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Principal directions and invariants in an arbitrary manifold

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1. – Let:

$$\varphi = \sum_{r,s=1}^{n} a_{rs}\, dx_r\, dx_s$$

be a positive quadratic differential form, and following the notations that I constantly use, let $a^{(pq)}$ denote the coefficients of the reciprocal form, and let $a_{pr,\,qs}$ denote the Riemann symbols relative to φ. Then let V_n be an arbitrary manifold whose metric can still be defined by φ, and let [1], [2], …, [n] represent n congruences of lines that constitute an orthogonal n-tuple on V_n , while $\lambda_{1|r}$, $\lambda_{2|r}$, …, $\lambda_{n|r}$ (r = 1, 2, …, n) are their systems of covariant coordinates.

The curvature of the geodetic surface that passes through an arbitrary point P of V_n tangent to the lines of the congruence [h] and [k] has the expression:

$$R_{hk} = \sum_{r,s,t,u=1}^{n} a_{rs,tu}\, \lambda_h^{(r)}\, \lambda_k^{(s)}\, \lambda_h^{(t)}\, \lambda_k^{(u)} . \tag{1}$$

Now recall ([1]) that if one sets:

$$\gamma_{hij} = \sum_{r,s=1}^{n} \lambda_{h|rs}\, \lambda_i^{(r)}\, \lambda_i^{(s)},$$

$$\gamma_{hi,kl} = \frac{\partial \gamma_{hik}}{\partial s_l} - \frac{\partial \gamma_{hil}}{\partial s_k} + \sum_{j=1}^{n} \left\{ \gamma_{hij}(\gamma_{jkl} - \gamma_{jlk}) + \gamma_{jhl}\gamma_{jik} - \gamma_{jhk}\gamma_{jil} \right\}$$

then one will have the relations:

$$\gamma_{hi,kl} = \sum_{r,s,t,u} a_{rs,tu}\, \lambda_h^{(r)}\, \lambda_i^{(s)} \lambda_k^{(t)} \lambda_l^{(u)}, \tag{2}$$

and one will obtain:

$$R_{hk} = \gamma_{hk,\,hk} .$$

One now puts:

([1]) Cf. RICCI, "Dei sistemi di congruenze ortogonali in una varietà qualunque," Mem. R. Accad. dei Lincei (5a), v. II^0.

(3) $$\alpha_{rs} = \sum_{p,q=1}^{n} a^{(pq)} a_{pr,qs}$$

and recalls the formula:

$$a^{(pq)} = \sum_{k=1}^{n} \lambda_k^{(p)} \lambda_k^{(q)}.$$

It follows from (1) that:

$$\sum_{k=1}^{n} R_{hk} = \sum_{r,s=1}^{n} \alpha_{rs} \lambda_h^{(r)} \lambda_h^{(s)},$$

which means that one has $R_{hh} = 0$, which would result from (1). That says that if one takes an arbitrary point P of V_n, along with a direction [h], and one considers any $n - 1$ other directions that constitute an n-tuple with the latter one, which is, in turn, associated with [h], then the sum of the curvatures that relate to the $n - 1$ adjacent planes that one just considered will be constant, i.e., independent of the choice of the $n - 1$ directions that one considered. That sum shall be called the *mean curvature* of the manifold V_n at the point P and the direction [h].

Suppose that the α_{rs} reduce to canonical form, i.e., to expressions of the type:

(4) $$\alpha_{rs} = \sum_{h=1}^{n} \rho_h \lambda_{h/r} \lambda_{h/s}.$$

We shall call any n-tuple in V_n for which the α_{rs} reduce to that form a *principal n-tuple*. *Principal congruences* are the ones that constitute a principle n-tuple, and the *principal directions* around a point P of V_n are the tangents at P to the lines of the principal congruences, which the *principal invariants* are $\rho_1, \rho_2, \ldots, \rho_n$. From some well-known considerations, one has that the principal directions are the ones that correspond to the maximum or minimum curvatures and are such that those curvatures can be represented by the principal invariants that correspond to those directions.

2. – In the case of $n = 2$, if one lets K denote the Gauss invariant that relates to V_2 then one will have:

$$\alpha_{rs} = K\, a_{rs},$$

as is easy to verify.

That says that in this case any direction can be regarded as principal and that the principal invariants are all equal to the Gauss invariant.

If $n = 3$, and one regards indices to be equivalent when they are congruent modulo 3, and one sets:

$$a \cdot \beta^{(rs)} = a_{r+1\; r+2,\; s+1\; s+2},$$

$$3K = \sum_{p,q=1}^{3} a_{pq} \beta^{(pq)},$$

then one will have the formula:

$$\alpha_{rs} = 3K\, a_{rs} - \beta_{rs}\ .$$

If one has canonical expressions of the form:

$$\beta_{rs} = \sum_{h=1}^{3} \omega_h\, \lambda_{h|r}\, \lambda_{h|s}$$

for the β_{rs} then it will follow that:

$$3K = \omega_1 + \omega_2 + \omega_3\ ,$$

$$\alpha_{rs} = \sum_{h=1}^{3} (3K - \omega_h)\, \lambda_{h|r}\, \lambda_{h|s}\ .$$

When one compares that with (4), it will follow that for $n = 3$, the principal congruences that are defined above for arbitrary n will coincide with the ones that I had referred to by the same name on other occasions – and for only that case ([1]) – and (always in that case) whose principal invariants ρ_h are coupled with the ω_h (which were then called "principal Riemannian curvatures") by the relations:

(5) $$\rho_h = \omega_{h+1} + \omega_{h+2}$$

and their equivalents:

(5′) $$2\omega_h = \rho_{h+1} + \rho_{h+2} - \rho_h\ .$$

3. – Now suppose that n is arbitrary and observe that (2) is equivalent to:

(2′) $$a_{pr,\,qs} = \sum_{h,i,k,l=1}^{n} \gamma_{hi,kl}\, \lambda_{h|p}\, \lambda_{i|r}\, \lambda_{k|q}\, \lambda_{l|s}\ ,$$

from which, one gets the expressions for the α_{rs} :

$$\alpha_{rs} = \sum_{h,i,l=1}^{n} \gamma_{hi,hl}\, \lambda_{i|r}\, \lambda_{l|s}\ .$$

When that is compared to (4), that will give:

([1]) Cf., RICCI, "Sui gruppi continui di movimenti in una varietà qualunque a tre dimensioni," in the Memorie della Società Italiana delle Scienze according to (*detta dei*) XL, series IIIa, Volume XII.

(6) $$\sum_{h=1}^{n} \gamma_{hi,hi} = \rho_i ,$$

(7) $$\sum_{h=1}^{n} \gamma_{hi,hi} = 0 \qquad (l \neq i) .$$

Whereas (6) can serve to define the ρ_i , (7) characterizes the principal n-tuple of an arbitrary V_n.

4. – We propose to determine the necessary and sufficient conditions for the Riemann coefficients to be regarded as second-order minors of a symmetric determinant, i.e., for them to admit expressions of the form:

(8) $$a_{pr,\,qs} = b_{pq}\, b_{rs} - b_{ps}\, b_{qr} ,$$

in which:

$$b_{rs} = b_{sr} .$$

For example, that will be true when the fundamental form φ defines the metric of a hypersurface, in which case:

$$\Psi = \sum_{r,s=1}^{n} b_{rs}\, dx_r\, dx_s$$

will be the expression for the second fundamental form for that hypersurface, which is considered to be immersed in a Euclidian manifold of dimension $n + 1$.

If one assumes that (8) is true then one next assumes that the b_{rs} have the canonical expressions:

$$b_{rs} = \sum_{h=1}^{n} \beta_h\, \mu_{h|r}\, \mu_{h|s} .$$

In that case, it will follow from (4) that:

(9) $$a_{pr,\,qs} = \sum_{h,k=1}^{n} \beta_h\, \beta_k\, \mu_{h|p}\, \mu_{h|r} (\mu_{h|q}\, \mu_{h|s} - \mu_{h|s}\, \mu_{h|q}) ,$$

and it will then follow from this, along with (3), that:

$$\alpha_{rs} = \sum_{h=1}^{n} (B - \beta_h)\, \mu_{h|r}\, \mu_{h|s} ,$$

in which:

$$B = \sum_{i=1}^{n} \beta_i \ .$$

That then says, above all, that when if is possible for the Riemann coefficients to be expressions of the type (8), the orthogonal n-tuples of V_n that give the canonical expressions for the b_{rs} will be found among the ones that give canonical expressions for the α_{rs} . In particular, if the fundamental form φ defines a hypersurface then any n-tuple that results from its lines of curvature will be a principle n-tuple for V_n .

A comparison of (9) with (2′) will then give:

$$\gamma_{hi,\,hi} = \beta_h \, \beta_i \qquad (i \neq h) \tag{10}$$

and:

$$\gamma_{hi,\,hi} = 0 \ , \tag{11}$$

when the simple combination ($k\,j$) of the indices 1, 2, …, n is distinct from ($h\,i$). We then conclude that:

In order for the Riemann symbols of a V_n to admit expressions like (8), it is necessary and sufficient that among the principal n-tuples of V_n , there is at least one of them that satisfies (11) [in which the simple combination ($k\,j$) must be distinct from ($h\,i$)] and (10), in which $\beta_1, \beta_2, \ldots, \beta_n$ represent indeterminates.

Observe that it would result from (7) that the (11) are not all independent when one takes into account the fact that [1], [2], …, [n] is a principal n-tuple of V_n .

5. – Elsewhere ([1]), I have called a manifold V_n that can be considered to be immersed in V_{n+m} and that enjoys the property that any geodetic line in V_n is also one in V_{n+m} a *geodetic manifold*. For m = 1, let:

$$\Psi = \sum_{u,v=1}^{n} c_{uv} \, dy_u \, dy_v$$

define the metric on V_{n+1} and let $\xi_1^{(u)}, \xi_2^{(u)}, \ldots, \xi_n^{(u)}$ denote contravariant coordinate systems of n orthogonal congruences of V_n , which are considered to belong to V_{n+1} . Let $z\,(u)$ denote the coordinate of a congruence of V_{n+1} that is normal to V_n , and let $c_{uu',\,vv'}$ be the Riemann coefficients relative to Ψ. In the cited reference, I proved the equations:

([1]) Cf., RICCI, “Sulle superficie geodetiche in una varietà qualunque, etc.,” Rend. R. Accad. dei Lincei, Session on 6 June 1903.

$$\sum_{uu',vv'=1}^{n} c_{uu',vv'}\, z^{(u)}\, \xi_i^{(u')}\, \xi_j^{(v)}\, \xi_k^{(v')} = 0 .$$

Set:

$$k = i ,$$

$$\gamma_{uv} = \sum_{u',v'=1}^{n+1} c^{(u'v')}\, c_{uu',vv'} ,$$

so the equation that was referred to will give:

$$\sum_{u,v=1}^{n} \gamma_{uv}\, z^{(u)}\, \xi_j^{(v)} = 0$$

when summed over i .

The following general theorem, which was proved in the cited article, will then follow easily:

If a V_{n+1} contains a simply-infinite family of geodetic hypersurfaces V_n then their orthogonal trajectories will constitute a principal congruence in V_{n+1} .

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