"Das Jacobische Kriterium der Variationsrechnung und die Oszillationseigenschaften linearer Differentialgleichungen 2 Ordnung," Math. Ann. **68** (1910), 279-304.

The Jacobi criterion in the calculus of variations and the oscillatory properties of second-order linear differential equations.

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Introduction.

The significance of the calculus of variation for the various domains of mathematics has emerged ever more clearly in recent times. **Hilbert** has discovered the connection between the theories of differential equations, integral equations, and the calculus of variations (*) and emphasized that the last of those theories is a more general discipline than the first one.

Hilbert had resolved the question of the existence of eigenvalues and eigenfunctions of a self-adjoint second differential equation that contains a parameter by means of his new theory of linear integral equations in general. The Sturm-Liouville theory of the oscillatory properties of an ordinary differential equation of that kind, which **Bôcher** (**) gave a rigorous basis for in recent times, plays a very important tole in applied mathematics and has been revised quite a bit. One can consider such a differential equation to be a Lagrange equation for a certain variational problem with a certain quadratic auxiliary condition or with the same quadratic condition and a certain linear auxiliary condition.

However, the questions of the necessary and sufficient conditions for the occurrence of a minimum for that variational problem have not been sufficiently clarified. In particular, the Jacobi criterion and its meaning for the problem have still not been represented in a consistent way (***). The goal of the present article is to fill in that gap, *and in particular, to exhibit the connection between the Jacobi criterion and the oscillatory properties of the solutions of the differential equation*.

§ 1 contains the formulation of the minimization problem and an overview of the tools from the theory of integral equations that are necessary. In § 2, we shall use Hilbert's theory of integral equations to prove that a solution of the variational problem exists. In § 3, we will represent the

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(*) "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," Göttinger Nachrichten, first and second communication in 1904, fourth and fifth communication in 1906.

(**) Translations of the American Mathematical Society, vol. 1, no. 4, pp. 414-420.

(***) The dissertation of Robert **König** (Göttingen 1907), which treated the same topic, contains some incorrect conclusions.

extremals of our variational problem explicitly, in order to derive the oscillatory properties of the eigenfunctions of our differential equation from the Jacobi criterion in §§ 4-6. In § 7, we will give a method for developing the Jacobi determinant of our problem into a Taylor series and in that way prove certain facts that will be used in §§ 4-6.

I hope to examine the corresponding problem for two differential equations with two parameters in a second communication. I am fulfilling a pleasant duty by expressing my thanks to Herrn Professor Hilbert at this point, at whose encouragement I undertook this work.

§ 1. – Statement of the minimization problem.

Let p be a function of x that is analytic inside of the interval from $x = 0$ to $x = 1$, and that proves to be positive inside that interval, moreover. Furthermore, let q be any function of x that is analytic and nowhere-positive on that interval. The integral:

(1) $$D(u) = \int_0^1 \left\{ p(x)\left(\frac{du}{dx}\right)^2 - q(x)\,u^2(x) \right\} dx$$

will certainly take on nowhere-negative values then. One must now determine that continuously-differentiable function u of x that satisfies the boundary conditions:

(2) $$u(0) = 0\,, \qquad u(1) = 0$$

and makes $D(u)$ a minimum, while the quadratic auxiliary condition:

(3) $$\int_0^1 k(x)\,u^2(x)\,dx = 1$$

is fulfilled, in which we would like to assume that $k(x)$ is a function of x that is likewise analytic in the entire interval including the boundary points. The calculus of variations then teaches us that the desired minimal function (whose existence will be proved in § 2) must satisfy the Lagrange second-order differential equation:

(4) $$L_1(u) = \frac{d(p\,u')}{dx} + q\,u + \lambda\,k\,u = 0\,.$$

We infer the following facts from Hilbert's theory of integral equations: If $k(x)$ is an everywhere-nonnegative function on the interval $0 \leq x \leq 1$ then there will be infinitely many positive parameter values $\lambda = \lambda_1, \lambda_2, \ldots$ that accumulate merely at infinity, namely, the so-called *eigenvalues*, for which the equation $L_1(u) = 0$ will possess one and only one solution $U_1(x)$, $U_2(x)$, … that satisfies the conditions (2) and (3). When $k(x)$ is not positive in the interval, we must pose the auxiliary condition:

$$\int_0^1 k(x)u^2\,dx = -1\,, \tag{5}$$

instead of (3). There will then be infinitely-many negative simple eigenvalues with the corresponding solution of the same equation (4). If $k\ (x)$ changes its sign then there will be infinitely-many positive and infinitely-many negative simple eigenvalues $\lambda_1, \lambda_2, \ldots; \lambda_{-1}, \lambda_{-2}, \ldots$, for which $L_1\ (u) = 0$ possess solutions $U_1\ (x), U_2\ (x), \ldots; U_{-1}\ (x), U_{-2}\ (x), \ldots$ that each satisfy the conditions (2) and (3) [(5), resp.]. Those solutions are called the *eigenfunctions* of the equation $L_1(u) = 0$.

If we understand the *kernel* $K\ (x, \xi)$ to mean the Green function of the self-adjoint second-order differential expression:

$$L\ (u) = \frac{d\,(p\,u')}{dx} + q\,u$$

multiplied by $k\ (x) \cdot k\ (\xi)$ then we will have:

$$k\ (x)\ u\ (x) = \lambda \int_0^1 K\,(x,\xi)\,u\,(\xi)\,d\xi\ . \tag{6}$$

That equation will be an orthogonal or polar integral equation according to whether $k\ (x)$ has the same sign over the entire interval or not, resp. From Hilbert's existence theorem on orthogonal (polar, resp.) integral equations, there are infinitely many parameter values $\lambda_1, \lambda_2, \ldots$ ($\lambda_1, \lambda_2, \ldots; \lambda_{-1}, \lambda_{-2}, \ldots$, resp.) for which equation (6) has the solutions $U_1\ (x), U_2\ (x), \ldots$ [$U_1\ (x), U_2\ (x), \ldots; U_{-1}\ (x), U_{-2}\ (x), \ldots$, resp.], and those simple eigenvalues and eigenfunctions coincide precisely with those of the differential equation (4).

In § 2, we will show that when $k\ (x) \geq 0$, the minimum of $D\ (u)$ exists under the boundary conditions (2) and the auxiliary condition (3) and is equal to the first eigenvalue λ_1 of the Lagrange equation (4) or the integral equation (6). When $k\ (x) \leq 0$, $-\lambda_{-1}$ will be the minimum of $D\ (u)$ under the conditions (2) and (5). However, when $k\ (u)$ has both signs in the interval, either λ_1 or $-\lambda_{-1}$ will prove to be a minimum of $D\ (u)$ according to whether the condition (3) or (5), resp., was posed.

If we modify our variational problem in such a way that we impose the linear auxiliary equation:

$$\int_0^1 k\,(x)U_1(x)\,u\,(x)\,dx = 0\,, \qquad \int_0^1 k\,(x)U_{-1}(x)\,u\,(x)\,dx = 0\,, \quad \text{resp.,} \tag{7}$$

on the desired function $u\ (x)$, in in addition to the quadratic auxiliary condition (3) or (5), where $U_1\ (x)$ [$U_{-1}\ (x)$, resp.] is the solution of the previous problem, then we will be led to the new Lagrange equation:

$$L_2\ (u) = \frac{d\,(p\,u')}{dx} + q\,u + \lambda\,k\,u + \mu\,k\,U_1 = 0\ .$$

We will show (§ 2) that a minimal function $U_2(x)$ [$U_{-2}(x)$, resp.] likewise exists in this case. It is clear that it is not equal to $U_1(x)$ [$U_{-1}(x)$, resp.], since such a solution cannot satisfy the two conditions (3) [(5) and (7), resp.]. As we will see (§ 3), moreover, the Lagrange factor m must be taken to be zero, and our minimal function will also be a solution to equation (4) then. In § 2, we will show that the minimal value of the integral $D(U_2)$ [$D(U_{-2})$, resp.] is λ_2 (λ_{-2}, resp.).

If we add even more linear auxiliary conditions:

$$\left\{\begin{array}{lll} \int_0^1 k\,U_2\,u\,dx = 0\,, & \int_0^1 k\,U_{-2}\,u\,dx = 0\,, & \text{resp.,} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ \int_0^1 k\,U_n\,u\,dx = 0\,, & \int_0^1 k\,U_{-n}\,u\,dx = 0\,, & \text{resp.,} \end{array}\right. \tag{8}$$

then that will imply that all of the associated Lagrange factors are equal to zero, and we will get the eigenfunctions U_{n+1} (U_{-n-1}, resp.) of the differential equation (4) as solutions to the variational problem and λ_{n+1} (λ_{-n-1}, resp.) as the minimal values. Due to the auxiliary condition (3), (5), (7), (8), the eigenfunctions will possess either orthogonality properties or polarity properties. Every four-times-differentiable function that fulfills certain conditions at the boundary points 0, 1 and the zeroes of k can be developed into a series of those successive eigenfunctions.

Obviously, we can restrict ourselves to the treatment of a series of positive eigenvalues λ_1, λ_2, …, since will arrive at positive eigenvalues again in the case of negative eigenvalues by the transformation $k(x) = -h(x)$.

§ 2. – Proof of the existence of a minimum.

We will now prove that our minimal problem (§ 1) actually has a solution by means of Hilbert's theory of integral equations.

Theorem 1:

The integral:

$$D(u) = \int_0^1 (p\,u'^2 - q\,u^2)\,dx$$

possesses a smallest value λ_n when $u(x)$ is a continuous function that satisfies the conditions:

$$(9)\qquad \begin{cases} u(0) = u(1) = 0, \\ \int_0^1 k(x)\, u^2\, dx = 1, \\ \int_0^1 k U_1\, u\, dx = 0, \quad \int_0^1 k U_2\, u\, dx = 0, \quad \cdots \quad \int_0^1 k U_{n-1}\, u\, dx = 0. \end{cases}$$

It will be assumed for $u(x) = \pm U_n(x)$.

The solutions $u(x) = U_1(x), U_{-1}(x), U_2(x), U_{-2}(x), \ldots$ to the integral equation (6) in § 1 satisfy the polarity properties:

$$(10)\qquad \begin{cases} \int_0^1 k(x) U_p U_q = 0, & p \neq q \\ \int_0^1 k(x) U_n^2 = 1, \qquad \int_0^1 k(x) U_{-n}^2 = -1, & n = 1, 2, 3, \cdots \end{cases}$$

They will also satisfy the differential equations:

$$(11)\qquad L(U_i) + \lambda_i\, k(x)\, U_i \equiv (p U_i')' + q U_i + \lambda\, k(x) U_i = 0, \quad i = 1, 2, \ldots; -1, -2, \ldots$$

We would now like to develop $u(x)$ in the eigenfunctions $U_1, U_{-1}, U_2, U_{-2}, \ldots$ (*).

$$u(x) = c_1 U_1(x) + c_{-1} U_{-1}(x) + c_2 U_2(x) + c_{-2} U_{-2}(x) + \ldots,$$

$$c_i = \int_0^1 k(x) U_i\, u\, dx, \qquad i = 1, 2, 3, \ldots; -1, -2, -3, \ldots$$

When one recalls (9), one will then have:

$$(12)\qquad u(x) = c_n U_n(x) + c_{n+1} U_{n+1}(x) + \ldots + c_{-1} U_{-1}(x) + c_{-2} U_{-2}(x) + \ldots,$$

and therefore, from (9) and (10):

(*) As one easily convinces oneself, that process is permissible, since one can regularly approximate every arbitrary continuously-differentiable function, along with its derivative, by functions that admit a development into a series in the successive twice-differentiable eigenfunctions $U_1, U_{-1}, U_2, U_{-2}, \ldots$

(13) $$1 = \int_0^1 k\,u^2\,dx = c_n^2 + c_{n+1}^2 + \cdots - c_{-1}^2 - c_{-2}^2 - \cdots$$

From (12) and (11), that will give:

$$L(u) = -c_n\,\lambda_n\,k\,U_n - c_{n+1}\,\lambda_{n+1}\,k\,U_{n+1} - \ldots - c_{-1}\,\lambda_{-1}\,k\,U_{-1} - c_{-2}\,\lambda_{-2}\,k\,U_2 - \ldots$$

and since:

$$D(u) = -\int_0^1 u[(p\,u')' + q\,u]\,dx = \int_0^1 u\,L(u)\,dx\,,$$

one will have:

(14) $$D(u) = \lambda_n\,c_n^2 + \lambda_{n+1}\,c_{n+1}^2 + \cdots - \lambda_{-1}\,c_{-1}^2 - \lambda_{-2}\,c_{-2}^2 - \cdots$$

If we multiply equation (13) by λ_n and subtract it from (14) then we will get:

$$D(u) - \lambda_n = c_{n+1}^2(\lambda_{n+1} - \lambda_n) + c_{n+2}^2(\lambda_{n+2} - \lambda_n) + \cdots + c_{-1}^2(\lambda_n - \lambda_{-1}) + c_{-2}^2(\lambda_n - \lambda_{-2}) + \cdots$$

We remark that $\ldots > \lambda_3 > \lambda_2 > \lambda_1 > 0 > \lambda_{-1} > \lambda_{-2} > \ldots$, so every term on the right-hand side will be positive or zero. Hence, $D(u)$ will be a minimum when $c_n = 1$, $c_{n+1} = c_{n+2} = \ldots = c_{-1} = c_{-2} = \ldots = 0$. $u(x) = U_n(x)$ will then be the desired solution to the variational problem, and λ_n will be the minimal value.

When we replace the quadratic auxiliary condition (3) with the condition (5), we will find the minimal value $-\lambda_{-n}$ in just the same way and the solution $u(x) = U_{-n}(x)$ to the variational problem.

§ 3. – The Lagrange equations and their solutions.

Since the treatment of the Lagrange equation for the problem with one quadratic auxiliary condition alone is very simple, here we will exhibit the equations for the problem with the quadratic auxiliary condition and *one* linear one. The cases of several linear auxiliary conditions yield nothing essentially new.

If we set:

$$v_1(x) = \int_0^x k(x)\,u^2(x)\,dx\,, \qquad v_2(x) = \int_0^x k(x)\,u(x)\,U_1(x)\,dx\,,$$

where $U_1(x)$ is the solution to the problem with only the quadratic auxiliary condition, then we can explain our variational problem geometrically as follows: We must look for a space curve $u = u(x)$, $v_1 = v_1(x)$, $v_2 = v_2(x)$ in four-dimensional $x\,u\,v_1\,v_2$-space that connects the points:

$$x = 0,\ u = 0,\ v_1 = 0,\ v_2 = 0 \qquad \text{and} \qquad x = 1,\ u = 0,\ v_1 = 1,\ v_2 = 0$$

that makes the integral:

$$D(u) = \int_0^1 (p u'^2 - q u^2)\, dx \qquad [p(x) > 0,\ q(x) \leq 0]$$

a minimum, while the auxiliary conditions:

$$v_1'(x) - k(x) u^2(x) = 0\,, \qquad v_2'(x) - k(x) u(x) U_1(x) = 0$$

are fulfilled.

Upon composing it with the Lagrange factors:

$$\lambda = \lambda(x)\,, \qquad 2\mu = 2\mu(x)\,,$$

our relative minimum problem will go to the following absolute minimum problem:

$$\int_0^1 [p u'^2 - q u^2 + \lambda (v_1' - k u^2) + 2\mu (v_2' - k u U_1)]\, dx = \min. \tag{15}$$

with the same boundary conditions. The Lagrange equations arise from (15):

$$(p u')' + (q + \lambda k) u + \mu k U_1 = 0\,, \tag{16}$$

$$v_1' - k u^2 = 0\,, \qquad v_2' - k u U_1 = 0\,. \tag{17}$$

If we consider the homogeneous equation that corresponds to (16):

$$L_1(u) = (p u')' + (q + \lambda k) u = 0 \tag{18}$$

then we will see that arises from the same variational problem with only one (viz., the quadratic) auxiliary condition. However, the solution to that equation is $U_1(x)$, and the corresponding value of λ is λ_1 . A solution of the inhomogeneous equation (16) that satisfies the boundary conditions is then $-\dfrac{\mu U_1(x)}{\lambda - \lambda_1}$, as one will convince oneself by substitution.

The three Lagrange equations (16), (17) yield a six-parameter family of extremals:

$$u = \alpha\, u_1(x_1\ \lambda) + \beta\, u_2(x\ \lambda) - \frac{\mu U_1(x)}{\lambda - \lambda_1}\,,$$

$$v_1 = \int_0^x k(x) \left\{ \alpha u_1 + \beta u_2 - \frac{\mu U_1}{\lambda - \lambda_1} \right\}^2 dx + \gamma,$$

$$v_2 = \int_0^x k(x)\, U_1 \left\{ \alpha u_1 + \beta u_2 - \frac{\mu U_1}{\lambda - \lambda_1} \right\} dx + \delta,$$

in which u_1, u_2 are two linearly-independent particular solutions of the homogeneous equation (18) that do not vanish in the entire interval 0, 1. $\lambda(x)$, $\mu(x)$ are constant on each of those curves of that family (however, the values are not the same on two different curves).

If we establish, for the sake of simplicity, that the particular solution $u_1(x)$ vanishes at the point $x = 0$ and pose the initial conditions $v_1(0) = 0$, $v_2(0) = 0$ then we select the following three-parameter family that emanate from the starting point from the six-parameter $\alpha, \beta, \gamma, \delta, \lambda, \mu$ family of extremals:

$$(19) \qquad \left\{ \begin{aligned} u &= \alpha u_1(x, \lambda) - \frac{\mu U_1(x)}{\lambda - \lambda_1}, \\ v_1 &= \int_0^x k(x) \left\{ \alpha u_1 - \frac{\mu U_1(x)}{\lambda - \lambda_1} \right\}^2 dx, \\ v_2 &= \int_0^x k(x) U_1 \left\{ \alpha u_1 - \frac{\mu U_1}{\lambda - \lambda_1} \right\} dx. \end{aligned} \right.$$

Any two solutions $U_1(x)$, $u_1(x)$ of the homogeneous equation (18) that satisfy the boundary conditions and correspond to two different parameter values λ_1, λ_2 satisfy the condition:

$$(20) \qquad \int_0^1 k(x) U_1(x) u_1(x)\, dx = 0.$$

Namely, if we multiply:

$$(pU_1')' + (q + \lambda_1 k)\, U_1 = 0,$$
$$(pu_1')' + (q + \lambda_2 k)\, u_1 = 0$$

by u_1 (U_1, resp.), subtract, and integrate then we will get:

$$(\lambda_2 - \lambda_1) \int_0^1 k(x) U_1 u_1\, dx = \int_0^1 (pU_1' u_1 - pU_1 u_1')' dx = [pU_1' u_1 - pU_1 u_1']_0^1.$$

Due to the boundary conditions, (20) is therefore proved.

We can now show that $\mu = 0$ for the solution $u(x) = U_2(x)$ of our variational problem. Namely, as a result of the last equations (19) and (20), we have:

$$0 = v_2(1) = -\frac{\mu}{\lambda-\lambda_1}\int_0^1 k U_1^2\,dx.$$

Since the integral is non-zero, it will follow that $\mu = 0$ for our extremal.

The extremal family for the problem with two auxiliary condition is thus posed. The method will remain the same when a third, fourth, … auxiliary condition is added. For example, when we demand that the desired minimal function must fulfill the auxiliary condition:

$$\int_0^1 U_2(x)u(x)\,dx = 0,$$

along with the previous two conditions, the extremal family though the origin of the corresponding five-dimensional space will become:

$$\text{(21)}\qquad \begin{cases} u = \alpha\,u_1(x,\lambda) - \dfrac{\mu U_1(x)}{\lambda-\lambda_1} - \dfrac{\nu U_2(x)}{\lambda-\lambda_1}, \\ v_1 = \alpha\,u_1(x,\lambda) - \dfrac{\mu U_1(x)}{\lambda-\lambda_1} - \dfrac{\nu U_2(x)}{\lambda-\lambda_1}, \\ v_2 = \displaystyle\int_0^x k(x)U_1\left\{\alpha\,u_1 - \frac{\mu U_1}{\lambda-\lambda_1} - \frac{\nu U_2}{\lambda-\lambda_1}\right\}dx, \\ v_3 = \displaystyle\int_0^x k(x)U_2\left\{\alpha\,u_1 - \frac{\mu U_1}{\lambda-\lambda_1} - \frac{\nu U_2}{\lambda-\lambda_1}\right\}dx, \end{cases}$$

in which u_1 is a particular solution of the differential equation (18). It can be shown, analogous to the above, that $\mu = \nu = 0$ on the desired curve $u(x) = U_3(x)$ (but not on all curves of the family) (*).

(*) Exhibiting the Weierstrass criterion does not differ essentially from the case of a problem without auxiliary conditions. In fact, since the part $\lambda(v_1' - k u^2) + \mu(v_2' - k u U_1)$ of the integral (15) does not include the derivative u', the form of the Weierstrass E-functions will be no different than the E-function for the problem without auxiliary conditions. One can easily show by means of the Hilbert independence theorem that in the case of n auxiliary conditions:

$$\mathsf{E}(x, u, v_1, v_2, \dots; \lambda, \mu, \nu, \dots) = p(u' - \mathfrak{P})^2,$$

in which $\mathfrak{P}$ denotes the Hilbert field function. Due to the positive sign on p, one will always have $\mathsf{E} \geq 0$ then, and the Weierstrass condition will then be fulfilled.

§ 4. – The Jacobi criterion for the case of one auxiliary condition.

$$\int_0^1 k(x) u^2 \, dx = 1 .$$

Let the function $U_1(x) = u(x) = \alpha_1 u_1(x, \lambda)$ be the solution to our variational problem (§ 1) with the quadratic auxiliary condition. In § 2, we proved that a minimum actually exists. It follows from this that our function U_1 fulfills the Jacobi criterion. We will now exhibit the Jacobi criterion explicitly and *use it to prove that the function U_1 does not oscillate in the interval* 0, 1.

We assume that the particular solution $u_1(x)$ of the Lagrange equation at the point $x = 0$ has a positive derivative. For instance, let:

$$\left(\frac{du_1}{dx}\right)_{x=0} = \lambda . \tag{22}$$

That is by no means a restriction on the generality of the problem, since the value $u_1'(0) = 0$ is excluded, because the only solution to the differential equation (18) with the boundary conditions $u(0) = 0$, $u_1'(0) = 0$ is $u(x) \equiv 0$.

Expressed geometrically, the Jacobi condition demands that one and only one extremal of the two-parameter family that begins at the initial point (*):

$$\begin{cases} u = \alpha\, u_1(x, \lambda), \\ v_1 = \alpha^2 \displaystyle\int_0^x k(x) u_1^2 \, dx \end{cases} \tag{23}$$

can go through each point of a certain neighborhood of our extremal. That is, for no value $x > 0$ in the interval 0, 1 can the two equations:

$$u_1(x, \lambda_1)\, \delta\alpha + \alpha_1 \frac{\partial u_1(x, \lambda_1)}{\partial \lambda} \delta\lambda = 0 ,$$

$$\left[\alpha_1 \int_0^x k(x) u_1^2 \, dx\right] \delta\alpha + \left[\alpha_1^2 \int_0^x k(x) u_1 \frac{\partial u_1}{\partial \lambda} dx\right] \delta\lambda = 0$$

(*) Naturally, for the case of one auxiliary condition, the family above will enter in place of (19).

be fulfilled simultaneously, where λ_1 and α_1 mean the values of the parameter for the curve $u = U_1(x)$, $v_1 = \int_0^x k U_1^2 \, dx$, and $\delta\alpha$, $\delta\lambda$ are constants that lie below certain limits. If there is a point $X > 0$ – namely, a so-called "conjugate point" to the starting point $x = 0$ – where those equations are true simultaneously then that value will be a zero of the determinant $D_1(x, \lambda_1)$, where:

$$D_1(x, \lambda_1) = \begin{vmatrix} u_1(x,\lambda) & \dfrac{\partial u_1(x,\lambda)}{\partial \lambda} \\ \int_0^x k(x) u_1^2(x,\lambda)\, dx & \int_0^x k(x) u_1(x,\lambda) \dfrac{\partial u_1(x,\lambda)}{\partial \lambda}\, dx \end{vmatrix} . \tag{24}$$

The Jacobi criterion demands that the first zero (beside the point $x = 0$) of the determinant $D_1(x, \lambda_1)$ does not lie in the interval 0, 1. It would then follow from this that the function $u_1(x)$, and therefore the function $U_1(x) = \alpha_1 u_1(x)$ does not oscillate in the interval.

The discussion of the determinant $D_1(x, \lambda_1)$ demands that one must know something about the sign of the function $\partial u_1(x, \lambda) / \partial\lambda$; the following two theorems will give us information about that:

Theorem 2:

If a is a zero of the solution u (x) of the differential equation:

$$(p u'(x))' + q\, u(x) + \lambda\, k(x)\, u(x) = 0 \qquad [p(x) > 0,\ q(x) \le 0,\ \lambda > 0] \tag{25}$$

and $a_1 > a$ is a second zero of u (x) or a zero of $u'(x)$ then one will have:

$$\int_0^{a_1} k(u) u^2(x)\, dx > 0 .$$

Proof:

If we multiply (25) by u and then integrate then we will have (*):

$$\lambda \int_a^{a_1} k u^2\, dx = -\int_a^{a_1} \{(p u')' u + q u^2\}\, dx = \int_a^{a_1} \{p u'^2 - q u^2\}\, dx - [p u u']_a^{a_1} = \int_a^{a_1} [p u'^2 - q u^2]\, dx ,$$

and since $p u'^2 - q u^2 > 0$, the theorem is proved.

(*) One can easily confirm from this that the integral $D(u)$ (§ 2) actually has the value λ_1, since $\int_0^1 k U_1^2\, dx$ is equal to 1.

Theorem 3:

Let $u(x, \lambda)$, $u^*(x, \lambda^*)$ *be two solutions to the differential equation:*

$$(p u')' + q\, u + \lambda\, k(x)\, u = 0 \qquad [p(x) > 0,\ q(x) \leq 0]$$

that satisfy the initial conditions $u(0) = 0$, $u^*(0) = 0$ *and belong to the parameter values* λ, λ^*, *resp. If one then has* $\lambda^* > \lambda > 0$ *then the second, third, … zero of* u^* *will lie before the second (third, …, resp.) zero of* u (*).

Proof:

It is only necessary to show that when $\lambda^* = \lambda + \varepsilon$ ($\varepsilon > 0$ is arbitrarily small). If we multiply:

$$(p u')' + q\, u + \lambda\, k(x)\, u = 0\,,$$
$$(p u^{*\prime})' + q u^* + [\lambda + \varepsilon] k u^* = 0$$

by u^* (u, resp.), subtract, and integrate from 0 to any zero α of u then we will get:

$$p(\alpha) u'(\alpha) u^*(\alpha) = \varepsilon \int_0^\alpha k\, u\, u^* dx\ .$$

Since the solution to the differential equation is a continuous function of λ, we can express that as follows:

$$p(\alpha) u'(\alpha) u^*(\alpha) = \varepsilon \left[\int_0^\alpha k\, u^2\, dx + \varepsilon' \right],$$

in which ε' is an infinitesimal quantity of the same order as ε. We can then conclude from Theorem 2 that $u'(\alpha) \cdot u^*(\alpha) > 0$. If $\alpha = a_1$ is the first zero of u then $u_1'(a_1) < 0$ and therefore $u^*(a_1) < 0$. However, since u^* is positive in the neighborhood of the point $x = 0$, the zero of u^* must fall between 0 and a_1. When $\alpha = a_2$ is the second zero of u, we will have $u'(\alpha_2) > 0$ and therefore $u^*(\alpha_2) > 0$. However, since $u^*(a_1) < 0$, the second zero of u^* must lie between a_1 and a_2, and in general, the n^{th} zero of u^* will lie before the n^{th} root of u.

We now consider the zeros of u_1 on the positive real axis ($x > 0$), $a_1 > 0$, a_2, a_3, … Obviously, $u_1'(a_1) < 0$, $u_1'(a_2) > 0$, $u_1'(a_3) < 0$, and so on. Now:

(*) That fact is well-known for the case of $k(x) \geq 0$.

$$D_1(a_1, \lambda) = -\left(\frac{\partial u_1}{\partial \lambda}\right)_{x=a_1} \int_0^{a_1} k\, u_1^2\, dx .$$

The first factor is negative, which would follow from Theorem 3; the second factor is positive (Theorem 2), and therefore $D_1(a_1, \lambda) > 0$. It can be shown in the same way that $D_1(a_2, \lambda) < 0$, $D_1(a_3, \lambda) < 0$, and so on. The continuous function $D_1(x, \lambda)$ must then have zeroes between a_1 and a_2, between a_2 and a_3, and so on. However, since $D_1(x, \lambda)$ has no zero in the interval 0, 1, one must have $a_1 = 1$, and the function U_1 will not oscillate in the interval.

It is useful to examine how the curve $u_1(x, \lambda)$ lies in relation to a neighboring curve with the parameter $\lambda + \delta\lambda$ that differs by the small positive quantity $\delta\lambda$. In order to have a simple picture in mind, we indicate the particular solution of the differential equation (18) by $u_1(x, \lambda)$, as below. Since all curves of the one-parameter family $u_1(x, \lambda)$ start from the point $x = 0$, we have $\left(\frac{\partial u_1}{\partial \lambda}\right)_{x=0} = 0$. We now assert that:

$$\left(\frac{\partial u_1'}{\partial \lambda}\right)_{x=0} = \left[\left(\frac{\partial u_1}{\partial \lambda}\right)'\right]_{x=0}$$

is always positive. In order to prove that it cannot be zero, it is only necessary to derive the relation:

$$\left(p\frac{\partial u'}{\partial \lambda}\right)' + q\frac{\partial u}{\partial \lambda} + \lambda k\frac{\partial u}{\partial \lambda} + k\,u = 0 \tag{26}$$

from equation (18) by differentiation with respect to λ and to remark that when we have:

$$\left(\frac{\partial u_1}{\partial \lambda}\right)_{x=0} = 0, \qquad \left(\frac{\partial u_1'}{\partial \lambda}\right)_{x=0} = 0,$$

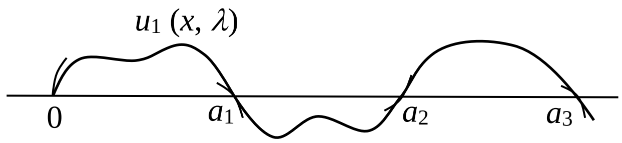
we must also have:

$$\left(\frac{\partial u_1''}{\partial \lambda}\right)_{x=0} = 0 .$$

We conclude analogously that:

$$\left(\frac{\partial u_1'''}{\partial \lambda}\right)_{x=0} = 0 ,$$

and so on, such that it would follow that $\partial u_1 / \partial\lambda \equiv 0$. However, it is impossible for the solutions to the differential equation (18) to coincide for two different parameter values λ, $\lambda + \delta\lambda$ (Theorem 3), and therefore $\left(\frac{\partial u_1'}{\partial \lambda}\right)_{x=0}$ is nowhere zero. The solution is zero over the entire interval for the parameter $\lambda = 0$; the solution is positive for $\lambda > 0$. The function $\left(\frac{\partial u_1'}{\partial \lambda}\right)_{x=0}$, which is continuous in λ, is then positive for at least one value λ, and since it cannot be zero anywhere, it must always be positive.



We can now say something about the relative position of the curve $u_1\,(x, \lambda)$ and the neighboring curve $u_1\,(x, \lambda + \delta\lambda)$ that is sketched in the figure. The curve $u_1\,(x, \lambda + \delta\lambda)$ leaves $u_1\,(x, \lambda)$ with a steep tangent at the starting point and runs (Theorem 3) beneath that curve at the first zero a_1, above it at the second one a_2, beneath it at the third a_3, and so on. As a result, the two curves must meet at least once in each subinterval $0a_1$, $a_1\,a_2$, $a_2\,a_3$, and so on.

When $k\,(x) \geq 0$, we can show that they meet only once in each subinterval. Namely, if we multiply (18), (26):

$$(p\,u')' + q\,u + \lambda\,k\,u = 0\,,$$

$$\left(p\,\frac{\partial u'}{\partial \lambda}\right)' + q\,\frac{\partial u}{\partial \lambda} + \lambda\,k\,\frac{\partial u}{\partial \lambda} + k\,u = 0$$

by $\partial u / \partial \lambda$ ($-\,u$, resp.), add them and integrate then we will get:

$$\int_0^x k\,u^2\,dx = p\left(u'\,\frac{\partial u}{\partial \lambda} - u\,\frac{\partial u'}{\partial \lambda}\right). \tag{27}$$

At the point of the subinterval $0, a_1$ where the curves meet, we have $\partial u / \partial \lambda = 0$, $u > 0$, and therefore $\partial u' / \partial \lambda < 0$. That is, the neighboring curve *always* intersects from above to below. Obviously, the curve will meet only once in the interval $0, a_1$ then. We show, in the same way, that the curves intersect only once in the other subintervals.

One easily sees that the zeroes (*) of the functions u, $\partial u / \partial \lambda$ can never coincide from (27) and Theorem 2.

We can now also say something about the course of the function $D_1\,(x, \lambda)$. From (24), we have:

$$D_1'(x, \lambda) = \frac{dD_1}{dx} = \begin{vmatrix} u_1'(x) & \dfrac{\partial u_1'(x)}{\partial \lambda} \\ \int_0^x k\,u_1^2\,dx & \int_0^x k\,u_1\,\dfrac{\partial u_1}{\partial \lambda}\,dx \end{vmatrix} + \begin{vmatrix} u_1(x) & \dfrac{\partial u_1(x)}{\partial \lambda} \\ k\,u_1^2 & k\,u_1\,\dfrac{\partial u_1}{\partial \lambda} \end{vmatrix} = \begin{vmatrix} u_1'(x) & \dfrac{\partial u_1'(x)}{\partial \lambda} \\ \int_0^x k\,u_1^2\,dx & \int_0^x k\,u_1\,\dfrac{\partial u_1}{\partial \lambda}\,dx \end{vmatrix},$$

(*) Here, as in what follows, we will speak of only the zeroes that are different from $x = 0$.

$$D_1' u_1 - D_1 u_1' = \begin{vmatrix} u_1'(x) & \dfrac{\partial u_1'(x)}{\partial \lambda} \\ \int_0^x k\, u_1^2\, dx & \int_0^x k\, u_1 \dfrac{\partial u_1}{\partial \lambda}\, dx \end{vmatrix} u_1 - \begin{vmatrix} u_1(x) & \dfrac{\partial u_1(x)}{\partial \lambda} \\ \int_0^x k\, u_1^2\, dx & \int_0^x k\, u_1 \dfrac{\partial u_1}{\partial \lambda}\, dx \end{vmatrix} u_1'$$

$$= \left[u_1'(x) \frac{\partial u_1(x)}{\partial \lambda} - u_1(x) \frac{\partial u_1'}{\partial \lambda} \right] \int_0^x k\, u_1^2(x)\, dx \,.$$

From (27), one then has:

$$\frac{d}{dx}\left(\frac{D_1}{u_1}\right) = \frac{D_1' u_1 - D_1 u_1'}{u_1^2} = \frac{\left(\int_0^x k\, u_1^2\, dx\right)^2}{p(x)\, u_1^2} \geq 0 \,.$$

The function D_1 / u_1 is always non-decreasing then and will become infinite at the points a_1, a_2, a_3, …, which are the zeroes of u_1 for $x > 0$, because:

$$D_1(a_i, \lambda) = \frac{\partial u_1(a_i)}{\partial \lambda} \int_0^{a_i} k\, u_1^2\, dx$$

is always non-zero at those points. D_1 has at least one six-fold zero (see § 7) at the point $x = 0$, and therefore D_1/u_1 will have the value zero. D_1 / u_1 assumes all values between 0 and $+\infty$ between 0 and a_1, and D_1 / u_1 assumes all values between $-\infty$ and $+\infty$ between a_1 and a_2, a_2 and a_3, and so on. Therefore, S_1 will have a zero in every subinterval $a_1\, a_2$, $a_2\, a_3$, …

In the same way, it follows from the fact that:

$$\frac{d}{dx}\left(\frac{u_1}{D_1}\right) = -\frac{\left(\int_0^x k\, u_1^2\, dx\right)^2}{p(x)\, D_1^2} \leq 0$$

that u_1 / D_1 will assume all values from $-\infty$ to $+\infty$ between two zeroes of D_1, and therefore u_1 will assume the value zero. Since u_1 and D_1 are both positive in the vicinity of the starting point $x = 0$, one will have:

$$\left(\frac{u_1}{D_1}\right)_{x=0} = \infty \,,$$

and u_1 must vanish between 0 and the first zero of D_1.

Theorem 4:

The zeroes of the functions u_1 (x, λ), D_1 (x, λ), *which vanish at the starting point, separate each other, and the first zero of* u_1 (x, λ) *lies before the first zero of* D_1 (x, λ) .

§ 5. – The Jacobi criterion for the case of two auxiliary conditions.

$$\int_0^1 k(x) u^2 \, dx = 1 , \qquad \int_0^1 k(x) U_1 \, u \, dx = 0 .$$

One sees immediately from the form of the auxiliary conditions that $u = U_1$ is not a solution to the problem. The minimum value λ_2 of the integral D (u) is therefore greater than λ_1 now, and as a result, from Theorem 3, the minimal function u $(x) = U_2$ (x) must vanish at least once in the interval 0, 1. From Theorem 4, the determinant D_1 (x, λ_2) will then have at least one zero in the interval. We will show that is has only one, and that U_2 (x) cannot vanish twice then. In other words: We will show that *the Jacobi criterion demands that the minimal function* $U_2(x)$ *must be a function that oscillates once in the interval* 0, 1 *in this case*.

In § 3, we exhibited the three-parameter family of extremals (19) in which the solution of our variational problem is included for certain parameter values $\alpha = \alpha_2$, $\lambda = \lambda_2$, $\mu = 0$ (*). The projection of that curve in four-dimensional space onto the (xu)-plane is the curve U_2 (x). The Jacobi criterion demands that our space curve must not meet any of its neighboring curves of the family (19) in the interval $0 < x < 1$. If we consider that $\mu = 0$ then we will easily see that this requirement is equivalent to the one that the three equations:

$$\delta\alpha \{u_1(x, \lambda_2)\} + \delta\lambda \left\{\alpha_2 \frac{\partial u_1(x, \lambda_2)}{\partial \lambda}\right\} - \delta\mu \left\{\frac{U_1(x)}{\lambda_2 - \lambda_1}\right\} = 0 ,$$

$$2\alpha_2 \, \delta\alpha \int_0^x k \, u_1^2 \, dx + 2\alpha_2^2 \, \delta\lambda \int_0^x k \, u_1 \frac{\partial u_1}{\partial \lambda} dx - \frac{2\alpha_2 \, \delta\mu}{\lambda_2 - \lambda_1} \int_0^x k \, u_1 \, U_1 \, dx = 0 ,$$

$$\delta\alpha \int_0^x k \, u_1 \, U_1 \, dx + \alpha_2 \, \delta\lambda \int_0^x k \, u_1 \frac{\partial u_1}{\partial \lambda} dx - \frac{\delta\mu}{\lambda_2 - \lambda_1} \int_0^x k \, U_1^2 \, dx = 0$$

cannot be satisfied by three constant quantities $\delta\alpha$, $\delta\lambda$, $\delta\mu$ for any value x in the interval 0, 1. After one neglects the constant factor be $-\dfrac{2\alpha_2^2}{\lambda - \lambda_1}$, the Jacobi determinant will then be:

(*) Naturally, the function u_1 that appears there should probably be distinguished from the one that was considered in the previous section.

(28) $$D_2(x, \lambda) = \begin{vmatrix} u_1(x,\lambda) & \dfrac{\partial u_1(x,\lambda)}{\partial \lambda} & U_1(x) \\ \int_0^x k\,u_1^2\,dx & \int_0^x k\,u_1 \dfrac{\partial u_1}{\partial \lambda}\,dx & \int_0^x k\,u_1 U_1\,dx \\ \int_0^x k\,U_1 u_1\,dx & \int_0^x k\,U_1 \dfrac{\partial u_1}{\partial \lambda}\,dx & \int_0^x k\,U_1^2\,dx \end{vmatrix}.$$

The Jacobi criterion demands that the first zero (beyond the point $x = 0$) of the determinant $D_2(x, \lambda_2)$ must not lie in the interval 0, 1.

Due to the boundary and auxiliary conditions, one has:

$$D_2(1, \lambda_2) = -\left(\frac{\partial u_1}{\partial \lambda}\right)_{x=1} \int_0^x k\,u_1^2\,dx .$$

If the function u_1 has an even (odd, resp.) number of zeroes between 0 and 1 then, due to Theorems 2 and 3, $D_2(1, \lambda_2)$ will be positive (negative, resp.). Later on (§ 7), we will develop D_2 into a Taylor series and show by means of that series that $D_2 < 0$ in a neighborhood of the point $x = 0$. Assuming that u_1 oscillates an even number of times in the interval, we likewise see that the continuous function D_2 has a zero between 0 and 1. However, that is impossible, and we conclude that the eigenfunction oscillates an odd number of times.

One can now show by continuity considerations that it oscillates only once. However, it is more satisfying to prove that fact by discussing the determinant D_2, as we would now like to do by an extension of the method that was given at the conclusion to section 3. We will first show that $\dfrac{d}{dx}\left(\dfrac{D_1}{D_2}\right)$ is nowhere negative in the entire interval $0 \le x \le \infty$.

Two of the solutions $U_n(x)$, $U(x)$ to the differential equation $(p\,u')' + q\,u + \lambda\,k\,u = 0$ with the initial conditions $U_n(0) = 0$, $U(0) = 0$ that correspond to the parameter values λ_n, λ will satisfy the identity:

(29) $$U_n' U - U_n U' = \left(\frac{\lambda - \lambda_n}{p}\right) \int_0^x k\,U_n U\,dx ,$$

whose proof is analogous to that of equation (20). It follows from (29) upon differentiating with respect to λ that:

(30) $$U_n' \frac{\partial U}{\partial \lambda} - U_n \frac{\partial U'}{\partial \lambda} = \frac{\lambda - \lambda_n}{p} \int_0^x k\,U_n \frac{\partial U}{\partial \lambda}\,dx + \frac{1}{p} \int_0^x k\,U_n U\,dx .$$

We remark that:

$$\frac{d}{dx} D_2(x,\lambda) = D_2' = \begin{vmatrix} u_1'(x,\lambda) & \dfrac{\partial u_1'(x,\lambda)}{\partial\lambda} & U_1'(x) \\ \int_0^x k\, u_1^2\, dx & \int_0^x k\, u_1 \dfrac{\partial u_1}{\partial\lambda} dx & \int_0^x k\, u_1 U_1\, dx \\ \int_0^x k\, U_1\, u_1\, dx & \int_0^x k\, U_1 \dfrac{\partial u_1}{\partial\lambda} dx & \int_0^x k\, U_1^2\, dx \end{vmatrix},$$

or more briefly:

$$D_2' = \begin{vmatrix} u_1'(x,\lambda) & \dfrac{\partial u_1'(x,\lambda)}{\partial\lambda} & U_1'(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

then due to (27), (29), and (30), we will get:

$$D_2\, D_1' - D_2'\, D_1$$

$$= \begin{vmatrix} u_1 & \dfrac{\partial u_1}{\partial\lambda} & U_1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} u_1' & \dfrac{\partial u_1'}{\partial\lambda} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} u_1'(x,\lambda) & \dfrac{\partial u_1'(x,\lambda)}{\partial\lambda} & U_1'(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} u_1 & \dfrac{\partial u_1}{\partial\lambda} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{aligned} &= (u_1\, u_1' - u_1'\, u_1 = 0)\, A_{11}\, a_{22} - \left(u_1 \frac{\partial u_1'}{\partial\lambda} - u_1' \frac{\partial u_1}{\partial\lambda} = -\frac{a_{21}}{p} \right) A_{11}\, a_{21} + \left(\frac{\partial u_1}{\partial\lambda} u_1' - \frac{\partial u_1'}{\partial\lambda} u_1 = \frac{a_{21}}{p} \right) A_{12}\, a_{22} \\ &+ \left(\frac{\partial u_1}{\partial\lambda}\frac{\partial u_1'}{\partial\lambda} - \frac{\partial u_1'}{\partial\lambda}\frac{\partial u_1}{\partial\lambda} = 0 \right) A_{12}\, a_{21} + \left(U_1\, u_1' - U_1'\, u_1 = -\frac{\lambda-\lambda_1}{p} a_{31} \right) A_{13}\, a_{22} \\ &+ \left(U_1 \frac{\partial u_1'}{\partial\lambda} - U_1' \frac{\partial u_1}{\partial\lambda} = -\frac{\lambda-\lambda_1}{p} a_{31} - \frac{a_{23}}{p} \right) A_{13}\, a_{21}\ , \end{aligned}$$

in which A_{11}, A_{12}, A_{13} are the sub-determinants that belong to the first rows of D_2 . Since $A_{11}\, a_{21} + A_{13}\, a_{23} = -A_{12}\, a_{22}$, we will have:

$$D_2\, D_1' - D_2'\, D_1 = \frac{(\lambda-\lambda_1)\, A_{13}}{p}\, (a_{21}\, a_{32} - a_{22}\, a_{31}) = \frac{(\lambda-\lambda_1)\, A_{13}^2}{p}$$

and

$$\frac{d}{dx}\frac{D_1(x,\lambda)}{D_2(x,\lambda)} = \frac{D_2\,D_1' - D_2'\,D_1}{D_2^2} = \frac{(\lambda-\lambda_1)\,A_{13}^2}{p(x)\,D_2^2}.$$

Since $p(x)$ is a positive function, and $\lambda_2 - \lambda_1 > 0$, we have $\frac{d}{dx}\frac{D_1(x,\lambda_2)}{D_2(x,\lambda_2)} > 0$. In § 7, we will show that D_1 and D_2 will have opposite signs and D_2 will vanish to a higher order than D_1 in a neighborhood of the starting point $x = 0$. The function D_1 / D_2 will then have the value $-\infty$ at the point $x = 0$. Since D_2 has no zero in 0, 1, D_1 can no longer have a zero. The function u_1 has at least one zero in the interval 0, 1, and as a result, we conclude from Theorem 4 that D_1 also has at least one. Therefore, $D_1(x, \lambda_2)$ vanishes exactly once in the interval, and we see from Theorem 4 that the function u_1, and therefore the function U_2, oscillates once and only once.

§ 6. – Adding more linear auxiliary conditions.

The addition of new linear auxiliary conditions:

$$\int_0^1 U_2(x)u(x)\,dx = 0\,, \quad \int_0^1 U_3(x)u(x)\,dx = 0, \quad \text{etc.},$$

will offer no essential complexities and will yield nothing new, in principle. For the case of one new condition, the Jacobi determinant, except for the constant factor $\frac{2\alpha_3^2}{(\lambda-\lambda_1)(\lambda-\lambda_2)}$, is written as follows with the use of the solutions (21):

$$D_3(x,\lambda) = \begin{vmatrix} u_1(x,\lambda) & \dfrac{\partial u_1}{\partial\lambda} & U_1(x) & U_2(x) \\ \int_0^x k(x)u_1^2\,dx & \int_0^x k\,u_1\dfrac{\partial u_1}{\partial\lambda}dx & \int_0^x kU_1u_1\,dx & \int_0^x kU_2u_1\,dx \\ \int_0^x k(x)U_1u_1\,dx & \int_0^x kU_1\dfrac{\partial u_1}{\partial\lambda}dx & \int_0^x kU_1^2\,dx & \int_0^x kU_2U_1\,dx \\ \int_0^x k(x)U_2u_1\,dx & \int_0^x kU_2\dfrac{\partial u_1}{\partial\lambda}dx & \int_0^x kU_2U_1\,dx & \int_0^x kU_2^2\,dx \end{vmatrix}.$$

The functions U_1 and U_2 are the first two eigenfunctions that were considered above. The law for constructing the determinants $D_4(x, \lambda)$, $D_5(x, \lambda)$, … is easy to recognize.

The Jacobi criterion demands that $D_3(x, \lambda)$ must have no zero in 0, 1. The fact that $u_1(x)$ must oscillate at least twice will indeed follow from Theorem 5, since the parameter λ_3 that belongs to the present variational problem is greater than λ_2 . We prove that u_1 (x) cannot oscillate an odd number of times by considering the determinant D_3 (x, λ). Namely, at the endpoint x = 1, we have:

$$D_3\,(1, \lambda) = -\left(\frac{\partial u_1}{\partial \lambda}\right)_{x=1} \int_0^1 k\, u_1^2\, dx \; .$$

If u_1 were to oscillate an odd number of times then we would have (Theorem 2 and 3) $\left(\frac{\partial u_1}{\partial \lambda}\right)_{x=1} >$ 0, and therefore D_3 (1, λ) < 0. However, when we develop D_3 (x, λ) into a Taylor series (§ 7), that will imply that D_3 (x, λ) > 0 in a neighborhood of x = 0. Therefore, the continuous function D_3 must have a zero in the interval 0, 1, which contradicts the Jacobi condition. It is then excluded that u_1 (x, λ_3) oscillates an odd number of times.

In order to prove our assertion that u_1 (x) oscillates exactly twice, we again consider the function $\frac{D_2(x,\lambda)}{D_3(x,\lambda)}$. Analogously to the notation (page 18) of the previous section, we write:

$$D_3\, D_2' - D_3'\, D_2$$

$$= \begin{vmatrix} u_1 & \frac{\partial u_1}{\partial \lambda} & U_1 & U_2 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \cdot \begin{vmatrix} u_1' & \frac{\partial u_1'}{\partial \lambda} & U_1' \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} u_1' & \frac{\partial u_1'}{\partial \lambda} & U_1' & U_2' \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \cdot \begin{vmatrix} u_1 & \frac{\partial u_1}{\partial \lambda} & U_1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

If α_{11}, α_{12}, α_{13}, α_{14} are the sub-determinants that belong to the first row in D_3 then:

$$D_3\, D_2' - D_3'\, D_2$$

$$= \begin{pmatrix} u_1\, u_1' - u_1'\, u_1 \\ = 0 \end{pmatrix} A_{11}\, \alpha_{11} + \begin{pmatrix} u_1 \frac{\partial u_1'}{\partial \lambda} - u_1' \frac{\partial u_1}{\partial \lambda} \\ = 0 \end{pmatrix} A_{12}\, \alpha_{11} + \begin{pmatrix} u_1\, U_1' - u_1'\, U_1 \\ = \frac{(\lambda - \lambda_1)\, a_{31}}{p} \end{pmatrix} A_{13}\, \alpha_{11}$$

$$+ \begin{pmatrix} \frac{\partial u_1}{\partial \lambda} u_1' - \frac{\partial u_1'}{\partial \lambda} u_1 \\ = \frac{a_{21}}{p} \end{pmatrix} A_{11}\, \alpha_{12} + \begin{pmatrix} \frac{\partial u_1}{\partial \lambda} \frac{\partial u_1'}{\partial \lambda} - \frac{\partial u_1'}{\partial \lambda} \frac{\partial u_1}{\partial \lambda} \\ = 0 \end{pmatrix} A_{12}\, \alpha_{12} + \begin{pmatrix} \frac{\partial u_1}{\partial \lambda} U_1' - \frac{\partial u_1'}{\partial \lambda} U_1 \\ = \frac{\lambda - \lambda_1}{p} a_{32} + \frac{a_{23}}{p} \end{pmatrix} A_{13}\, \alpha_{12}$$

$$+\begin{pmatrix} U_1\,u_1' - U_1'\,u_1 \\ = -\dfrac{\lambda-\lambda_1}{p}\,a_{31}\end{pmatrix} A_{11}\,\alpha_{13} + \begin{pmatrix} U_1\,\dfrac{\partial u_1'}{\partial\lambda} - U_1'\,\dfrac{\partial u_1}{\partial\lambda} \\ = -\dfrac{\lambda-\lambda_1}{p}\,a_{32} - \dfrac{a_{23}}{p}\end{pmatrix} A_{12}\,\alpha_{13} + \begin{pmatrix} U_1\,U_1' - U_1'\,U_1 \\ = 0\end{pmatrix} A_{13}\,\alpha_{13}$$

$$+\begin{pmatrix} U_2\,u_1' - U_2'\,u_1 \\ = \dfrac{\lambda-\lambda_2}{p}\,a_{41}\end{pmatrix} A_{11}\,\alpha_{14} + \begin{pmatrix} U_2\,\dfrac{\partial u_1'}{\partial\lambda} - U_2'\,\dfrac{\partial u_1}{\partial\lambda} \\ = -\dfrac{\lambda-\lambda_2}{p}\,a_{42} - \dfrac{a_{24}}{p}\end{pmatrix} A_{12}\,\alpha_{14} + \begin{pmatrix} U_2\,U_1' - U_2'\,U_1 \\ = \dfrac{(\lambda-\lambda_1)\,a_{34}}{p} - \dfrac{(\lambda-\lambda_2)\,a_{43}}{p}\end{pmatrix} A_{13}\,\alpha_{14}\,.$$

Since $-(\alpha_{11}\,a_{31} + \alpha_{13}\,a_{23} + \alpha_{14}\,a_{24}) = \alpha_{12}\,a_{22}$, the sum of the terms with the coefficients $(\lambda - \lambda_1)\,/\,p$ will be $-\,\alpha_{13}\,(a_{31}\,A_{11} + a_{42}\,A_{12} + a_{43}\,A_{13}) = 0$. We therefore finally have:

$$D_3\,D_2' - D_3'\,D_2 = -\,\frac{\lambda-\lambda_2}{p}\;\alpha_{14}\,(a_{41}\,A_{11} + a_{42}\,A_{12} + a_{43}\,A_{13}) = \frac{\lambda-\lambda_2}{p}\,\alpha_{14}^2\,,$$

and

$$\frac{d}{dx}\left[\frac{D_2(x,\lambda)}{D_3(x,\lambda)}\right] = \frac{(\lambda-\lambda_2)\,\alpha_{14}^2}{p\,D_3^2}\,.$$

The calculations in the cases of 4, 5, … auxiliary conditions take an entirely analogous form, such that we can state the following theorem:

Theorem 5:

If D_{n-1}, D_n mean the Jacobi determinants for the minimization problem with $n-1$ (n, resp.) auxiliary conditions then:

$$\frac{d}{dx}\left[\frac{D_{n-1}(x,\lambda)}{D_n(x,\lambda)}\right] = \frac{(\lambda-\lambda_{n-1})\,(A_{1,n+1})^2}{p\,(x)\,D_n^2}\,,$$

in which $A_{1,\,n+1}$ is a certain sub-determinant of D_n .

One will see from the development of the determinants D_2 and D_3 (§ 7) that the value of the increasing function $\dfrac{D_2(x,\lambda)}{D_3(x,\lambda)}$ is $-\infty$ at the starting point. Since $D_3\,(x, \lambda_3)$ has no zero in the interval 0, 1, $D_2\,(x, \lambda_2)$ can have at most one. The same considerations will suffice to show that in this case $D_1\,(x, \lambda_3)$ will have no more than two zeroes in the intervals. We ultimately conclude from Theorem 4 that u_1 will vanish *no more* than twice. However, since Theorem 3 says that the function $u_1\,(x)$ vanishes *at least* twice, we will finally get the desired result. The solution to the minimization problem with one quadratic and two linear auxiliary condition is a twice-oscillating function.

The arguments will be entirely the same when we have n auxiliary conditions. We know that the function u_1 must oscillate *at least* $n - 1$ times. Since the Jacobi determinant $D_n(x, \lambda_n)$ has no zero in the interval 0, 1, $D_{n-1}(x, \lambda_n)$ can vanish at most once, $D_{n-2}(x, \lambda_n)$, at most twice, and finally, $D_1(x, \lambda_n)$, at most $n - 1$ times. From Theorem 4, u_1 can then oscillate *at most* $n - 1$ times. The function $u_1(x, \lambda_n)$ must then vanish *exactly* $n - 1$ times in the interval 0, 1, $D_1(x, \lambda_n)$, exactly $n - 2$ times, $D_2(x, \lambda_n)$, exactly $n - 3$ times, and so on.

Main theorem:

The Jacobi criterion for the calculus of variations says that the solution $u(x) = U_1(x)$ *of the minimization problem* (§ 1):

$$\int_0^1 (p\,u'^2 - q\,u^2)\,dx = \text{min.} \qquad [p(x) > 0,\; q(x) \leq 0,\; u(0) = u(1) = 0]$$

with the quadratic auxiliary condition $\int_0^1 k(x)\,u^2\,dx = 1$ *will not oscillate in the interval* 0, 1, *that the solution* $u(x) = U_2(x)$ *of the same problem with the quadratic and one linear auxiliary condition* $\int_0^1 U_1(x)\,u(x)\,dx = 0$ *will oscillate once in the interval, and that in general the solution* $u(x) = U_{n+1}(x)$ *to the problem with the quadratic and n linear auxiliary conditions:*

$$\int_0^1 k(x)U_1(x)u(x)\,dx = 0\,, \qquad \int_0^1 k(x)U_2(x)u(x)\,dx = 0\,, \qquad \ldots, \qquad \int_0^1 k(x)U_n(x)u(x)\,dx = 0$$

will oscillate exactly n times in the interval.

§ 7. – Developing the determinants $D_1, D_2, \ldots$

In this section, we will give a method for developing the determinants $D_1(x, \lambda)$, $D_2(x, \lambda)$, … into Taylor series. We will carry it out for the three-rowed determinant D_2. The other cases would give nothing new, in principle.

We would like to distinguish two cases in the development of D_2 (28) according to whether:

1. $k(0) \neq 0$

or

2. $k(0) = k'(0) = \ldots = k^{(n-1)}(0) = 0, \qquad k^{(n)}(0) \neq 0$.

We develop the four functions $\partial u_1 / \partial\lambda$, $U_1(x)$, $u_1(x)$, $k(x)$ in the first case, as follows [see (22)]:

$$\text{(31)} \qquad \begin{cases} \dfrac{\partial u_1}{\partial \lambda} = x + \cdots, \\ U_1(x) = m_1\, x + \cdots, \\ u_1(x) = m_2\, x + \cdots, \\ k(x) = m_3 + \cdots \end{cases}$$

If we replace those values in the determinant D_2 (28) then we will easily see that the first terms of the corresponding functions in the first and second columns differ by only the multiplicative constant m_2 . If we multiply the first one by $1 / m_2$ and subtract it from the second then the first terms in the functions in the second column will drop out. In fact, as will be shown later, not only the first ones, but also the second terms in the functions in the second column will drop out. The third column can be treated the same as the second one. We can treat the new second and third column in the same way and eliminate the first two terms of each function in the third. That process is best performed in the following way:

If one sets:

$$\text{(32)} \qquad s(x) = \frac{\partial u_1}{\partial \lambda} - \frac{u_1}{m_2}, \qquad t(x) = U_1 - \frac{m_1 u_1}{m_2}$$

then it will follow, upon considering (31), that $s(0) = t(0) = s'(0) = t'(0) = 0$. One immediately produces the equations for $s(x)$, $t(x)$ from the equations (18) and (26) for the functions u_1, U_1, $\partial u_1 / \partial\lambda$:

$$\text{(33)} \qquad (p\,s')' + q\,s + \lambda_2\,k\,s + k\,u_1 = 0\,,$$

$$\text{(34)} \qquad (p\,t')' + q\,t + \lambda_1\,k\,t + (\lambda_1 - \lambda_2)\frac{m_1}{m_2} k\,u_1 = 0\,.$$

It follows immediately from (33) and (34) that $s''(0) = t''(0) = 0$. Upon differentiating the two equations (33), (34), we will get:

$$p\,s''' + 2p'\,s'' + p''s' + q\,s' + s\,q' + \lambda_2(k'\,s + k\,s') + k\,u_1' + k'\,u_1 = 0\,,$$

$$p\,t''' + 2p'\,t'' + p''t' + q\,t' + t\,q' + \lambda_1(k'\,t + k\,t') + (\lambda_1 - \lambda_2)\frac{m_1}{m_2}(k\,u_1' + k'\,u_1) = 0\,,$$

and

$$s'''(0) = -\frac{k(0)\,u_1'(0)}{p(0)} = -\frac{m_3\,m_2}{p(0)}, \qquad t'''(0) = -\frac{m_3\,m_1(\lambda_1 - \lambda_2)}{p(0)}\frac{x^3}{3!} + \cdots$$

One can write the determinant D_2 in a different form by means of the transformations (32):

$$D_2(x,\lambda) = \begin{vmatrix} u_1(x,\lambda) & s(x,\lambda) & t(x,\lambda) \\ \int_0^x k\,u_1^2\,dx & \int_0^x k\,u_1\,s\,dx & \int_0^x k\,u_1\,t\,dx \\ \int_0^x k\,t\,u_1\,dx & \int_0^x k\,t\,s\,dx & \int_0^x k\,t^2\,dx \end{vmatrix} . \tag{36}$$

The first terms in the development of each function in the second and third columns behave like $m_2 : m_1\,(\lambda_1 - \lambda_2)$. If we now set:

$$r(x) = t(x) - \frac{m_1(\lambda_1-\lambda_2)}{m_2}\,s(x) \tag{37}$$

then we will get the equation for $r(x)$ from (33) and 34):

$$(p\,r')' + q\,r + \lambda_1\,k\,r + (\lambda_1-\lambda_2)^2\,\frac{m_1}{m_2}\,k\,s = 0 . \tag{38}$$

It follows from (35) and (37) that:

$$r'''(0) = r''(0) = r'(0) = r(0) = 0 ,$$

and one will get that $r^{\mathrm{IV}}(0) = 0$ by differentiating equation (38) twice. It will follow upon repeated differentiation that:

$$p\,r^{\mathrm{V}} + 4p'\,r^{\mathrm{IV}} + 6p''\,r''' + 4p'''\,r'' + p^{\mathrm{IV}}\,r' + q'''\,r + 3q''\,r' + 3q'\,r'' + q\,r''$$

$$\lambda_1\,(k'''\,r + 3k''\,r + 3k'\,r' + k\,r''') + (\lambda_1-\lambda_2)^2\,\frac{m_1}{m_2}\,(s'''\,k + 3s''\,k' + 3s'\,k'' + k'''\,s) = 0 ,$$

and

$$r^{\mathrm{V}}(0) = -\frac{(\lambda_1-\lambda_2)^2\,m_1}{m_2\,p'(0)}\,s'''(0)\,k(0) = \frac{(\lambda_1-\lambda_2)^2\,m_3^2\,m_1}{(p(0))^2} .$$

Therefore:

$$r(x) = \frac{(\lambda_1-\lambda_2)^2\,m_3^2\,m_1}{(p(0))^2}\,\frac{x^5}{5!} + \cdots \tag{39}$$

If we now perform the transformation (37) on the determinant (36) then we will get a new determinant in which the functions u_1, s, r, k appear, and upon substituting (31), (35), (39):

$$D_2(x,\lambda_2) = \begin{vmatrix} m_2\,x+\cdots & -\dfrac{m_3\,m_1}{p(0)}\dfrac{x^3}{3!}+\cdots & \dfrac{(\lambda_1-\lambda_2)^2 m_3^2\,m_1}{(p(0))^2}\dfrac{x^5}{5!}+\cdots \\ m_2^2\,m_3\,\dfrac{x^3}{3}+\cdots & -\dfrac{m_3^2\,m_2^2}{3!\,p(0)}\dfrac{x^5}{5}+\cdots & \dfrac{m_3^3\,m_2\,m_1(\lambda_1-\lambda_2)}{5!\,(p(0))^2}\dfrac{x^7}{7}+\cdots \\ -\dfrac{m_1^2\,m_2^3\,m_3^6(\lambda_1-\lambda_2)}{5!\,3!\,3!\,(p(0))^4}\dfrac{x^5}{5}+\cdots & \dfrac{m_3^3\,m_2\,m_1(\lambda_1-\lambda_2)}{3!\,3!\,(p(0))^2}\dfrac{x^7}{7}+\cdots & -\dfrac{m_3^4\,m_1^2(\lambda_1-\lambda_2)^3}{5!\,3!\,(p(0))^3}\dfrac{x^9}{9}+\cdots \end{vmatrix}$$

$$= \frac{m_1^2\,m_2^3\,m_3^6(\lambda_1-\lambda_2)^3}{5!\,3!\,3!\,(p(0))^4}\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} \end{vmatrix} x^{15}+\cdots$$

Since m_2 and the numerical determinant are positive and $\lambda_1 - \lambda_2$ is negative, $D_2(x, \lambda_2)$ will be negative in a neighborhood of $x = 0$.

The method remains entirely the same in the case of:

$$k(0) = k'(0) = \ldots = k^{(n-1)}(0) = 0\,, \quad k^{(n)}(0) \neq 0$$

as it was in the simplest cast, although the calculations will be more complicated. One finds that (*):

$$D_2(x,\lambda_2) = \frac{{}^{2n+3}C_{n+3}\,m_1^2\,m_2^3\,m_3^6(\lambda_1-\lambda_2)^3(n+1)^3}{2n+5!\,2n+3!\,2n+3!\,(p(0))^4}\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{n+3} & \frac{1}{2n+5} & \frac{1}{3n+7} \\ \frac{1}{2n+5} & \frac{1}{3n+7} & \frac{1}{4n+9} \end{vmatrix} x^{6n+15}+\cdots$$

Therefore, $D_2(x, \lambda_2)$ is also negative in a neighborhood of the point $x = 0$ in this case.

In the case of the determinant D_3 (§ 6), we add the development:

$$U_2 = m_4\,x + \ldots$$

to the series (31). When $k(0) \neq 0$, we will then get:

(*) The sign ${}^hC_k$ means the number of combinations of h elements k at a time.

$$D_3(x, \lambda_3) = \frac{B^2 m_2^4 m_3^{12} m_1^2 (\lambda_1-\lambda_3)^4(\lambda_2-\lambda_3)^3(\lambda_1-\lambda_2)}{(p(0))^9 3!5!7!3!5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \frac{1}{13} \end{vmatrix} x^{28} + \cdots$$

by calculation, where is a positive constant. Since m_4 is positive and $\lambda_1 - \lambda_2$ and the numerical determinant are negative, $D_3(x, \lambda_3)$ will then be positive in a neighborhood of $x = 0$.

For the case of $k(0) \neq 0$, the first terms in the developments of the determinants:

$D_4, D_5, \ldots$ will be $-c_4^2\, x^{1+5+9+13+17}$, $c_5^2\, x^{1+5+9+13+17+21}$, …, resp.,

in which $c_4, c_5, \ldots$ are constants. Naturally, when $k(0) \neq 0$, the first terms will be of higher orders. As one can easily show, the numerical determinant:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+1} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n-1} & \frac{1}{2n+1} & \frac{1}{2n+3} & \cdots & \frac{1}{4n-3} \end{vmatrix}$$

will be positive or negative according to whether n is odd or even, resp. Corresponding to that change of sign in the numerical determinant, the $(n + 1)$-rowed determinant $D_n(x, \lambda_n)$ will be positive or negative in a neighborhood of $x = 0$ according to whether n is odd or even, resp.

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