the Pfaff form that is defined on W_{n+1} in the following fashion: The restriction $\omega(x, p)$ of ω to the tangent space to W_{n+1} at (x, p) is equal to $P^*(p)$, where P^* is the transposed map to the canonical projection P of W_{n+1} onto V_{q+1} [therefore, $P(x, p) = x$]. Let r_i ($i = 1, 2, \ldots, q + 1$) be local coordinates in V_{q+1}. One can prolong the coordinate system r to a coordinate system (r_i, s_i) on W_{n+1} upon letting s_i denote the covariant components of the covariant vector (x, s) that is attached to the point x whose coordinates are r_i. The form ω will then have the following structure:

$$\omega = \sum_i s_i \, dr_i \; .$$

It will then result that ω verifies the relation (6).

Being given a *Finsler space* structure on V_{q+1} is equivalent to being given a manifold V_n that is embedded in W_{n+1} and verifies the following conditions:

α) The restriction P' of P to V_n has rank n.

β) $P_{(x)}^{-1} \cap V_n$ (where $x \in V_{q+1}$) is a compact, convex, $(n - 1)$-dimensional manifold that contains the origin $(x, 0)$ in its interior. [$P_{(x)}^{-1} \cap V_n$ is the *figuratrix* at x.]

The form π that is induced by ω in V_n is non-zero at all points. One effortlessly verifies that *the form π defines an* I.D.S. *on V_n. It then results from E. Cartan's theory of the integral invariant* (and one can prove this directly) *that the trajectories of that* I.D.S. *project onto V_{q+1} along geodesics of the Finsler space V_{q+1}*. [**8**, **9a**, **39a**].

c. Consider a dynamical system (of rational mechanics) with perfect holonomic constraints that are independent of time. Suppose that the known forces derive from a force function. From the analytical viewpoint, that system is characterized perfectly by the Hamilton function $H(y)$ that is defined on the phase space. Moreover, H is a first integral. Suppose that the manifold V that is defined by the equation $H(y) = h_0$ (constant) is regular (i.e., that $dH = 0$ at every point of V). It results from É. Cartan's classical theory of the integral invariant that the trajectories that are traced out in V are trajectories of an I.D.S. [**9a**].

A remarkable analogy is valid for the regular problems of the calculus of variations [**8**, **9a**].

3.2. Nonexistence of a compact transversal manifold for an I.D.S. –

Definition 3. – Let X be a vector field on an n-dimensional manifold. An $(n - 1)$-dimensional manifold W_{n-1} that is embedded in V_n by a map φ of rank $n - 1$ is a *transversal manifold* for X if for any $x \in W_{n-1}$, the vector X $(\varphi(x))$ of the field X is not contained in the subspace $\varphi(T_x)$, where T_x is the vector space that is tangent to W_{n-1} at x.

Theorem 1:

The field E *that is associated with an* I.D.S. *does not admit a compact transversal manifold.*

The proof is quite simple: Let (φ, W_{n-1}) be a possible transversal manifold of E that is compact, connected, and oriented. The form φ^* ($[d\pi]^q$) is non-zero at any point of W_{n-1} . Hence:

$$\int_{W_{n-1}} \varphi^*([d\pi]^q) \neq 0. \tag{7}$$

However, the left-hand side of (7) is zero, from Stokes's formula; one has thus reached a contradiction.

We remark that the proof in Theorem 1 supposes only the existence of a form Ω (= $[d\pi]^q$) that verifies the following conditions:

Ω is homologous to 0 ; i.e., $\Omega = d\beta$.

Ω has degree $n - 1 = 2q$, and $\Omega \neq 0$ at any point.

i (E) $\cdot$ $\Omega = 0$.

Theorem **1** is valid for more general D.S.'s than the I.D.S.'s then.

3.3. Consequences of Theorem 1 and remarks. –

a. One knows the importance of sectional surfaces in the global study of the trajectories of dynamics [**5**, Chap. V]. Theorem **1** explains why the sectional surfaces that are used in the study of conservative systems have a boundary. See [**2**], as well.

b. Theorem **1** admits some applications to the study of second-order contact elements to a Finsler space [**39b**]. Let us explain that with the aid of a very special example. A Finsler space structure on the two-dimensional torus T^2 is associated with an I.D.S. (E, T^3) on the space T^3 of oriented tangent directions to T^2. A direction field X on T^2 corresponds to a section of the fiber bundle T^3. From Theorem 1, that section will be tangent to a trajectory of E at a non-vacuous set A of points: The trajectory of X that issues from x will then present a second-order contact with the geodesic that issues from x that is it tangent to. (One can say that x is a geodesic inflection point for the trajectory X that issues from that point.)

c. Theorem **1** can be extended to the regular problems of the calculus of variations that pertain to a multiple integral [**39d**]. The preceding remark (*b*.) thus extends to general Cartan spaces [**39d**].

3.4. Other problems relating to I.D.S.'s. – If the manifold V_n ($n = 2q + 1$) is given, and if V_n is compact then one knows that there exists a vector field without singularities on V_n . One is then led to pose the following problem:

Problem 1. – *If one is given a compact manifold* V_n ($n = 2q + 1$) *then does there exist an* I.D.S. *on* V_n ?

The remark (*a*.) in **3.1** shows that there exists a Pfaff form π that verifies the relation $[d\pi]^q \neq 0$ on any manifold V_n that is regularly embedded in a manifold W_{n-1} that is endowed with a Pfaff form ω that verifies (6).

Problem **1** leads to the following more-general problem, which is a problem of constructing a section for a fiber bundle:

Problem 2. – *Does there exist an exterior differential form* Ω *of degree two on* V_n ($n = 2q + 1$) *that verifies the relation:*

$$[\Omega]^q \neq 0$$

at every point?

One remarks that the form Ω defines a direction field $\mathcal{E}$ on V_n that one calls the *associated field*. Indeed, one can associate Ω with the equation $i(\mathcal{E}) \cdot \Omega = 0$, which is analogous to (1).

Problem **2** can be approached by the usual methods of the theory of obstructions. We shall forgo such a study here. We nonetheless remark that if V_n is the topological product of a compact manifold V_{n-1} with the circle S_1 , and if the associated field E is homotopic to the direction field that is tangent to the fibers $\{x\} \times S_1$ ($x \in V_{n-1}$) then the manifold V_{n-1} will admit an almost-complex structure [**16**]. Consequently, *a direction field on the topological product* $S_4 \times S_1$ *that is homotopic to the direction field that is tangent to the fiber cannot generate an* I.D.S.

One should point out that the form Ω that was considered in Problem **2** generates an integral invariance relation [**48**] for the D.S. that was envisioned. Problem **2** then makes sense for a very broad class of D.S.'s that are governed by equations of Euler-Lagrange type.

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CHAPTER IV

TOPOLOGICAL PROPERTIES OF I. F. D. S.’S AND I. F. D. S.’S.

4.1. Introduction and definition. – In this chapter, we propose to study the topological properties of the I.D.S.’s that are, at the same time, properties of F′.D.S.’s. Theorems **1** and **3** show that the corresponding fiber structured is quite complex (e.g., a section does not exist when V_n is compact).

Definition 1. – An *I.F′.D.S.* is a D.S. that is simultaneously an F′.D.S. and an I.D.S. One defines an *I.F.D.S.* and an *I.F″.D.S.* similarly (Definitions **1.2** and **3** in **2.1**).

Paragraphs **4.2** and **4.3** are concerned with a fundamental property of I.F′.D.S.’s (viz., Theorem **1**) that makes Theorem **1** in **3** more precise. In **4.4**, we shall recall some classical properties of F′.D.S.’s (meanwhile, to simplify the presentation, we shall always consider the case of an I.F′.D.S.). Paragraph **4.5** is devoted to the special properties of I.F′.D.S.’s. One will find some examples of I.F′.D.S.’s in **5.1**. (One should compare them with the results of that chapter in [**11**].)

4.2. Topological properties of an I.F′.D.S. –

Definition 2: A form α that is defined in the manifold of an F′.D.S. (E, V_n) will be called an *invariant form* when:

$$\theta(\mathrm{E}) \cdot \alpha = 0. \tag{1}$$

An invariant form α will be called a *basic* form if it verifies:

$$i\,(\mathrm{E}) \cdot \alpha = 0, \tag{2}$$

in addition to (1).

Proposition 1:

The form $d\pi$ on an I.F′.D.S. *is a basic form; the form π is invariant.*

Remark: If we had adopted the more general definition of an F′.D.S. (cf., **2.2**) then Proposition **1** would permit us to show that condition **3** in Definition **2** of **2.2**. is verified automatically in the case of an I.F′.D.S.

Proposition 2:

In the particular case of an F.D.S., *any basic form* α *is the image of a form* β *on* V_{n-1} *under the transpose of the projection* γ_1 – viz., $\alpha = \gamma^*(\beta)$. *Conversely, if* β *is a form on* V_{n-1} *then* $\gamma^*(\beta)$ *will be a basic form.*

We further point out that in the case of an F″.D.S., there also exists a bijective correspondence between basic forms and forms on V_{n-1} .

Proposition 3:

If α_1 *and* α_2 *are invariant [basic, resp.] forms then* $d\alpha_1$, $\alpha_1 + \alpha_2$, $\alpha_1 \wedge \alpha_2$ *will also be invariant [basic, resp.] forms. In other words, the invariant [basic, resp.] forms define a differential ring.*

Proposition **3** is an immediate consequence of (1) and (2) and of [**2.1**, (8) and (12)].

Proposition 4:

If V_n *is compact then the cohomology ring that is associated with the differential ring of the invariant forms is isomorphic to the cohomology ring of* V_n .

Indeed, let Γ be the group of transformations that is generated by the infinitesimal transformation E. Since V_n is compact, one can suppose that Γ is compact (cf., **2.2**). Let dg be the invariant measure on Γ such that $\int_\Gamma dg = 1$. Any differential form α on V_n can be associated with the invariant form:

$$\tilde{\alpha} = \int_\Gamma g^*(\alpha)\,dg\,. \tag{3}$$

One verifies that $\tilde{\alpha}$ is homologous to α; one then has Proposition **4**.

Definition 3. – The cohomology ring of the differential ring of basic forms will be called the *basic cohomology ring;* one defines the *basic Betti numbers* analogously.

Theorem 1:

If V_n *is compact and if* V_n *is the defining manifold of an* I.F′.D.S. *then the basic forms:* $d\pi$, $[d\pi]^2$, …, $[d\pi]^q$ *will not be homologous to zero in the differential ring of basic forms.*

Indeed, suppose that there exists a basic form α such that $d\alpha = [d\pi]^q$. The form $\pi \wedge [d\pi]^q$ is not homologous to zero in V_n since it is not annulled at any point of V_n :

$$\int_{V_n} \pi \wedge [d\pi]^q \neq 0. \tag{4}$$

However, by hypothesis, $[d\pi]^q = d\alpha$, so:

$$[d\pi]^q \wedge \pi = d\,(\alpha \wedge \pi) - \alpha\, d\pi\,. \tag{5}$$

The first term in the left-hand side of (5) is homologous to zero; the second term is equal to zero. That contradicts (4). Hence, $[d\pi]^q$ is not basically homologous to zero. It will then result that $[d\pi]^q$ (where $1 \leq r \leq q$) is not basically homologous to zero.

4.3. Consequences of Theorem 1 and Proposition 2 in the case of an I.F.D.S. **or an** I.F″.D.S.

Proposition 5:

In the particular case of an I.F.D.S., *there exists an exterior differential form* Ω *on the base* V_{n-1} *of* V_n *that verifies the following property:*

$$d\pi = \gamma^*(\Omega) \qquad \text{(cf., } \mathbf{2.2}\text{)}\,. \tag{6}$$

Furthermore, the basic cohomology ring is identified with that of V_{n-1} .

This proposition is obvious. It admits the following consequences:

a. $d\,\Omega = 0$.

b. $[\Omega]^q \neq 0$ *at any point.*

c. *If* V_n *is compact then* Ω *will not be homologous to zero, nor will* $[\Omega]^2, \ldots, [\Omega]^q$.

The consequences (*a*.) and (*c*.), and the relation (6) show that the characteristic class of the fiber structure imagined [**13**] is the class of Ω.

In the case of an I.F″.D.S., it is possible to prove the existence of a form Ω on V_{n-1} that verifies (6) at any point where that formula makes sense.

4.4. Relations between the Betti numbers of a fiber bundle and its base. – (These relations are classical: cf., [**11**]).

Proposition 6:

Let α *be an invariant form. There exists a unique decomposition of* α *of the following type:*

$$\alpha = \alpha_1 \wedge \pi + \alpha_0\,, \tag{7}$$

in which α_0 *and* α_1 *are basic forms. Moreover, if* α *is closed (i.e., if* $d\alpha = 0$) *then the same thing will be true for* α_1 .

In order to verify that proposition, it is sufficient to set $\alpha_1 = i\,(\mathrm{E}) \cdot \alpha$ and use [**2.1**, (13)].

Proposition **7** is a simple corollary to the Proposition **6**.

Theorem 2:

Suppose that the defining manifold of an I.F′.D.S. *is compact. Under those conditions:*

a. The fibers of V_n *are homologous to zero in* V_n *(homology with real coefficients).*

b. The Betti numbers b_i *of* V_n *and* p_i *of the base are coupled by the relation:*

$$b_i = p_i - p_{i-2} + \chi_{i-1} + \chi_{i-2}\,, \tag{8}$$

in which χ_i *denotes the dimensions of the vector subspace of the vector space of basic cohomology classes whose product with the cohomology class of* $d\pi$ *is zero.* (*If* $i < 0$ *then one sets* $p_i = 0$, *by convention*.)

In fact, Theorem **2** is classical. It is true for more general F′.D.S.'s, but we shall apply to only I.F′.D.S.'s.

The first part (*a*.) results from Proposition **7**. Indeed, any closed form of degree 1 in V_n will be homologous to a basic form, so its integral over a fiber of V_n will be zero, and it will then result that the fiber is homologous to zero.

In order to prove (*b*.), one first remarks that the map $\alpha \to \alpha_1$ (cf., Proposition **6**) induces a homomorphism φ_i of the Betti group B_i of V_n into the Betti group P_{i-1} of the base. Indeed, if α is an invariant form that is homologous to zero (in V_n) then there will exist an invariant form β such that $d\beta = \alpha$. From (7), one can decompose β into $\beta = \beta_0 + \beta_1 \wedge \pi$; hence:

$$\alpha = d\beta = (d\beta_0 + \varepsilon\,\beta_1 \wedge d\pi) + d\beta_1 \wedge \pi \qquad (\varepsilon = \pm 1)$$

and

$$\alpha_1 = i\,(\mathrm{E}) \cdot \alpha = d\beta_1\,.$$

Q.E.D.

If the form α_1 in (7) is homologous to zero (i.e., if there exists a basic form β such that $d\beta = \alpha_1$) then:

$$\alpha = \alpha_0 + d\beta \wedge \pi = (\alpha_0 + \varepsilon\,\beta \wedge d\pi) + d\,(\beta \wedge \pi) \qquad (\varepsilon = \pm 1)\,;$$

hence, α is homologous to a basic form. The dimension r_i of the kernel of φ_i is then equal to the dimension of the space of i-dimensional cohomology classes of V_n that contain basic forms. Let α be a basic form that is homologous to zero in V_n ; there will then exist an invariant γ such that $\alpha = d\gamma$. Let $\gamma = \gamma_0 + \pi \wedge \gamma_1$ be the decomposition (7) of γ.

$$\alpha = (d\gamma_0 + \gamma_1 \wedge d\pi) + \pi \wedge d\gamma_1 .$$

It will then result that $d\gamma_1 = 0$ and that α is homologous to the product of a cocycle in dimension $i - 2$ with $d\pi$. The dimension of the subspace of P of classes that are homologous to zero in V_n will then be equal to $p_{i-2} - \chi_{i-2}$, so:

$$r_i = p_i - (p_{i-2} - \chi_{i-2}) .$$

If the form α in (7) is closed then one will get the following relation upon exterior differentiating the two sides of (7):

$$\gamma = d\alpha_0 + \varepsilon\, \alpha_1 \wedge d\pi, \qquad \varepsilon = \pm 1.$$

That relation shows that $\alpha_1 \wedge d\pi$ is basically homologous to zero. Conversely, if $\alpha_1 \wedge d\pi$ is basically homologous to zero then there will exist a closed form α such that $\alpha_0 + \varepsilon\, \alpha_1 \wedge \pi$, where α_0 is a basic form. The image of B_i under φ_i will then be χ_{i-1}-dimensional. The relation (8) is obtained remarking that the dimension of B_i is the sum of the dimensions of the kernel and image of φ_i .

Proposition 8:

If $b_i = 0$ *then* $\chi_{i-1} = 0$.

Indeed, from (8):

$$b_i = [p_i - (p_{i-2} - \chi_{i-2})] + \chi_{i-1} .$$

However, the two terms on the right-hand side are positive, so the proposition is proved.

Proposition 9:

If q is odd ($n = 2q + 1$) *and if* $p_{q-2} = \chi_{q-2}$ *then* $p_q = b_q - \chi_{q-1}$ *will be even.*

Since $p_{q-2} - \chi_{q-2} = 0$, any basic cocycle α_q that is homologous to zero in V_n is likewise homologous to zero in the base. Therefore, let α_q be a basic cocycle in dimension q that is not homologous to zero. The Poincaré duality theorem permits one to associate α_q with a cocycle α_{q+1} on V_n such that:

a. α_{q+1} *is not homologous to zero.*

b. The correspondence $\alpha_q \to \alpha_{q+1}$ *is an isomorphism.*

The cocycle α_{q+1} admits the decomposition (7):

$$\alpha_{q+1} = \beta_q \wedge \pi + \gamma_{q+1}\ .$$

Hence, $\alpha_q \wedge \alpha_{q+1} = \alpha_q \wedge \beta_q \wedge \pi$, so it will result that $\alpha_q \wedge \beta_q$ is not homologous to zero. The map that associates the cohomology class of α_q with that of β_q is therefore an isomorphism with no real proper values. Q.E.D.

Remark: *In the particular case of an* I.F.D.S. *or an* I.F″.D.S., *the numbers p_i are the Betti numbers of the differentiable manifold V_{n-1} . Therefore, if q is odd then p_q will be even* (Poincaré).

4.5. On the Betti numbers of an I.F′.D.S. and some manifolds of class F′. –

Proposition 10:

A compact manifold V_n ($n = 2q + 1$, q odd) whose Poincaré polynomial is:

$$1 + t^{q} + t^{q+1} + t^n$$

cannot be endowed with the structure of an I.F′.D.S.

More generally, one can show that *if the Poincaré polynomial of the space V_n of an* I.F.D.S. *is:*

$$1 + s\,(t^{q} + t^{q+1}) + t^n, \qquad q \text{ odd},$$

and if V_n is compact then s will be even. Indeed:

$$b_1 = b_2 = \ldots = b_{q-1} = 0, \qquad \text{so} \qquad \chi_0 = \chi_1 = \ldots = \chi_{q-2} = 0\ .$$

Hence, from (8): $p_0 = p_2 = \ldots = p_{q-1} = 1$. Since $p_{q-1} = 1$, and since $[d\pi]^q$ is not basically homologous to zero, $\chi_{q-1} = 0$. Therefore: $s = b_q = p_q$. Proposition **10** will now be an immediate consequence of Proposition **9**.

Definition 4. – A compact manifold V_n is said to have *class* F′ (F, resp.) if V_q can be endowed with the structure of a Finsler space such that the I.D.S. of the geodesic lines (which are defined in the phase space V_{2q-1} ; **3.1**) is an I.F′.D.S. (I.D.F.S., resp.).

Proposition 11:

If the compact manifold V_q has class F′ *then the Betti number in dimension* 1 *of V_q will be zero.*

Indeed, the fibers of the associated I.F′.D.S. are homologous to zero (Theorem **2**). Hence, the geodesics of V_q are homologous to zero in V_q . However, if there exists a one-dimensional cycle in V_q that is not homologous to zero then there will exist a closed geodesic that is homologous to that cycle, which is then a contradiction.

One will find some properties that are peculiar to I.F.D.S.'s and manifold of class F in [**39a**]). Those properties easily extend to I.F″.D.S.'s and manifolds of class F″. Some examples are given in [**39a**] of simply-connected manifolds that do not have class F (since the topological product $S_2 \times S_3$ of the spheres S_2 and S_3 does not have class F). The only work that I know of that is concerned with manifolds of class F is in [**45**] and [**18**].

CHAPTER V

H. POINCARÉ'S METHOD OF SMALL PARAMETERS

5.1. Examples of I.F.D.S.'s –

a. Harmonic oscillators. – Consider q + 1 independent harmonic oscillators whose fundamental frequencies ω_i (i = 1, …, q) are commensurable and proportional to integers N_i. The differential system that governs the motion can be put into the form:

$$(1)\qquad \frac{dx_i}{dt} = \omega_i\, u_i\,, \qquad \frac{du_i}{dt} = -\,\omega_i\, x_i\,,$$

in which x_i, u_i are linear coordinates in the numerical space $\mathbb{R}^{2(q+1)}$. The system (1) admits a first integral $H(x, u) = \sum_i (u_i^2 + x_i^2)$. The equation $H(x, u) = 1$ defines a sphere S_n ($n = 2q + 1$). The system (1) defines an I.D.S. on S_n. The trajectories of (1) are closed, so one easily verifies that (1) indeed defines an I.F′.D.S., and more precisely, an I.F″.D.S. In the particular case where $\omega_1 = \omega_2 = \ldots = \omega_q$, the I.D.S. that one has in mind is an I.F.D.S. The corresponding base space is the complex projective space in q complex dimensions.

b. Examples of manifolds of class F′ (**4.5**, Definition **4**). – Élie Cartan [**9b**] gave the complete list of compact spaces that can be endowed with the structure of a symmetric Riemannian space whose geodesics are all closed. One can verify that those manifolds belong to the class F (cf., **4.5**, Definition **4**). The simply-connected coverings of those manifolds are:

Spheres, complex and quaternionic projective spaces, and the projective plane of the octaves. It seems that no other manifolds of class F′ are known; on the other hand, it is not known whether those manifolds exhaust the class F′.

c. Elliptic trajectories of a planet around the Sun.

Remarks: These few examples show why I.F.D.S.'s are so interesting. Theorems **2** and **4** exhibit the complexity of the fiber structure that is associated with an I.F.D.S. The problem of the perturbations of an I.F.D.S. is then particularly interesting (**1.3**).

5.2. Principle of the method of small parameters [37, 29]. – Let (E_0, V_n) be an F.D.S. on the manifold V_n whose base is V_{n-1}. Let E_μ be a vector field on V_n that depends upon the real, positive parameter μ such that: $(E_\mu)_{\mu=0} = E_0$.

The field E_μ admits a limited development in the powers of μ:

(1) $$\mathrm{E}_\mu = \mathrm{E}_0 + \mu\,\mathrm{E}' + \ldots$$

Definition 1. – Let Γ be the structure group of an F.D.S., and let dg be an invariant measure on Γ. One sets:

(2) $$\bar{\mathrm{E}}_\mu(z) = \int_\Gamma g\,(\mathrm{E}_\mu(g^{-1}(z))\,dg\,.$$

$\bar{\mathrm{E}}_\mu(z)$ admits the limited development:

$$\bar{\mathrm{E}}_\mu = \mathrm{E}_0 + \mu\,\bar{\mathrm{E}}' + \ldots,$$

in which:

$$\bar{\mathrm{E}}' = \int_\Gamma g\,(\mathrm{E}'(g^{-1}(z))\,dg\,.$$

The field $\bar{\mathrm{E}}'$ is invariant under Γ. There then exists a field $\tilde{\mathrm{E}}$ on V_{n-1} such that:

(3) $$\gamma(\bar{\mathrm{E}}') = \tilde{\mathrm{E}}\,.$$

One proposes to study the relations between the closed trajectories of E_μ and the singularities of $\tilde{\mathrm{E}}$ for small values of μ. Let (x, θ) be a local coordinate system in V_n such that x = constant represents a fiber of V_n and θ varies from 0 to 2π along the fibers of V_n.

Definition 2. – A trajectory of E_μ (for given μ) is called *simply closed* if it admits a parametric representation of the form:

$$x = \varphi_\mu(x_0, \theta), \qquad \text{in which} \qquad 0 \leq \theta \leq 2\pi,$$

and

$$\varphi_\mu(x_0, 0) = \varphi_\mu(x_0, 2\pi) = x_0\,.$$

One can associate a simply-closed trajectory with the index of the fixed point x_0 of the transformation $x \to \varphi(x, 2\pi)$.

The following lemmas are almost obvious and classical, and one will find proofs in [**39b**].

Lemma 1:

1. Let $\psi(\mu, x_0) = \varphi_\mu(x_0, 2\pi)$, in which $x = \varphi_\mu(x_0, 0)$ is the equation of the trajectory of E_μ that issues from $(x_0, 0)$. Under those conditions:

(3) $$\left[\frac{\partial}{\partial\mu}\psi(\mu, x_0)\right]_{\mu=0} = \tilde{\mathrm{E}}(x_0)\,.$$

Lemma 2:

If V_n is compact and if the singularities of $\tilde{\mathrm{E}}$ are isolated or if they are internal points of the simplexes of a convenient simplicial subdivision of V_n then there will exist $\mu' > 0$ such that for $\mu < \mu'$, the trajectories of the field E_μ enjoy the following properties:

a. If all of the simply-closed trajectories of E_μ are isolated then the sum of their indices will be equal to the Euler-Poincaré characteristic χ of V_{n-1} .

b. If $\chi \neq 0$ then the field E_μ will admit simply-closed trajectories.

In [**39b**], one will find various consequences of Lemma **2** and some applications to concrete problems (notably to relaxation oscillation).

We remark that Lemma **2** does not permit one to conclude the existence of simply-closed trajectories for small values of μ ($\mu \neq 0$) in every case. Meanwhile, Lemma **1** gives some useful information about the behavior of the trajectories of E_μ even in the case where the singularities of $\tilde{\mathrm{E}}$ are not isolated.

5.3. Particular case of an I.F.D.S. – Suppose that the D.S. that is defined by E_μ is an I.D.S., i.e., that E_μ admits the relative integral invariant π_μ . The form π_μ admits the limited development:

$$\pi_\mu = \pi_0 + \mu\,\pi' + \dots$$

Set:

$$\bar{\pi}_\mu = \int_\Gamma g^*(\pi')\,dg\,. \tag{4}$$

Since π' is an invariant form, it will admit the canonical decomposition (cf., Proposition **6** in Chap. IV):

$$\bar{\pi}' = \bar{\pi}'_0 + f \cdot \pi_0\,, \tag{5}$$

in which f is a numerical function on V_n . One deduces from (5) that:

$$d\bar{\pi}' = (d\bar{\pi}'_0 + f\,d\pi_0) + df \wedge \pi_0\,. \tag{6}$$

On the other hand:

$$0 = \int_\Gamma g^*[i(\mathrm{E}_\mu)\cdot d\pi_\mu]\,dg = \int_\Gamma \{g^*[i(\mathrm{E}_\mu)\cdot d\pi_\mu]\,dg + \mu g^*[i(\mathrm{E}')\cdot d\pi_0] + g^*[i(\mathrm{E}_0)\cdot d\pi']\}\,dg\,,$$

in which the terms that have been neglected have order greater than 1 in μ .

Hence:

$$i(\bar{\mathrm{E}}')\cdot d\pi_0 + i(\mathrm{E}_0)\cdot d\bar{\pi}' = 0$$

and if one takes (6) into account then:

(7) $$i(\bar{\mathrm{E}}') \cdot d\pi_0 = df\,.$$

The relation (7) determines the field $\bar{\mathrm{E}}'$ perfectly, and therefore the field E, as well. The field $\tilde{\mathrm{E}}$ is annulled exactly at points where:

(8) $$df = 0.$$

These results permit one to state:

Theorem 1:

If the D.S. *that is defined by* E_μ *is an* I.D.S. *that admits the relative integral invariant* π_μ *then the field* $\tilde{\mathrm{E}}$ *will admit the critical points of the function f that is defined by* (6) *for its singular points.*

In the particular case of a conservative dynamical system, one generally knows that function H_μ [**9a**]. It is then important to known how to calculate f upon starting with H_μ. To that effect, one easily sees that the function f is identified with the function $\tilde{H}$ in [**39b**]. (One recalls that $\tilde{H}$ is the integral of $H' = \left(\dfrac{\partial H_\mu}{\partial \mu}\right)_{\mu=0}$ over the fibers of the I.F.D.S. considered.)

One can refer to [**39b**] for some concrete applications of the preceding results. In order to apply the theory of M. Morse, one must know the Betti numbers of V_{n-1}. One will find that information in [**16b**] in the case of an I.F.D.S. that cited in **5.1**. The Lyusternik-Schnirelmann theory gives some indications in regard to the minimum number of critical points [**42a**].

Without stopping to review the applications that were treated in [**39b**], we rapidly discuss the following example:

On the Euclidian sphere S_2, we consider: The natural structure of a Riemann space that is defined by its line element $ds_0 = \mathcal{F}_0\,(x, dx)$ and a closely-related Finsler space structure that is defined by its line element $ds_\mu = \mathcal{F}_\mu\,(x, dx)$. [One supposes that the function $\mathcal{F}_\mu\,(x, dx)$ verifies the symmetry relation $\mathcal{F}_\mu\,(x, dx) = \mathcal{F}_\mu\,(x, -dx)$.] The structure $\mathcal{F}_0\,(x, dx)$ is associated with an I.F.D.S. In that case, $V_n = V_3 = P_3$ (viz., three-dimensional real projective space) and $V_{n-1} = V_2 = S_2$ (viz., the two-dimensional sphere). However, the symmetry condition on $\mathcal{F}_\mu$ shows that the D.S. that is associated with $\mathcal{F}_\mu$ can be considered to be an I.D.S. on the manifold $\tilde{V}_3$ of unoriented tangent directions to S_2. (The manifold $\tilde{V}_3$ admits P_3 as a two-sheeted covering.) The structure that is associated with $\mathcal{F}_0$ defines an I.F.D.S. on $\tilde{V}_3$ whose base is homeomorphic to the real projective plane P_2 . The Betti numbers of P_2 are 1, 1, 1, resp. The category of P_2 is 3; hence, the function f admits at least three distinct critical points. One sees the relation that is involved with those remarks from the classical results of Lyusternik-Schnirelmann (which are obviously much stronger).

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CHAPTER VI

FINITE PERTURBATIONS OF A FIBERED DYNAMICAL SYSTEM

6.1. Statement of the problem. – Consider an F.D.S. (E_0 , V_n) that is defined in a compact manifold V_n . Let $E = E_0 + \mu X$ be a vector field that is close to E_0 . In Chapter V, we established certain properties of the simply-closed trajectories of the field E for small values of the parameter μ by means of certain supplementary hypotheses on the field X. The properties that are established by that method are valid only for values of μ that are less than a real number $\varepsilon > 0$ that depends upon (V_n , E_0 , X). H. Seifert stated the following theorem [**41a**]:

(Seifert's) theorem:

Let E_0 *be a vector field that defines an* F.D.S. *on the compact (three-dimensional) manifold* V_n *. Furthermore, suppose that* V_n *is endowed with a Riemann space structure, and let* $\| E(x) \|$ *denote the norm of the vector* $E(x)$*. Under those conditions, there exists an* $\varepsilon > 0$ [*where* ε *depends upon only the pair* (V_n , E_0)] *such that if* $\| E(z) - E_0(z) \| < \varepsilon$ *for any* $z \in V_n$ *then the sum of the indices of the simply-closed trajectories (which are assumed to be isolated) of the field* E *will be equal to the Euler-Poincaré* χ *of the base* V_{n-1} *of the* F.D.S. *In particular, if* $\chi \neq 0$ *then the field* E *will possess simply-closed trajectories.*

We propose to show how Seifert's method permits us to prove an analogous theorem for certain four-dimensional manifolds V_4 with boundaries. That generalization will be sufficient to permit us to study the simply-closed trajectories in certain dynamical problems.

One can also refer to [**39b**], in which some analogous results are proved.

In order to show why Seifert's theorem (and some analogous theorems) is so interesting, it is convenient to explicitly determine an acceptable value of the number ε that appears in its statement. We shall not go into that problem, but only point out the following result:

If V_3 denotes the three-dimensional Euclidian sphere of radius 1 in Seifert's theorem, and E_0 is a unit vector field on S_3 that generates the usual fibration of S_3 over S_2 (cf., **5.1**) then an acceptable value for ε will be $\varepsilon = 1/2$.

That result shows that the scope of the fields E and E_0 is quite considerable and covers a great number of nonlinear phenomena.

6.2. The generalized Seifert theorem. – Let V_3 be a compact three-dimensional manifold on which an F.D.S. (V_3 , E_0) is defined whose base is V_2 . One supposes that V_3 is endowed with the structure of a Riemann space. Let $\mathbb{R}$ be the Euclidian number line.

The topological product $V_3 \times \mathbb{R}$ is endowed with the structure of a Riemann product space and the structure of an F.D.S. that is defined by the field E whose components at the point (x, t) are $(\mathrm{E}_0(x), 0)$.

We associate any point $(x, t) \in V_3 \times \mathbb{R}$ with the plaque $Q(x, t)$ that is composed of the geodesic segments of length l that issue from (x, t) and are orthogonal to $\mathrm{E}(x, t)$. We suppose that l is very small in such a fashion that $Q(x, t)$ will be a local section of the fiber space (i.e., that $Q(x, t)$ will meet each fiber of $V_3 \times \mathbb{R}$ at no more than one point).

Let E′ be a second vector field on $V_3 \times \mathbb{R}$. One lets $\| \mathrm{E}(x, t) \|$ denote the norm of the vector $\mathrm{E}(x, t)$. The following Lemma is almost obvious:

Lemma 1:

One can associate the pair $(V_3 \times \mathbb{R}, \mathrm{E})$ *with a real number* $\varepsilon > 0$ *such that the relation* $\| \mathrm{E}(x, t) - \mathrm{E}'(x, t) \| < \varepsilon$ *will imply the following consequences:*

a. After one circuit, the trajectory of E′ *that issues from* (x, t) *will meet the plaque* $Q(x, t)$ *again at a point* $z = (x', t') \in V_3 \times \mathbb{R}$.

b. The point z is a continuous function of (x, t).

If $z = (x, t)$ *then the trajectory of* E *that issues from* (x, t) *will be simply closed* (cf., Chap. V).

Definition 1. – If the field E′ verifies $\| \mathrm{E}(x, t) - \mathrm{E}'(x, t) \| < \varepsilon$ then one can associate it with the field U, which is defined in the following manner:

The vector $\mathrm{U}(x, t)$ is tangent to the arc of the geodesic that issues from (x, t) whose extremity is z. It points in the same direction as that arc and has a modulus that equals the length of that arc. One lets $\mathrm{U}'(x, t)$ denote the projection of the vector $\mathrm{U}(x, t)$ onto $V_2 \times \mathbb{R}$.

Let I be the interval $[-1, +1]$ in $\mathbb{R}$.

Hypothesis 1: *One supposes that the restriction of the field* U *to the boundary* $V_3 \times (\{-1\} \cup \{1\})$ *of* $V_3 \times I$ *points towards the interior of* $V_3 \times I$.

Definition 2: Let $w \in V_2$, and let G_w be the inverse image of w under the canonical projection of $V_3 \times I$ onto V_2 .

Suppose that G_w does not meet any simply-closed trajectory of E′. Associate the point $z \in G_w$ with the vector $\mathrm{U}'(z) \neq 0$ and the oriented direction U″ that is defined by that vector at the point (w, 0). (One points out that the vector spaces that are tangent to the points of $w \times \mathbb{R}$ are endowed with a natural parallelism.) One thus defines a map Π of G_w into the sphere S (w) that corresponds to the two tangent directions to $w \times I$. If Hypothesis **1** is verified then the map Π will map the boundary $G_{+1} = G_w \cap (V_3 \times 1)$ to the northern hemisphere of S (w) and $G_{-1} = G_w \cap (V_3 \times \{1\})$ to the southern hemisphere. Complete G_w into a sphere Σ_2 by considering G_{+1} and G_{-1} to be the boundary discs D and D'. Prolong Π to a map Π' of Σ_2 into S_2 (w) such that D maps to the northern hemisphere and D' maps to the southern hemisphere. One verifies that the degree of that map Π' depends upon only the map Π.

Definition 3: One denotes the degree of the map Π' by N (G_w).

Lemma 2:

*Suppose that Hypothesis **1** is verified and that N (G_w) is defined for all $w \in V_2$. If* $N(G_w) = 0$ *then the Euler-Poincaré χ of V_2 is zero.*

[One should remark that if E′ does not admit simply-closed trajectories, and if N (G_w) = 0 for some w then N ($G_{w'}$) = 0 for any w'. On the other hand, in order for N (G_w) to be defined for any w, it is necessary and sufficient that E′ should not admit simply-closed trajectories.]

Let $\tilde{V}_2$ be a manifold with a boundary S that is homeomorphic to the circle, such that there exists a map φ of $\tilde{V}_2$ onto V_2 that enjoys the following properties:

a. $\varphi(S) = w \in V_2$.

b. The restriction of φ of $\tilde{V}_2 - S$ is a homeomorphism.

It is easy to construct such a manifold and such a map. One again lets φ denote the product map of φ with the identity map of I to I. The theory of the construction of a section of a fiber bundle shows that it is possible to find a map f of $\tilde{V}_2 \times I$ into $V_3 \times I$ such that:

$$\gamma \cdot f = \varphi \qquad \text{(in which } \gamma \text{ is the projection of } V_3 \times I \text{ onto } V_2 \times I\text{).}$$

Let f' be the restriction of f to $S \times I$. One can define the degree N' of the map $\Pi \cdot f'$ in the same way that one defined the degree N (G_w) above. By hypothesis: N (G_w) = 0; it will then result that N' = 0. The map $\Pi \cdot f$ defines a direction U″ at every point of φ ([$V_2 - S$) $\times I$]). One then deduces that there exists a direction field U‴ on $V_2 \times I$ whose restriction to the boundary $V_2 \times (\{-1\} \cup \{+1\})$ of $V_2 \times I$ points to the interior of $V_2 \times I$; hence, Lemma **2**.

Lemma 3:

One supposes that Hypothesis ***1*** *is verified and that the field* E′ *does not admit any simply-closed trajectory. Under those conditions,* $N(G_w) = 0$.

The proof of that lemma is sketched out in paragraph **6.3**. Lemmas **2** and **3** have the generalized Seifert theorem as an immediate consequence:

Theorem 1:

Suppose that Hypothesis ***1*** *is verified. If the Euler-Poincaré characteristic of* V_2 *is not zero then the field* E′ *will admit at least one simply-closed trajectory.*

Indeed, a deeper study will permit one to show that χ is equal to the sum of the indices of the simply-closed trajectories. We shall not go into that study because its detailed presentation would be too long.

6.3. Proof of Lemma 3. – The hypotheses are the ones in Lemma **3**. One can then find a system of three open neighborhoods $B_1 \subset B_2 \subset B_3$ of the point w in V_2 and a field E″ on V_n that verifies the following conditions (recall that γ denotes the canonical projection of V_3 onto V_3 and γ' is the canonical projection of $V_3 \times I$ onto $V_2 \times I$):

1. $\| \mathrm{E}(z) - \mathrm{E}(z) \| < \varepsilon$ for $z \in V_3 \times \mathbb{R}$.

2. Under the map φ, the system of neighborhoods $B_1 \subset B_2 \subset B_3$ is homeomorphic to the system of three concentric balls in the Euclidian plane $\mathbb{R}^2$ that are centered at $\varphi(w) = \bar{w}$.

3. The field E″ admits a finite number of simply-closed trajectories.

4. The restriction of E″ to $\gamma'^{-1}_{(B_1 \times I)}$ is identical to the restriction of E′.

5. The restriction of E″ to the boundary of $V_3 \times I$ is identical to the restriction to E′.

6. The simply-closed trajectories of E are contained in $\gamma'^{-1}(B_2 - B_1)$.

7. The simply-closed trajectories of E″ are homotopic to the fibers of $\gamma'^{-1}(B_2 - B_1)$ in $\gamma'^{-1}_{(B_2 - B_1)}$.

8. The field E″ projects onto $B_3 - B_2$ along a vector field Γ. (The field U projects onto $B_3 - B_2$ along a field of parallel directions.)

The proofs of these assertions are very simple and almost obvious; however, the details are very lengthy. We remark only that the proofs will indeed possibly demand a new choice of the constant ε in Lemma **1**. It will suffice for us to know that we can attach an $\varepsilon > 0$ to the pair $(V_3 \times \mathbb{R}, \mathrm{E}')$ as in Lemma **1**, and the preceding eight properties will be verified. The proof of Lemma **3** is easily achieved then:

Indeed, let $w' \in B_3 - B_2$. The manifolds with boundary G_w and $G_{w'}$ define two cycles modulo $V_3 \times (\{1\} \cup \{-1\})$. The property **7** will permit one to show that there exists a chain Γ whose boundary [modulo $V_3 \times (\{1\} \cup \{-1\})$] is $G_w - G_{w'}$ and whose intersection with the simply-closed trajectories of E'' is vacuous. [Indeed, the intersection number of a simply-closed trajectory of E'' and a chain whose boundary is contained in $V_3 \times (\{1\} \cup \{-1\})$ is zero.] It then results that $N(G_{w'}) = N(G_w)$; however, from the property **8**, $N(G_{w'}) = 0$, which gives Lemma **3**.

6.4. Applications. – Here are two examples of dynamical systems that lead immediately to some applications of Seifert's theorem:

a. The dynamical system is composed of a material point that moves without friction on a Euclidian sphere S_2 subject to the action of very weak perturbing forces that depend upon the position and velocity in such a way that the system admits a first integral of *vis viva*. In this particular case, V_3 is homeomorphic to three-dimensional real projective space (cf., **5.4**), and V_2 is homeomorphic to the sphere S_2 . Hence, $\chi(V_2) = -2$.

b. The dynamical system is composed of two harmonic oscillators under the action of perturbing forces of the same type as in the preceding example (*a.*). Now, V_3 will be homeomorphic to the sphere S_3 , and V_2 will be homeomorphic to the sphere S_2 .

We shall now give an application of the generalized Seifert theorem. The differential system:

$$\text{(1)} \qquad \begin{cases} x'' + f(x,x')\,x' + x + g(x) = \varphi(x,x',y,y'), \\ y'' + h(y,y')\,y' + y + i(y) = \psi(x,x',y,y') \end{cases}$$

admits an obvious geometric interpretation [cf., Example (*b.*) above]. The system (1) can be replaced with:

$$\text{(2)} \qquad \begin{cases} \dfrac{dx}{dt} = u, \\ \dfrac{du}{dt} = -x + [\varphi(x,u,y,v) - g(x) - f(x,u)\cdot u], \\ \dfrac{dy}{dt} = v, \\ \dfrac{dv}{dt} = -y + [\psi(x,u,y,v) - i(y) - h(y,v)\cdot v]. \end{cases}$$

If the functions φ, ψ, f, h, g, i are zero then the system (2) will define an F.D.S. (E_0 , $\mathbb{R}_4^0$) in the space $\mathbb{R}_4^0$ that is obtained by removing the point (0, 0, 0, 0) from the space $\mathbb{R}^4$ of the variables x, u, y, v. Hence, $\mathbb{R}_4^0$ is homeomorphic to $S_3 \times \mathbb{R}$, and E_0 defines an F.D.S. on the spheres whose equations are:

$$x^2 + u^2 + y^2 + v^2 = \text{constant}.$$

We are then within the scope of the general hypotheses of **6.2**. Let Δ (M, m) be the hollow sphere whose equation is:

$$0 < m \leq x^2 + u^2 + y^2 + v^2 \leq M, \qquad \text{in which} \qquad M > m.$$

Δ (M, m) is homeomorphic $S_3 \times I$. Let E be the vector field that is defined by (2). In order for us to be able to apply the generalized Seifert theorem to (Δ (M, m), E_0, E), it is sufficient for us to suppose that:

a. The functions f, g, φ, ψ, h, i are bounded on Δ (M, m) by a convenient positive number ε that depends upon only M, m. (Under those conditions, it is possible to construct the field U, cf., Definition **1**.)

b. Hypothesis **1** is verified by the field U.

It is easy to indicate the analytical conditions on the functions f, g, φ, ψ, h, i that ensure condition (*b*.). We simply remark that condition (*b*.) expresses the idea that the energy ($x^2 + y^2 + u^2 + v^2$) of the dynamical system increases in time when the energy is weak and, on the contrary, diminishes when the energy is large. That is the general situation that is produced in the case of relaxation oscillations.

The same method will likewise permit one to establish the existence of the periodic solutions of period T (forced oscillations) for differential equations of the type:

$$x'' + f(x, x')\, x' + x = \varphi(t, x, x'),$$

in which the function admits the period T in t and in which the function f (x, x') verifies the general hypotheses of relaxation. (Of course, one supposes that the perturbing terms and f are not too large.) Cf., [**46a**].

CHAPTER VII

ON THE STABILITY OF PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION

$$X(x, y)\,dx + Y(x, y)\,dy = 0.$$

7.1. Introduction. – Some classical formulas will permit one to calculate the characteristic exponent of a closed trajectory C of the differential equation $X\,dx + Y\,dy = 0$ [**47**, **31**]. Those formulas will then permit one to study the stability of C. *Levinson has shown what one can infer from those formulas in the study of a number of those closed trajectories* [**31**, **47**]. One knows the importance of those questions in the study of the Liénard equation or some analogous equations of mathematical physics.

We propose to restate those two questions by using the methods of differential algebra. In the first place, we shall prove the following two lemmas:

Lemma 1:

Let V_2 be a two-dimensional numerical manifold, and let ω be a Pfaff form without singularities on V_2. There exist a Pfaff form $\bar{\varpi}$ on V_2 that is defined modulo ω such that $d\omega = \bar{\varpi} \wedge \omega$. Let C be an integral curve of the equation $\omega = 0$. The form ϖ that is induced in C by $\bar{\varpi}$ is closed. The cohomology class of ϖ depends upon only the differential equation $\omega = 0$, but it does not depend upon the particular choice of ω (which can be replaced by $\lambda\,\omega$, in which λ is a numerical function that is not annulled at any point of V_2).

Lemma 2:

*In addition to the hypotheses of Lemma **1**, one also supposes that C is compact and oriented; one sets $I(C) = \int_C \omega$. The characteristic exponent of C is equal to $-I(C)$.*

*It will then result from Lemma **2** that if $I(C) < 0$* [$I(C) > 0$, resp.] *then the trajectory C will be stable (unstable, resp.).*

One should point out that the statements in Lemma **1** are all obvious, except for the last one. However, Lemma **1** can be stated in a more general form:

Lemma 3:

Let V_n be an n-dimensional numerical manifold, and let ω be a completely-integrable Pfaff form without singularities on V_n. There will then exist a Pfaff form $\bar{\varpi}$ on V_n that is defined modulo ω and is such that $d\omega = \bar{\varpi} \wedge \omega$. Let V_{n-1} be an integral manifold of the equation $\omega = 0$. The form ϖ that is induced in V_{n-1} by $\bar{\varpi}$ is closed. The cohomology class of ϖ depends upon only the equation $\omega = 0$ [**39**].

Lemma **2** admits an analogous generalization [**39**].

7.2. Proofs of Lemmas 1, 2, and 3. – We immediately prove Lemma **3**, which has Lemma **1** as a consequence. Since ω is completely integrable, $\omega \wedge d\omega = 0$; i.e., $d\omega$ is divisible by ω, which implies the existence of $\bar{\omega}$ (which is defined modulo ω). However, $dd\omega = 0$, so:

$$d(\bar{\omega} \wedge \omega) = 0 = d\bar{\omega} \wedge \omega - \bar{\omega} \wedge \bar{\omega} \wedge \omega = d\bar{\omega} \wedge \omega .$$

Now, $d\bar{\omega} \wedge \omega = 0$ is equivalent to $d\bar{\varpi} = 0$. One then replaces ω with $\omega_1 = \lambda\omega$. One can write:

$$d\omega_1 = d\lambda \wedge \omega + \lambda\, d\omega = (d\lambda / \lambda + \bar{\omega}) \wedge \omega,$$

with some obvious notations. Hence:

$$\bar{\omega}_1 = \bar{\omega} + d\lambda / \lambda \bmod \omega \qquad \text{and} \qquad \bar{\varpi}_1 = \bar{\varpi} + \widetilde{d\lambda / \lambda}\,,$$

in which $\widetilde{d\lambda / \lambda}$ is the form on V_{n-1} that is induced by $d\lambda / \lambda$. Q.E.D.

It should be remarked that one can replace the form ω with a "twisted" form (i.e., one that is defined up to sign) in the statement of Lemma **3**.

In order to prove Lemma **2**, one introduces local coordinates (r, θ) in an annular neighborhood of C, in which r and θ are some real numbers, and θ is defined modulo 2π, such that the equation of C is $r = 0$, and θ increases when C is described in the positive sense. Under those conditions, the proposed equation will be transformed into:

$$\omega_1 \equiv dr + \varphi(r, \theta)\, d\theta = 0,$$

in which $\varphi(r, \theta)$ is a periodic function of period 2π in θ, and $\varphi(0, \theta) = 0$.
Here, one has:

$$d\omega_1 = \varphi_r(r, \theta)\, dr \wedge d\theta \qquad \text{and} \qquad \bar{\omega}_1 = -\varphi_r(r, \theta)\, d\theta \qquad\qquad (\text{mod } \omega_1).$$

The function $\varphi(r, \theta)$ admits the limited development (in r):

$$\varphi(r, \theta) = r\, \varphi_0(\theta) + \ldots, \qquad \text{in which} \qquad \varphi_0(\theta) = \varphi'_r(0, \theta)\,.$$

The solutions of (2) have the form $r = \Psi(r_0, \theta)$, in which $0 \le \theta \le 2\pi$, and $\Psi(r_0, \theta)$ admits the limited development $\Psi(r_0, \theta) = r_0\, \Psi_0(\theta) + \ldots$ One then concludes that Ψ_0 verifies the equation:

$$d\,\Psi_0 + \Psi_0\, \varphi_0(\theta)\, d\theta,$$

so

$$\Psi_0(\theta) = \Psi_0(0)\exp\left[-\int_0^\theta \varphi_0(\xi)\,d\xi\right].$$

The conclusions of Lemma **2** result immediately.

Remarks:

1. A particularly simple form for the equation $X\,dx + Y\,dy = 0$ is the following one:

$$\omega \equiv \cos\Psi\,dx + \sin\Psi\,dy = 0,$$

in which Ψ is a (possibly multi-valued) function in a domain Δ of the Euclidian plane (x, y) whose various determinations differ by a multiple of 2π. Under those conditions:

$$d\omega = (\sin\Psi\cdot\Psi_y + \cos\Psi\cdot\Psi_x)\,dx \wedge dy,$$

and one can set:

$$\varpi = \Psi_y\,dx - \Psi_x\,dy.$$

The form $d\varpi$, which will play an important role in what follows, will have a remarkable structure then:

$$d\varpi = (\Psi_{x^2} + \Psi_{y^2})\,dx \wedge dy = \Delta\Psi\,dx \wedge dy.$$

2. It should be pointed out that the form ϖ is coupled with the integrating factors of ω. Indeed, if φ is an integrating factor of ω then:

$$0 = d(\varphi\cdot\omega) = d\varphi \wedge \omega + \varphi\,d\omega,$$

so

$$d\omega = -(d\varphi/\varphi) \wedge \omega \quad \text{and} \quad \varpi = -d\varphi/\varphi \ \ (\text{modulo } \omega).$$

If ϖ is a closed form then there will (locally) exist a function f that $df = \varpi$. One can then deduce that $e^f\omega$ is a closed form, and that the equation $\omega = 0$ can be integrated by quadratures.

Examples:

a. Consider the linear equation. $\omega \equiv dy + [f(x)\,y + g(x)]\,dy = 0$. Here, $d\omega = f(x)\,dy \wedge dx$ and $\varpi = -f(x)\,dx$; hence, $\exp\int f(x)\,dx$ is an integrating factor.

b. Let $x'' + f(x)\,x'^2 + g(x) = 0$ be a differential equation. Its integration comes down to the integration of the system:

$$dx = y\,dt, \qquad \omega \equiv y\,dy + [f(x)\,y^2 + g(x)]\,dx = 0 \qquad \text{(Bernoulli equation)}.$$

Thus:

$$d\omega = 2\, f(x)\, y\, dy\, dx\,; \qquad \bar{\omega} = -2\, f(x)\, dx\,.$$

One then sees that the proposed differential equation can be integrated by quadratures. One should note the analogy between the proposed equation and the Liénard equation (cf., **7.4**).

7.3. Applications to the equation $\omega \equiv dp - f(p, \theta)\, d\theta = 0$. – We propose to show how Levinson's ideas permit one to establish very simply the uniqueness of a closed trajectory in certain vector fields that are defined in the $\mathbb{R}^2$ [**31**].

To that effect, we recall certain classical results on the integral curves of a vector field that is defined in the plane $\mathbb{R}^2$.

We suppose that the vector field E verifies the following hypotheses:

Hypothesis 1: *The field* E *admits only one singular point, namely, the origin* 0.

Under those conditions, the closed trajectories of E are arranged in the manner of concentric circles around the origin 0. Each trajectory is obviously oriented in the natural fashion by E.

Hypothesis 2: *The sense of rotation around* 0 *of a moving point that describes a closed trajectory of* E *in the positive sense does not depend upon the particular choice of that closed trajectory.*

Under those conditions, one can state the following classical result:

Lemma 4:

Let C and C′ be two closed trajectories of E *that verify the following conditions:*

a. There exists no closed trajectory between C and C′ that is either stable or unstable.

b. C is stable (unstable, resp.).

Under those conditions C′ will not be stable (unstable, resp.).

Let (ρ, θ) be a polar coordinate system with its pole at O in the plane $\mathbb{R}^2$. Consider the differential equation:

$$(1) \qquad \omega \equiv d\rho - f(\rho, \theta)\, d\theta = 0, \qquad \text{in which} \qquad f(\rho, \theta + 2\pi) = f(\rho, \theta)\,,$$

and the corresponding vector field whose components along the polar axes are $(-f, 1)$. It is clear that this field E verifies Hypotheses **1** and **2**.

The exterior differential of ω is:

(2) $$d\omega = f_\rho(\rho, \theta)\, d\rho \wedge d\theta\,;$$

hence:

(3) $$\bar{\omega} = -f_\rho(\rho, \theta)\, d\theta$$

and

(4) $$d\bar{\omega} = -f_{\rho^2}(\rho, \theta)\, d\rho \wedge d\theta\,.$$

One should point out that in the particular case of $f_{\rho^2} = 0$, the form $\bar{\omega}$ will be closed; equation (1) will then be very easy to discuss.

We examine the case in which f_{ρ^2} keeps a constant sign in the entire plane ($f_{\rho^2} > 0$, or $f_{\rho^2} < 0$).

Theorem 1:

If the function $f_\rho(\rho, \theta)$ *keeps a constant sign in the plane* $\mathbb{R}^2$ *then equation* (1) *will admit at most three closed trajectories. Moreover,* (1) *admits at most one closed trajectory C for which* $I(C) > 0$ [$I(C) = 0$ and $I(C) < 0$, resp.].

Indeed, let C be a closed trajectory of (1) such that $I(C) = 0$. If C' is another closed trajectory of (1) then Stokes's formula will show that:

(5) $$I(C) - I(C') = \int_\Delta d\bar{\omega} \neq 0,$$

in which Δ is the annulus that is bounded by C and C' and endowed with a convenient orientation. Hence, $I(C') \neq 0$. Let $\varepsilon(C')$ be the function that takes the value + 1 or – 1 according to whether $I(C') > 0$ or $I(C') < 0$. Let C'' be a closed trajectory that is distinct from C and C'. One lets $\varepsilon(C', C)$ denote the function that takes the value + 1 or – 1 according to whether C' is interior to C'' or exterior to it, resp. When Stokes's formula is applied to the domain that is bounded by C' and C'', that will permit one to establish the following property:

(6) $$\varepsilon(C') \cdot \varepsilon(C', C'') \cdot \varepsilon(f) > 0 \quad \text{implies that} \quad \varepsilon(C) \cdot \varepsilon(C') > 0\,,$$

in which:

$$\varepsilon(f) = +1 \quad \text{if} \quad f_{\rho^2} > 0 \qquad \text{and} \qquad \varepsilon(f) = -1 \quad \text{if} \quad f_{\rho^2} < 0.$$

However, it is impossible to have $\varepsilon(C') \cdot \varepsilon(C'') > 0$ for all closed trajectories C'' that verify $\varepsilon(C', C'')$ = constant. (It will suffice to apply Lemma **4**.) Hence:

(7) $$\varepsilon(C') \cdot \varepsilon(C', C'') \cdot \varepsilon(f) < 0.$$

Similarly:

(8) $$\varepsilon(C'') \cdot \varepsilon(C'', C') \cdot \varepsilon(f) < 0.$$

Hence, $\varepsilon(C'', C') < 0$, which completes the proof.

Remarks:

1. Consider the class Δ of functions $f(r, \theta)$ that verify the following condition:

The function $f(r, \theta)$ will verify the inequality $-f_{\rho^2} \leq \rho \cdot \varepsilon$ for a given $\varepsilon > 0$ in the circular annulus D (which is defined by the inequality $0 < m < \rho < M$, in which M and m are given).

Let C_1, C_2, … be the closed trajectories of equation (1) that are contained in D and have a positive characteristic exponent $-I(C)$. [One supposes that $f(\rho, \theta)$ belongs to the class Δ.] There are a finite number of those trajectories, so one can suppose that C_i is interior to C_{i+1}. There then exists a closed trajectory Γ_i between C_i and C_{i+1} such that $I(\Gamma_i) \geq 0$. When Stokes's theorem is applied to the trajectories Γ_i and C_i, that will show that:

$$|I(C_i)| = -I(C_i) \leq I(\Gamma_i) - I(C_i) = -\int_{\Delta_i} \left(\frac{1}{\rho} f \rho^2\right) \rho \, dr \, dq \leq \varepsilon \cdot A_i ,$$

in which A_i denotes the area of the annulus Δ_i that is bounded by C_i and Γ_i.

Hence:

$$\text{area of } D = A \geq \sum_i A_i \geq \sum_i I(C_i)/\varepsilon \qquad \text{because} \qquad A_i \geq I(C_i)/\varepsilon .$$

Let $I = \inf_i |I(C_i)|$, and let n be the number of closed trajectories C_i :

$$\boxed{A \geq \frac{nI}{\varepsilon}.}$$

The latter inequality shows that if the trajectories C_i are strongly stable then the number of those trajectories will be small.

It goes without saying that inequalities that are analogous to the preceding one can be proved for equations of a more general type than the proposed equation.

2. Recall the form $\omega = \cos \Psi\, dx + \sin \Psi\, dy$ (cf., Remark **1** in **7.2**). We suppose that Ψ is defined in the entire plane (x, y), except for the point 0.

Let $\varphi(x, y)$ be the polar angle of the radius vector whose origin is at 0 and whose extremity is (x, y). One easily verifies that if:

$$\Psi = \varphi + \log \sqrt{x^2 + y^2}$$

then the form ω will enjoy the following properties:

a. There exists an infinitude of closed trajectories that verify $\omega = 0$; the set of those trajectories admits the point (0, 0) as its accumulation point.

b. $\Delta\Psi = 0$.

The results that were stated above (Theorem **1** and Remark **1**) will therefore no longer be valid when Hypothesis **1** is not verified.

7.4. Applications to the Liénard equation. – In this paragraph, we shall once again use the method of N. Levinson (cf., **7.3**).

Before discussing the Liénard equation:

$$x'' + f(x)\, x' + g(x) = 0, \tag{9}$$

let us make some very simple remarks about the analogous equation ([1]):

$$x'' + \varepsilon(x')\, f(x)\, x'^2 + g(x) = 0, \tag{10}$$

in which:

$$\varepsilon(x') = +1 \quad \text{if } x' > 0 \qquad \text{and} \qquad \varepsilon(x') = -1 \quad \text{if } x' < 0 .$$

(Cf., **7.2.**, Remark **2**, Example **2**) Equation (10) has a mechanical interpretation that is analogous to that of (9).

The equation:

$$\omega = y\, dy + [\varepsilon(y)\, f(x)\, y^2 + g(x)]\, dx = 0 \tag{11}$$

that is associated with (10) verifies Hypothesis **1** of **7.3**.

$$\bar{\varpi} = -2\, \varepsilon(y)\, f(x)\, dx \qquad \text{for} \qquad y \neq 0. \tag{12}$$

The form $\bar{\varpi}$ is a closed form for $y \neq 0$.

([1]) Contrary to our conventions, the function $\varepsilon(x')$ is not indefinitely differentiable; meanwhile, our statements are easy to justify.

We make the following natural hypotheses on (10):

Hypothesis 2:

a. $f(x) = 0$ if $x = \alpha$ or $x = \beta$, where $\alpha < 0 < \beta$.

b. $f(x) < 0$ if $\alpha < x < \beta$.

c. $f(x) > 0$ if $x < \alpha$ or $x > \beta$.

d. *Each closed trajectory C of* (11) *meets the axes* $x = \alpha$ *and* $x = \beta$.

One should note that a closed trajectory of (11) necessarily meets one of the axes $x = \alpha$ and $x = \beta$. Indeed, the integral of $d\omega = 2 f(x)\, \varepsilon(y)\, y\, dx \wedge dy$ over the domain that is bounded by C must be zero, from Stokes's formula. The condition (*d*.) in Hypothesis **2** is then verified when $f(x)$ is an even function and $g(x)$ is an odd function, because in that case, equation (11) will be invariant under the symmetry $x \to -x$ and $y \to -y$.

Suppose that (11) admits closed trajectories. Let C be the closed trajectory of (11) that contains no other closed trajectory in its interior. C is not unstable, because (0, 0) is an unstable stationary position. Therefore, $I(C) \leq 0$. Let C' be another closed trajectory of (11), and let u, v [u', v', resp.] be the abscissa of the points where C [C', resp.] meets the $y = 0$ axis, with $u < u' < 0 < v < v'$, so:

$$I(C') = -\,4\int_{v}^{u} f(x)\, dx, \qquad I(C'') = -\,4\int_{v'}^{u'} f(x)\, dx.$$

Hence, when one takes Hypothesis **2** into account, one will see that $I(C) > I(C')$. Hence:

Theorem 2:

If equation (10) *verifies Hypothesis* **2** *then that equation will admit at most a single stable periodic solution and possibly one periodic solution whose characteristic exponent is zero.*

Let us now examine the Liénard equation (9). Introduce the variable:

$$x' + F(x) = u, \qquad \text{in which} \qquad F(x) = \int_0^x f(\xi)\, d\xi.$$

Equation (9) then comes down:

$$\omega \equiv g(x)\, dx + [u - F(x)]\, du = 0. \tag{13}$$

Therefore:

$$d\omega = -f(x)\, dx \wedge du,$$

and one can set:

$$\bar{\varpi} = f(x) / g(x)\, du, \quad \text{if} \quad g(x) \neq 0,$$

and

$$\bar{\varpi} = -[f(x) / (u - F(x))]\, du, \quad \text{if} \quad [u - F(x)] \neq 0.$$

Hypothesis 3:

a. $x \cdot g(x) > 0$ if $x \neq 0$; $g(0) = 0$.

b. The function $f(x)$ *is annulled for two values* x_1 *and* x_2 *of the variable* x *such that:*

$x_1 < 0 < x_2$; $f(x) < 0$ *if* $x_1 < x < x_2$ *and* $f(x) > 0$ *if* $x < x_1$ *or* $x > x_2$.

c. Each closed trajectory C_i *of* (3) *meets the axis* $x = x_2$ *at* P_i *and* S_i *and the axis* $x = x_1$ *at* Q_i *and* R_i *; those points follow in the order* P_i , Q_i , R_i , S_i *on* C_i .

One remarks that the condition (*c.*) is necessarily verified when the function $f(x)$ is even and $g(x)$ is odd. (Cf., the preceding example.) If (13) admits closed trajectories then there will exist a closed trajectory C_1 that contains no other closed trajectory in its interior (indeed, the origin is an unstable focus). It will then result that $I(C_1) \leq 0$. Let C_2 be another closed trajectory of (13). Let $I(P_i\, Q_i)$, … denote the integrals of $\bar{\varpi}$ along the arcs PQ :

$$\text{(14)} \qquad \begin{aligned} I(P_2\, Q_2) &< I(P_1\, Q_1), \\ I(R_2\, S_2) &< I(R_1\, S_1)\, . \end{aligned}$$

Indeed, one can verify the first of these inequalities by applying Stokes's formula to the form $\bar{\varpi} = \dfrac{-f(x)}{u - F(x)}\, dx$ and to the contour L that is composed of the arcs $P_2\, Q_2$ and $P_1\, Q_1$ (of C_1 and C_2) and some rectilinear segments $P_2\, P_1$ and $Q_2\, Q_1$ upon remarking that the integral of $d\bar{\varpi} = \dfrac{[u - F(x)]^2}{-f(x)}\, dx \wedge du$ over the interior of L is negative. (One verifies that $[u - F(x)] \neq 0$ in the useful region.]

In order to establish the analogous inequalities:

$$\text{(15)} \qquad \begin{aligned} I(Q_2\, R_2) &< I(Q_1\, R_1), \\ I(S_2\, P_2) &< I(S_1\, P_1)\, , \end{aligned}$$

it is convenient to set:

$$\bar{\varpi} = -f(x) / g(x)\, du = -\Psi(x)\, du, \quad \text{in which} \quad \Psi(x) = +f(x) / g(x) \quad (x \neq 0).$$

Consider the contour $F = Q_2\, R_2\, R_1\, Q_1$, which is composed of the arcs $Q_2\, R_2$ and $Q_1\, R_1$ and the segments $R_2\, R_1$ and $Q_1\, Q_2$. Stokes's formula will show that:

$$\int_F \bar{\omega} = \int_\Delta d\bar{\omega},$$

in which Δ is the area that is bounded by *F*. Now:

$$d\bar{\omega} = -\Psi(x)\, dx \wedge du .$$

The inequalities (15) will then be valid when $\psi'(x) > 0$ (for $x \neq 0$, or even better, for $x < x_1$ or $x > x_2$). It results from (14) and (15) and the fact that $I(C_1) \leq 0$ that $I(C_2) < 0$. Upon taking into account some results that were obtained above, one will finally see that C_1 is the only closed trajectory except for possibly when $I(C_1) = 0$. In the latter case, there will possibly be yet another closed trajectory. One can summarize these results as:

Theorem 3:

If equation (13) *verifies Hypothesis* ***3****, and if the function* $\psi(x) = f(x) / g(x)$ *verifies* $\psi'(x) > 0$ *for* $x < x_1$ or $x > x_2$ *, moreover, then equation* (9) *will admit at maximum one stable periodic solution, and it will admit no unstable periodic solution.*

One should note that if $f(x) = \varepsilon(x^2 - 1)$, $g(x)$ (viz., the Van der Pol equation) then $\psi(x) = x - 1/x$ and $\psi'(x) = 1 + 1/x^2 > 0$. One then recovers that classical result.

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CHAPTER VIII

ON THE NATURE AND DISTRIBUTION OF THE PERIODIC TRAJECTORIES OF CERTAIN DYNAMICAL SYSTEMS

8.1. Introduction.

8.1.1. The dynamical systems that are envisioned in this article: A dynamical system is the pair (V_n , E) that consists of an n-dimensional numerical manifold and a vector field E that is defined on V_n . Here, we shall consider some particular dynamical systems (P.D.S.) that verify the following properties:

a. The field E *admits no singularities. The field* E *admits a finite number of closed trajectories* C_i ($i = 1, \ldots, N$). *The characteristic exponents that relate to any one of those closed trajectories are all different and their real parts are non-zero. The qualitative behavior of the trajectories of* E *in the neighborhood of* G_i *is the behavior that one will observe upon replacing* E *with the differential system that is defined by the equations of first variation.*

b. Let $x(t, x_0)$ (in which $x, x_0 \in V_n$) *be the trajectory of* E *that passes through the point* x_0 *at the instant* $t = 0$. *For any open neighborhood* Ω *of the set* $\bigcup_i C_i$ *, there exists a* $T(x_0) > 0$ *such that:* $x(t, x_0) \in \Omega$ *for* $|t| > T$.

8.1.2. Objective of this chapter. – The hypotheses (*a.*) and (*b.*) in **1.1** do not seem very natural *a priori*. One hardly knows any criteria that will permit one to recognize whether a given dynamical system (V_n, E) will verify those hypotheses. Meanwhile, it is clear that one knows numerous dynamical systems (i.e., nonlinear mechanical ones) that verify properties (*a.*) and (*b.*), and the physical behavior of such systems is particularly simple and agreeable (e.g., the absence of recurrent motions other than periodic motions). We propose to show that the distribution of periodic motions of such a system will obey some simple laws.

We remark that the known laws of the distribution of periodic motions are, for the most part, coupled with some theorems on the fixed points of a topological transformation. They are valid by means of some hypotheses on the topology of V_n (for example: V_n is a topological product $V_{n-1} \times S$ of a manifold V_{n-1} with a circle S, or rather: V_n admits the structure of a fiber bundle whose fiber is S) and on the field E (for example: the field E has a non-zero component along the fibers of V_n). In our article, we make the hypotheses (*a.*) and (*b.*), but no other hypotheses of the same type as the ones that we just recalled.

Later on (**8.5**), we shall recall the justification for that study.

8.2. Preliminaries. Definition of a P.D.S.

Definition: A dynamical system (V_n , E) will be called a *P.D.S.* if:

1. The phase space V_n is compact.

2. (V_n , E) enjoys the properties (*a*.) and (*b*.) in **1.1**.

8.2.1. On the closed trajectories of a P.D.S. – Consider a closed trajectory C_i of a P.D.S. (V_n , E). We let $\alpha_1^i, \ldots, \alpha_{n-1}^i$ denote the $n - 1$ characteristic exponents of C_i and let $\beta_1^i, \ldots, \beta_{n-1}^i$ denote the real parts of $\alpha_1^i, \ldots, \alpha_{n-1}^i$. We can suppose:

$$\beta_1^i \le \beta_2^i \le \ldots \beta_{s_i}^i < 0 < \beta_{s_i+1}^i \le \ldots \le \beta_{n-1}^i .$$

The whole number s_i is called the *character* of C_i , and $(-1)^{s_i} = \gamma_i$ is called the *index* of C_i.

In this paragraph, we consider a particular trajectory C and thus suppress the index i. The hypothesis (*a*.) of **1.1** on the closed trajectory C is stated precisely as follows:

Hypothesis: *There exists a neighborhood Ω of C that is homeomorphic by a map h to the topological product* $\mathbb{R}^{n-1} \times S$ *of the* $(n - 1)$*-dimensional numerical space* $\mathbb{R}^{n-1}$ *with the circle S. The function h enjoys the following properties:*

The map h maps the trajectories of E *onto the curves that are defined by the equations:*

$$x = \exp A\theta,$$

in which $x \in \mathbb{R}^{n-1}$*, θ is the abscissa (defined modulo* 1) *on S, and A is a linear transformation whose characteristic roots are* $\alpha_1 , \ldots, \alpha_{n-1}$.

It is clear that h maps C to the trajectory $x = 0$.

The transformation A is completely reducible to the sum of two transformations A_1 and A_2 that operate on two complementary subspaces $\mathbb{R}_1^s$ and $\mathbb{R}_2^{n-1-s}$ of $\mathbb{R}^{n-1}$. Moreover, the characteristic roots of A_1 are $\alpha_1 , \ldots, \alpha_s$, and those of A_2 are $\alpha_{s+1}, \ldots, \alpha_{n-1}$.

Since:

$$A = A_1 \oplus A_2 ,$$

$$\exp A = \exp A_1 \oplus \exp A_2 .$$

Finally, one can endow $\mathbb{R}^n$ with a Euclidian metric such that the subspaces $\mathbb{R}_1^s$ and $\mathbb{R}_2^{n-1-s}$ are orthogonal and such that:

$$\| (\exp A_1)\, x \| \leq (1 - \varepsilon) \| x \| \qquad \text{if} \qquad x \in \mathbb{R}_1^s$$

$$\| (\exp A_2)\, x \| \geq (1 - \varepsilon) \| x \| \qquad \text{if} \qquad x \in \mathbb{R}_2^{n-1-s},$$

in which $\| u \|$ denotes the norm of u and ε ($\varepsilon > 0$) do not depend upon x.

Definition: Set:

$$P_1(C) = h^{-1}(\mathbb{R}_1^s \times S),$$
$$P_2(C) = h^{-1}(\mathbb{R}_1^{n-1-s} \times S).$$

8.2.2. On the canonical decomposition of the defining space of a P.D.S. – Let ρ denote the equivalence relation that is defined in V_n whose equivalence classes correspond to the trajectories of E. Suppose that (V_n , E) is a P.D.S. Let C_i be one of the closed trajectories of E. The spaces $P_1(C_i)$ and $P_2(C_i)$ are not defined in a canonical fashion; however, the following properties are obvious:

Definition 1: Let $Q_1(C_i)$ and $Q_2(C_i)$ be the sets that are obtained by saturating $P_1(C_i)$ and $P_2(C_i)$ for the equivalence relation ρ.

Proposition:

The sets $Q_1(C_i)$ and $Q_2(C_i)$ depend upon only (C_i, E, V_n). The sets $Q_1(C_i)$ and $Q_2(C_i)$ are homeomorphic to $\mathbb{R}_1^{s_i} \times S$ and $\mathbb{R}^{n-s_i-1} \times S$, respectively.

For what follows, it will be convenient to enumerate the trajectories C_i in a different way; to that effect, we set:

Definition 2: One divides the trajectories C_i into n classes, where each class Γ_j ($j = 0, \ldots, n-1$) includes all of the trajectories G_i that have a given character $s_j = j$; one enumerates the trajectories of the same class (arbitrarily). An arbitrary closed trajectory will be defined by its character j and its number i ; one then writes C_{ji} . The index j varies from 0 to $n-1$, and i varies from 1 to N_j , where N_j is the number of trajectories of character j .

Definition 3: One sets:

$$L_i = \bigcup_j Q_1(C_{ij}),$$

$$K_n = V_n ,$$

$$K_{n-1} = K_n - L_{n-1} ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$K_{r-1} = K_r - L_{n-1}\,,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$K_0 = K_1 - L_0\,.$$

One easily sees that $K_0 = \theta$, L_r is an open subset in K_{r+1}, and K_r is at most r-dimensional.

8.3. Some examples of P.D.S.'s. –

8.3.1. Dynamical systems that are defined in the plane $\mathbb{R}^2$. – Suppose then that $V_n = \mathbb{R}^2$, and let E be a vector field in $\mathbb{R}^2$ that admits a unique singular point 0 whose coordinates are $x = 0$, $y = 0$.

Under those conditions, the trajectories of E will either be closed trajectories C_i that surround the origin 0 or curves that unroll like spirals around those orbits. Hypotheses (*a*.) and (*b*.) in **1.1** imply the following properties:

1. There are a finite number of trajectories C_i .

2. Each trajectory C_i is completely stable or completely unstable.

A dynamical system that verifies the preceding two properties is not a P.D.S. (because V_n is not compact, and E admits a singular point), but we will see later on that it is possible to extend the properties of P.D.S.'s to such dynamical systems.

Let C_1 and C_2 be two completely-stable closed trajectories of E. Let s be the number of completely-stable closed trajectories between C_1 and C_2 , and let i be the number of completely-unstable closed trajectories between C_1 and C_2 . The following relation is obvious:

$$\boxed{i - s = 1.}$$

We propose to establish some relations of that type for P.D.S.'s [Cf., **8.4.3** (2) and **8.4.4**, Theorems **1** and **2**].

One will then be led to consider a P.D.S. when one considers a vector field without singularities E that is defined on the two-dimensional torus T^2. If E admits a (non-zero) finite number of closed trajectories, and if each of those trajectories is completely stable or completely unstable then (T^2, E) will be a P.D.S. With the preceding notations, one can write the following relation under those conditions:

$$\boxed{i - s = 0.}$$

8.3.2. Examples of P.D.S.'s. – The (three-dimensional) sphere S_3 can be considered to be a fiber space whose base is the two-dimensional sphere S_2 and whose fiber S is isomorphic to the one-dimensional torus T (T is the multiplicative group of complex numbers modulo 1).

Let P be the canonical projection of S_3 onto S_2. Let U be a vector field with no irregularities on S_3 that is tangent to the fibers of S_3. Finally, let V be a vector field on S_2 that has a finite number of singular points x_i ($i = 1, \ldots, s$) of the saddle or focus type, and enjoys the following property:

Any trajectory of V will tend to a singular point x_i (for $t \to +\infty$ and $t \to -\infty$).

There exists a field W on S_3 such that P (W) = U. Set E = U + W. It is clear that S_3, E) defines a P.D.S.

Let:

$$N_2 = \text{number of stable foci (or nodes)}$$

$$N_0 = \text{number of unstable foci (or nodes)}$$

$$N_1 = \text{number of saddles.}$$

Under those conditions:

$$N_2 - N_1 + N_0 = 2.$$

There will then exist a particular P.D.S. on S_3 that admits a single completely-stable periodic trajectory and a single completely-unstable periodic trajectory, with the exclusion of any other type of closed trajectory. It is remarkable that this phenomenon cannot be produced on S_{2q+1} when $q \geq 2$ (cf., **8.4.4**).

It is easy to give some examples of P.D.S.'s whose phase spaces are the n-dimensional sphere S_n (n odd).

8.3.3. Systems of q oscillators. – The study of a dynamical system that is composed of q oscillators frequently (in the case of relaxation) leads to the study of a dynamical system (Δ_{2q+1}, E), where Δ_{2q+1} is a manifold with boundary that is homeomorphic to the topological product $S_{2q+1} \times I$ of the $(2q + 1)$-dimensional sphere S_{2q+1} with the closed interval $I = [0, 1]$, and in which E is a vector field without singularities on Δ_{2q+1} whose restriction to the boundary of Δ_{2q+1} points to the interior of Δ_{2q+1}.

The study of dynamical systems of the type that verify properties (a.) and (b.) in **8.1** is therefore particularly important from the physical viewpoint.

8.4. On the distribution of periodic solutions of a P.D.S. –

8.4.1. On the homology of K_r / K_{r-1}. – In all of what follows, we shall let $\mathcal{H}^i(K_r)$ denote the cohomology group of K_r in dimension i when the coefficient ring is the ring of integers. The dimension of the Betti group of $\mathcal{H}^i(K_r)$ will be denoted by $p_i(K_r)$. Finally, if one takes the domain of the coefficients modulo 2 then one will denote the dimension of the Betti group in dimension i by $b_i(K_r)$.

In order to find the cohomology of K_r / K_{r-1} , it will suffice to note that $K_r - K_{r-1} = L_{n-1}$ is homeomorphic to the sum of N_{r-1} spaces that are homeomorphic to $\mathbb{R}^{r-1} \times S$. Now, $\mathbb{R}^{r-1}$ can be completed to an $(r - 1)$-dimensional topological sphere Σ_{r-1} by adding a point at infinity ω . The cohomology group of K_r / K_{r-1} will then be isomorphic to the cohomology group of:

$$\Sigma_{r-1} \times S / \{\omega\} \times S = (\Sigma_{r-1} / \{\omega\}) \times S.$$

It will then result that the torsion groups of $\mathcal{H}^i(K_r / K_{r-1})$ are all zero and that the Poincaré polynomials:

$$\sum_i p_i(K_r / K_{r-1}) X^i \qquad \text{or} \qquad \sum_i b_i(K_r / K_{r-1}) X^i$$

are equal to:

$$N_{r-1}(X^r + X^{r-1}) \equiv P_r(X).$$

In order to obtain the desired laws, it will now suffice to write down the well-known classical relation between the cohomology groups of the spaces K_r , K_{r-1}, and K_r / K_{r-1} . We shall look for relations between the numbers N_i (viz., the number of closed trajectories of character i) and the cohomology of V_n .

8.4.2. Review of the classical relations between the cohomologies of the spaces K_r, K_{r-1} , **and** K_r / K_{r-1} . – Those relations can be summarized thus:

There exist canonical homomorphisms I_i of $\mathcal{H}^i(K_r)$ into $\mathcal{H}^i(K_{r-1})$, and then ∂_i takes $\mathcal{H}^i(K_{r-1})$ into $\mathcal{H}^{i+1}(K_r / K_{r-1})$ and φ_i takes $\mathcal{H}^i(K_r / K_{r-1})$ into $\mathcal{H}^i(K_r)$, such that the sequence of homomorphisms:

$$\ldots \xrightarrow{\partial_{i-1}} \mathcal{H}^i(K_r / K_{r-1}) \xrightarrow{\varphi_i} \mathcal{H}^i(K_r) \xrightarrow{I_i} \mathcal{H}^i(K_{r-1}) \xrightarrow{\partial_i} \mathcal{H}^{i+1}(K_r / K_{r-1}) \xrightarrow{\varphi_{i+1}} \ldots$$

is exact (i.e., the image of each of the homomorphisms in the sequence is equal to the kernel of the following homomorphism).

One lets $\mathcal{H}'^i(K_r / K_{r-1})$, $\mathcal{H}'^i(K_r)$, and $\mathcal{H}''^i(K_{r-1})$, denote the images of ∂_{i-1} , φ_i , and I_i , respectively.

We now propose to exhibit some consequences of those relations.

8.4.3. A general formula. – With some obvious notations, the exact sequence in **4.2** will give the following relations between the Betti numbers:

$$a. \qquad p'_i(K_r) + p''_i(K_{r-1}) = p_i(K_r) ,$$

(1) $\quad b. \qquad p_i'(K_r / K_{r-1}) + p_i''(K_r) = p_i\,(K_r / K_{r-1})$,

$\qquad c. \qquad p_{i+1}'(K_r / K_{r-1}) + p_i''(K_{r-1}) = p_i\,(K_{r-1})$,

from which, it results that:

$$p_i\,(K_r / K_{r-1}) + p_i\,(K_{r-1}) - p_i\,(K_r) = p_i\,(K_r / K_{r-1}) + p_{i+1}\,(K_r / K_{r-1}) \geq 0.$$

Upon adding corresponding sides of the preceding n inequalities that are obtained by varying r from n to 0, one can write (taking the preceding paragraphs into account):

(2) $$\boxed{N_i + N_{i-1} - p_i(V_n) \geq 0.}$$

Obviously, one has the analogous relation between the Betti numbers (mod 2):

$$N_i + N_{i-1} - b_i\,(V_n) \geq 0.$$

These inequalities have a striking analogy with the inequalities of M. Morse, which are concerned with the distribution of critical points of a numerical function (cf., **8.5.2**).

8.4.4. Particular case in which V_n is a homology sphere:

Lemma 1:

If $p_q\,(K_r) = 0$ *for* $1 \leq q < r$ *(r fixed), and if* $p_r\,(K_r) \neq 0$ *then* $p_{r-2}\,(K_{r-2}) \neq 0$ *and* $p_{r-2}(K_{r-2}) = 0$ *when* $1 \leq q - 2 < r - 2$.

Proof:

Indeed, it results from formulas (2) of **4.3** that:

$$p_{q-2}\,(K_{r-1}) = p_{q-1}'(K_r / K_{r-1}) + p_{q-2}''(K_{r-1}) \qquad \text{[cf., (}c\text{.)].}$$

Now:

$$p_{q-1}\,(K_r / K_{r-1}) = 0 \qquad \text{if} \qquad q - 1 < r - 1$$

and:

$$p_{q-2}''(K_{r-1}) = p_{q-2}\,(K_r) - p_{q-1}'(K_r) = 0 \qquad \text{if} \qquad 3 \leq q < r$$

Hence:

$$p_{q-2}\,(K_{r-1}) = 0.$$

One likewise shows that:

$$p_{q-2}\,(K_{r-2}) = 0 \qquad \text{if} \qquad 1 \leq q - 2 < r - 2 .$$

Now, suppose that $p_{r-2}\,(K_{r-2}) = 0$. It will result from (1). c. that:

$$p'_{r-1}(K_r / K_{r-1}) = 0,$$

and as a result:

$$p'_{r-1}(K_r) = p_{r-1}(K_r / K_{r-1}) = N_{r-1},$$

but $p_{r-1}(K_r) = 0$, by hypothesis, so $N_{r-1} = p_{r-1}(K_r / K_{r-1}) = 0$.

Now, $p_{r-1}(K_r / K_{r-1}) = p_r(K_r / K_{r-1})$, so another application of (1).*b* will give $p'_r(K_r)$ = 0, but from (1).*c*, $p''_r(K_{r-1}) = p_r(K_{r-1})$, but dim $(K_{r-1}) \leq r - 1$, so $p_r(K_{r-1}) = 0$, and due to (1).*a*, one will arrive at a contradiction; thus:

$$p_r(K_r) = 0.$$

Consequence of Lemma 1:

If V_n is a homology sphere (n odd) then the Betti numbers:

$$p_n(V_n),\ p_{n-2}(K_{n-2}),\ \ldots,\ p_{n-2q}(K_{n-2q}),\ \ldots,\ p_1(K_1)$$

will all be non-zero.

Now, (2) shows that if $p_r(V_r) \neq 0$ then $p'_r(K_r) \neq 0$, and it will result from (1).*b* that $p_r(K_r / K_{r-1}) \neq 0$.

Theorem 1:

If V_n *is a homology sphere, and if* (V_n , E) *is a* P.D.S. *then the numbers* N_r *with odd index* r *will be non-zero.* [*In other words, there exists at least one closed trajectory that has a given even character,* i ($i \leq n - 1$).]

Corollary:

If V_n *is a homology sphere, and if* (V_n , E) *is a* P.D.S. *then the number of closed trajectories will be greater than or equal to* $(n + 1) / 2$.

The same method will permit one to prove the following property (which we shall state without proof):

Theorem 2:

If V_n *is a three-dimensional (homology) sphere, and if* (V_3 , E) *is a* P.D.S. *then:*

$$\boxed{N_2 - N_1 + N_0 = 2.}$$

8.5. Remarks on the nature and distribution of the trajectories of a dynamical system. –

8.5.1. Remarks on hypotheses (*a*.) **and** (*b*.) **in 1.1. –** One will be led naturally to consider dynamical systems that verify hypotheses (*a*.) and (*b*.) in **1.1**. However, it is appropriate to point out the following reservation: The hypothesis that was made about the qualitative behavior of the neighboring trajectories to a closed trajectory is neither natural nor consistent with the properties of the integral curves of a vector field. However, if we suppose that the characteristic exponents of a trajectory C_i are all distinct and have non-zero real parts then it will be possible to replace the field E with a field E′ that verifies the following properties:

1. – C_i is a trajectory of E′ and the characteristic exponents of C_i that relate to E′ are the same as the characteristic exponents that relate to E.

2. – The fields E and E are identical, except in every small neighborhood of C_i .

That remark justifies the consideration of P.D.S.'s. Finally, we point out that the results of **4.4** are "stable."

8.5.2. Another class of dynamical systems. – Basically, we know some dynamical systems that are even simpler, and in a certain sense more interesting, than the P.D.S.'s.

They amount to the dynamical systems (V_n , E) that verify the following properties:

1. – *The field* E *admits a finite number of singular points* (which have characteristic exponents that are all distinct and have non-zero real parts).

2. – *All of the trajectories of* E *converge to a singularity of* E.

One can study such dynamical systems by some procedures that are analogous to the ones that we just studied. Such a study is virtually carried out in [**49**] by Thom. Thom's results completely clarify the analogies with the M. Morse's theory of critical points.

8.5.3. On a general problem. – We remark that the preceding study shows that a dynamical system (V_n , E) (where V_n is compact) that does not admit "enough" periodic motions will admit certain stable motions, *à la* Poisson, of a type that is more complicated than the periodic motions. That is why we consider the preceding study to be a first step into the study of a general problem that can, in our opinion, be stated in a tractable form in the following manner:

Consider a dynamical system (V_n , E) with indeterminacy [**24a**]. One then studies the relations between the topology of V_n and the distribution and nature of the "stable states" of the dynamical system envisioned.

REMARKS ADDED DURING CORRECTION OF THE PROOFS

1. The article [**53**], which we were just informed about, treats some questions that are close to the ones that are examined in Chapter VIII.

2. It is appropriate to add the following hypothesis to Hypotheses (*a*.) and (*b*.) of **8.1.1**: V_n is orientable. Indeed, if V_n were not orientable then the statements of **8.2.1** might break down. It is nonetheless clear that the study in Chapter VIII can be extended to non-orientable manifolds.

3. The statement that was made at the end of **8.2.2** that would make L_r be open in K_{r+1} is not exact. However, the only case in which L_r cannot be an open subset of K_{r+1} is the one in which there exist trajectories of E that tend to closed trajectories that are neither stable nor unstable when $t \to +\infty$ and $t \to -\infty$. Now, the latter situation is highly exceptional, and it would then be legitimate to discard it by way of a supplementary hypothesis.

Indeed, in order to give a correct proof of the results of Chapter VIII, it would suffice to make the following very simple modification: Replace the sets L_r with the sets that are obtained by removing the closed trajectories that are contained in L_r. The vexations that were pointed out above (e.g., L_r is not an open subset of K_{r+1}) would no longer be produced; on the other hand, the proofs would mimic the ones that were made exactly.