“Zur Transformation von Raumkurven,” Math. Ann. **66** (1909), 517-557.

On the transformation of space curves

By

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Two space curves C and C_1 can be related to each other point-wise in many different ways. Every such association determines a rule by which the one space curve C will go to the other one C_1 . More or less simple relationships can exist between the determining data of the two curves according to the type of assignment whose analytical derivation from the given conditions would not cause any sort of tangible difficulties. If the rule is chosen to be too specialized then it can happen that it either gives n o curves that satisfy it or only a special class of space curves for which the prescription can then be regarded as a defining property. For example, if one arrives at a space curve C_1 that has all tangents in common with the curve C then C and C_1 must coincide: The chosen rule would not be suitable for relating two distinct curves to each other. By contrast, the requirement that two curves should possess common principal normals can be fulfilled by only a special class of curves and will therefore be employed as the definition of the **Bertrand** curves.

If one chooses a somewhat less specialized relationship then one will gain the advantage of connecting curves with unknown properties with ones that are known more precisely and thus gain a tool for research from that connection.

In particular, it would be natural to consult the geometrically easily-overlooked connection between the fundamental trihedra at corresponding points. One examines the relations between two curves whose tangents at corresponding points are parallel or perpendicular to each other. If one poses the same problem for binormals or principal normals then one will arrive at further laws of relationship that space curves can be subject to.

Those questions are found to be treated in the literature in scattered, isolated places. The relationship between space curves with parallel tangents is found in the French literature in the form of the *Combescure transformation* (*). **Aoust** (**) addressed those “courbes parallèles” in his

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(*) The author first pointed out that fact when he reported on some applications of those transformations to the Berliner Math. Gesellschaft. (Sitzungberichte der B. M. G, **4**, pp. 64-60.)

(**) **Aoust**, *Analyse infinitésimale des courbes dans l’espace*, Paris, 1876, pp. 382.

comprehensive textbook. More recently, N. J. Hatzidakis (*) has presented the relations between the curvature, torsion, and arc-length of two mutually-related curves and extended the formulas to structures in *n*-dimensional space.

Aoust (**) considered curves whose principal normals are parallel to each other for the first time. **L. Bianchi** (***) then employed a special case of that relation (when the line elements of the two curves are equal) in order to derive the equations for Bertrand curves from the ones for curves of constant curvature.

Relating two curves by orthogonality of the line elements was investigated by **G. Sannia** (†) using the methods of natural geometry.

However, all of those works proceed in a purely analytical fashion, so perhaps it would not be trivial to return to the simple geometric form of those transformations and read off their properties from that. The first section of the present investigation therefore poses the problem of systematically exhibiting *all relations that are made possible by parallelism or orthogonality of two edges of the fundamental trihedra of two curves*, determining a geometric way of generating each of them, and deriving the characteristic equations geometrically.

The *second section* applies the relations by parallel tangents to the theory of *loxodromes*. Once the loxodrome that is associated with an arbitrary curve by parallelism is determined by two quadratures, in particular, the problem of *loxodromic helices* will be reduced to quadratures, and the problem of *loxodromes of constant curvature*, as well as the *double loxodromes*, will be reduced to differential equations.

The *third section* represents an application of the transformation by orthogonal line elements. From the determination of the curves that are associated with an ordinary helix in the given way, by specializing the examination, one will be led *to a special type of curves of constant curvature, among which one finds infinitely-many algebraic ones.* That result seems to be interesting insofar as up to now no algebraic curves of constant curvature are known besides the circle. The curves that are investigated here offer, in addition, the appeal of an intuitive geometric manner of generation and possess a series of properties that are closely related, on the one hand, to the theory of second-order surfaces, and on the other, to the cyclic curves in the plane.

In the fourth section, in conclusion, the investigation will be generalized and extended to the curves that appear as geodetic lines on the tangent surfaces to helices. Their determination in terms of natural equation requires only eliminations, while their finite equations will be achieved by a quadrature. They possess the characteristic property that their principal normals envelop the cylinder on which the edge of regression of its rectifying surface is geodetic.

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(*) **N. J. Hatzidakis**, "Om nogle Konsekvenser af Frenet's of Brunel's Formler," Nyt Tidskr. f. Math. **13** (1902); also **G. Sannia**, "Trasformazione di Combescure ed altre analoghe per le curve storte," Rendiconti de Circolo Mat. di Palermo **20** (1902), 83-92.

(**) **Aoust**, *loc. cit.*, pp. 368.

(***) **L. Bianchi**, *Differentialgeometrie*, German by Lukat, Leipzig, 1896/99, pp. 32.

(†) **G. Sannia**, "Deformazioni infinitesime delle curve inestendibili e correspondenze per orthogonalità di elementi," Rendiconti de Circolo Mat. di Palermo **21** (1906), 229-256.

I. – Curve-pairs whose principal triehdra have parallel or orthogonal edges.

1. – *Let two curves C_1 and C_2 be related to each other point-wise in such a way that their tangents at corresponding points are parallel.*

Along with the tangents, the osculating planes that they determine are also parallel, so the altitudes to them, as well, i.e., the binormals to the curves. However, if the tangents and binormals are parallel then so are the principal normals, which must define a right-handed system of axes along with the first two lines.

Curves whose line-elements at corresponding points are parallel have fundamental trihedra that are parallel everywhere. Their contingency and osculation angles are equal to each other, and they will always then possess the same ratio of curvature and torsion at corresponding points.

That transformation plays a distinguished role in regard to the general helices. Since the ratio of curvature and torsion is constant for them, a helix will go to another curve of the same kind under any such transformation, and conversely there will always be a transformation by parallel line-elements that takes two arbitrarily-chosen helices with the same curvature ratio and parallel axes to each other. One can then easily derive the properties of the general helices from those of the usual ones that belong to cylinders of rotation.

Since any two corresponding line elements are parallel to each other, the lines that connect the corresponding points of both curves will generate a developable surface.

Geometrically, one generates the curves that are associated with a curve *C* when one lays an arbitrary developable surface through *C* and determines the curves on that surface that intersect the generating lines of the surface parallel to the given curve.

Since two arbitrary plane curves can always be associated with each other as being parallel, one can also realize the association as follows:

One connects those points of two arbitrary curves in a plane at which the tangents are parallel rectilinearly and then bends the plane in such a way that the connecting lines remain rectilinear. The two chosen curves will then give one of the desired curve-pairs in space.

2. *Two curves C and C′ shall possess parallel binormals at corresponding points.*

If the binormals are parallel then the same thing will be true of the osculating planes. However, that will imply the parallelism of the tangents, which can be regarded as the intersections of neighboring osculating planes. The transformation will then be identical to the previous one.

3. *If two curves possess parallel principal normals at corresponding points* then their rectifying planes will be parallel, so their rectifying lines will also be parallel. The rectifying curves are then associated with each other by parallel line elements.

Two curves whose principal normals are parallel at corresponding points are then geodetic lines on developable surfaces whose edges of regression possess parallel line elements.

In particular, as one should have expected to begin with, the geodetic lines of one and the same developable surface go to each other under transformations by means of parallel principal normals. However, as is well-known and geometrically obvious, the tangents to the geodetic lines of a developable surface at two corresponding (i.e., lying on the same generator) points define a constant angle ϑ on the surface. However, if the curves C_1 and C_2 are geodetics on two different developable surfaces F_1 and F_2 with parallel generators then one can associate the curve C_2 on F_1 with a line C_2' by parallel line elements. Since C_2' defines the same angle with C at corresponding points, the same thing will also be true for C_1 and C_2 .

If two curves can be related to each other by parallel principal normals then their tangents at corresponding points will define the same angle ϑ everywhere (*).

The curvature ratio of the curves depends upon the angle ϑ. Initially, the angle of the total curvature $d\omega''$ is the same for all curves with parallel principal normals. If one now denotes the angle that the curve C_1 defines with its rectifying lines by φ_1 and the corresponding angle between the curve C_2 and its rectifying lines by φ_2 then if κ_i , τ_i denote the curvature and torsion of the curve C_i , one will have:

$$\tan \varphi_1 = \frac{\kappa_1}{\tau_1}, \quad \tan \varphi_2 = \frac{\kappa_2}{\tau_2},$$

and

$$\varphi_2 - \varphi_1 = \vartheta .$$

That will then imply that:

$$\frac{\kappa_2 \tau_1 - \kappa_1 \tau_2}{\tau_1 \tau_2 + \kappa_1 \kappa_2} = \tan \vartheta ,$$

i.e., the ratio of curvature and torsion of a curve is a rational-linear function of the same quantities for the other curve (**).

For $\vartheta = 0$, one returns to the transformation by parallel line elements, and for $\vartheta = \pi / 2$, one will get the special type of association for which the tangents to one curve are parallel to the binormals of the other.

From now on, we shall consider the transformations for which corresponding determining data are perpendicular to each other.

4. *Let two curves C_1 and C_2 be related to each other by orthogonality of their line-elements.*

One generally realizes that transformation in the following way:

One chooses any filar evolvent (*Filarevolvente*) on the tangent surface to C_1 and determines the edge of regression of the developable surface, which will be defined by the normals to that

(*) **Aoust**, *Analyse infinitésimale des courbes dans l'espace*, 1876, pp. 369-370.
(**) **Aoust**, *loc. cit.*, pp. 370.

tangent surface along the filar evolvent, so the curves that are parallel to that edge of regression will be associated with the curve C by orthogonal line elements.

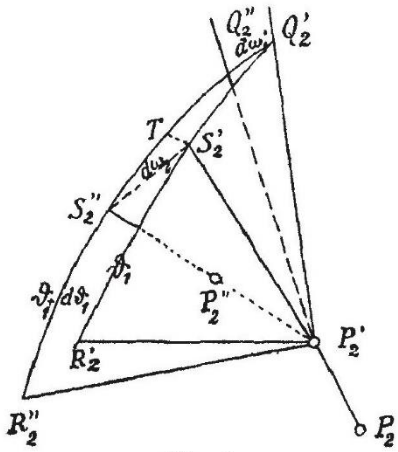

Fig. 1.

The dependency that exists between the curvatures and torsions of the two curves C_1 and C_2 depends upon the arbitrarily-prescribed law by which the angle ϑ_1 between the tangent to C_2 and the principal normal of C_1 changes.

Let P_2, P_2', P_2'' be neighboring points of the curve C_2 . Lay the plane that is perpendicular to the tangent $P_1 P_1'$ to the curve C_1 through $P_2 P_2'$, and likewise let the normal plane to the line element $P_1' P_1''$ to the first curve be laid through $P_2' P_2''$. The two planes intersect the neighboring tangents to C_1 at an angle of $d\omega_1$ in a line $P_2' Q_2'$ that is parallel to the binormal to C_1 at P_1. Furthermore, in the first plane, lay the parallel $P_2' R_2'$ to the principal normal of C_1 at P_1, and in the second plane, law the parallel $P_2' Q_2''$ to the principal normal and the parallel to the binormal of C_1 at P_1'. If one now constructs the unit sphere around P_2' (which might intersect all lines in the endpoints that were just denoted, as well as intersecting $P_2 P_2'$ at S_2', and $P_2' P_2''$ at S_2'') then that will yield the following relations in the figure with no further analysis:

$$\begin{aligned}
&\sphericalangle S_2' Q_2' S_2'' = d\omega_1 , \\
&\text{arc } S_2' Q_2' = \frac{\pi}{2} - \vartheta_1 , \\
&\text{arc } S_2'' Q_2' = \frac{\pi}{2} + d\omega_1' - (\vartheta_1 + d\vartheta_1) , \\
&\text{arc } S_2' S_2'' = d\omega_2 .
\end{aligned}$$

If one now drops the altitude $S_2' T$ to S_2' to $S_2'' Q_2'$ then it will be equal to $d\omega_1 \cos \vartheta$, and the infinitesimal triangle $S_2' S_2'' T$ implies that:

$$d\omega_2^2 = d\omega_1^2 \cos^2 \vartheta_1 + (d\vartheta_1 - d\omega_1')^2 .$$

If one now establishes an arbitrary relation between the line-elements ds_1 and ds_2 of the two curves then that will give:

$$ds_2 = f_2 \, ds_1 ,$$
$$f_2^2 \kappa_2^2 = \kappa_1^2 \cos^2 \vartheta_1 + (d\vartheta_1 - d\omega_1')^2$$

as the characteristic system of equations for the transformation. In them, κ_1 and κ_2 mean the curvatures of the two curves, while τ_1 and τ_2 mean their torsions.

In order to also relate τ_2 to the determining data of C_1, one employs the reciprocity between the properties of the two curves. Since, in fact, the tangents to C_2 are also perpendicular to those of C_1, if ϑ_2 means their angle with the principal normals to C_1 then the equation:

$$d\omega_1^2 = d\omega_2^2 \cos^2 \vartheta_2 + (d\vartheta_2 - d\omega_2')^2$$

must be fulfilled.

If the tangents to C_1 define the angle ϑ_2 with the principal normal to C_2 then their angle with the binormal to C_2 will be equal to its complement. However, the angle between the normal plane of C_1 and the osculating plane of C_2 will then be equal to $\pi / 2 - \vartheta_2$. Nonetheless, the angle between those two planes is denoted by $T\, S_2'' S_s'$ in the figure, so one will have:

$$\sin \vartheta_2 = \frac{d\vartheta_1 - d\omega_1'}{d\omega_2},$$

and correspondingly:

$$\sin \vartheta_1 = \frac{d\vartheta_2 - d\omega_2'}{d\omega_1}.$$

That extremely general transformation (in whose equations the two functions f and ϑ are subject to no sort of restriction, so it is completely arbitrary) encompasses certain special cases that are especially important.

Some of those cases are the following ones:

a. *The principal normals to C_2 are parallel to the tangents of C_1.*

In this case, one will have:

$$\vartheta_2 = 0,$$

so

$$d\vartheta_1 = d\omega_1',$$

as well as:

$$d\omega_2 = d\omega_1 \cos \vartheta_1$$

and

$$d\omega_1' = d\omega_1 \cos \vartheta_1 .$$

The geometric interpretation of that transformation, which will be proved in the Part Three of the present investigation as a convenient tool for treating some special problems, is simple.

If the principal normal to C_2 is parallel to the tangent of C_1 then the rectifying plane of C_2 will also be parallel to the normal plane of C_1, i.e., the rectifying curve of C_2 will be associated with the curve of the center of the osculating sphere of C_1 by parallel line-elements.

b. Should the *binormals to* C_2 *be parallel to the tangents of* C_1 then the principal normals to the curve must necessarily be parallel to each other, and one will, in turn, be dealing with a case that is of lesser interest and shall be addressed with all due brevity later on.

5. *The relation between two curves by perpendicular binormals* offers no intrinsic interest, since one will be led directly to the transformation by orthogonal line elements that was just dealt with by going to the curve of the centers of the osculating spheres.

6. *The curves* C_1 *and* C_2 *have perpendicular principal normals at corresponding points.*

In this case, the rectifying planes are perpendicular to each other, and parallels to the principal normals h_2 to the curve C_2 lie in the rectifying plane r_1 of the curve C_1 .

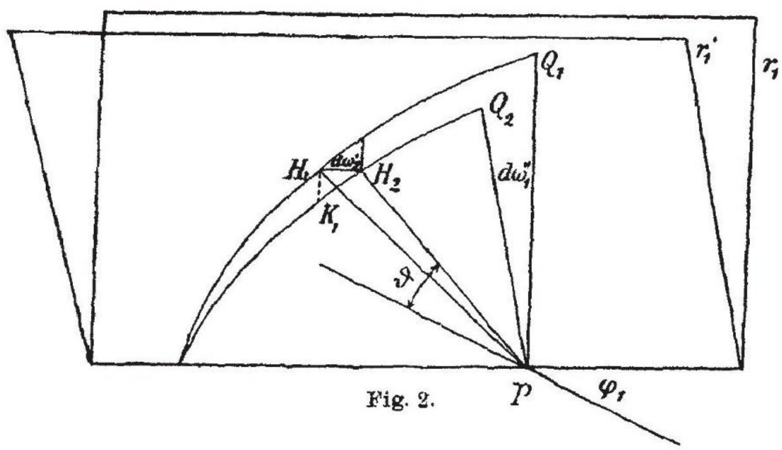

Fig. 2.

If one erects the altitudes PQ_1 and PQ_2 to the line of intersection between two neighboring rectifying planes then they will subtend the angle $d\omega_1''$ of the total curvature to the curve C_1 . Two successive tangents to C_1 define the same angle φ_1 with the rectifying lines, which is given by the known equations:

$$\sin \varphi_1 = \frac{d\omega_1}{\sqrt{d\omega_1^2 + d\omega_1'^2}} ,$$

$$\cos \varphi_1 = \frac{d\omega_1'}{\sqrt{d\omega_1^2 + d\omega_1'^2}} .$$

The successive principal normal directions PH_1 and PH_2 define angles ϑ_1 and $\vartheta_1 + d\vartheta_1$ with the tangent directions. If one again constructs the unit sphere around P and drops the altitude H_1K_1 from H_1 to $Q_2 H_2$ then one will have:

$$H_1 K_1 = \sin (\varphi_1 + \vartheta_1) ,$$
$$K_1 H_2 = d\vartheta_1 ,$$
$$H_1 H_2 = d\omega_2' ,$$

such that when one considers the defining equations for φ_1, that will yield:

$$d\omega_2''^2 = (d\omega_1 \cos \vartheta_1 + d\omega_1' \sin \vartheta_1)^2 + d\vartheta_1^2 .$$

Analogously, one also naturally has the equation:

$$d\omega_1''^2 = (d\omega_2 \cos\vartheta_2 + d\omega_2' \sin\vartheta_2)^2 + d\vartheta_2^2 ,$$

in which ϑ_2 means the angle between the principal normals to C_1 and the tangents to C_2 .

The extremely general association between curves C_1 and C_2 will become easier to see when one represents it by a sequence of simple transformations that are known already.

If the principal normals to the curves C_1 and C_2 are perpendicular to each other then the rectifying planes will be, as well, and the edges of regression C_3 and C_4 for the developable surface that they envelop will possess orthogonal binormals. If one goes from the curves C_3 and C_4 to the curves that are defined by the centers of their osculating spheres then those lines C_5 and C_6 will be associated with each other by perpendicular tangents. Instead of the curve of the centers of the osculating spheres, one can also go from C_3 and C_4 to any two planar evolvents of those curves, which can indeed be obtained from the former by a transformation that is mediated by parallel line-elements. In other words:

If two space curves are connected by orthogonal line-elements then the geodetic lines of their polar surfaces relate to each other by perpendicular principal normals,

and conversely:

If two space curves possess perpendicular principal normals at corresponding points everywhere then their rectifying surfaces will be polar surfaces of curves that correspond to each other by perpendicular line-elements.

A slight modification of that line of reasoning will give the following general construction of a curve C_2 that corresponds to an arbitrarily-given curve C_1 in the required way:

One associates the curve C_1 with a curve C' in such a way that its tangents are parallel to the principal normals of C_1 . If one now lays an arbitrary developable surface through C' then its geodetic lines C_2 will be associated with the curve C_1 by orthogonal principal normals.

For the sake of completeness, and with all due brevity, let us examine those transformation for which the inequivalent edges of the fundamental trihedron of the curves C_1 and C_2 are associated with each other by parallelism or orthogonality and reduce them to the cases that are known already.

Should the *tangent to C_1 be parallel to the binormal to C_2*, then one would have to associate C_1 with the edge of regression of the polar surface by parallel tangents. That transformation is then characterized by the equations:

$$d\omega_1 = d\omega_2' ,$$

$$d\omega_1' = d\omega_2 .$$

The case in which *the tangent to* C_1 *is parallel to the principal normal of* C_2 was treated before at a previous location as a noteworthy special case of the transformation by orthogonal line-elements.

If the *binormals to* C_1 *are supposed to be parallel to the principal normals to* C_2 then the tangents to the curves of the centers of their osculating spheres will be associated with principal normals to C_1 by parallels. The roles of $d\omega_1$ and $d\omega_1'$ are then switched in the characteristic equations for the aforementioned transformation without any further alterations occurring.

Should *the tangents to* C_1 *be perpendicular to the binormals to* C_2, then C_1 would be associated with the curve of the centers of the osculating spheres, or what amounts to the same thing, the planar evolvent of C_2 by orthogonal line-elements.

If *the tangents to* C_1 *are perpendicular to the principal normals to* C_2 then C_1 will be associated with the rectifying curve of C_2 in the manner that was just dealt with, i.e., the tangents to C_1 will be perpendicular to the planar evolvents of the rectifying curve of C_2 .

Finally, when *the binormals to* C_1 *are supposed to be perpendicular to the principal normals to* C_2, the same argument as before will imply that the curve of centers of the osculating spheres for C_1 must be associated with the edge of regression of the polar surface to the rectifying curve of C_2 by orthogonal line elements.

II. – Loxodromes.

One refers to curves that intersect the parallel circles of a surface of revolution at a constant angle as *loxodromes*, or what amounts to the same thing, curves that pierce the planes of a pencil isogonally (*).

One can associate any curve in space with a loxodrome by parallel tangents, and indeed for any arbitrarily-prescribed direction of the carrier of the pencil of planes and for every value ε of the constant angle.

Geometrically, that association can be realized as follows: For every point P of the given space curve C, one constructs the line PQ that defines an angle of $\pi / 2 - \varepsilon$ with the tangent to the curve PP_1 in the plane that is perpendicular to the direction of the chosen carrier t. One lays a plane perpendicular to that line through t. Now, if the plane E corresponds to the point P of the given curve and the plane E_1 corresponds to the neighboring point P_1 then one can start from an arbitrary point R of E and construct the parallel to PP_1, which meets E_1 at R_1 . If one continues the construction of R_1 that was just set down then one will obtain further points R_2, R_3, … The polygonal path that is constructed in that way will give a loxodrome when one passes to the limit and it will be associated with the curve P by parallel line elements. Since the determination of the line PQ is generally double-valued, there will always be two real or imaginary loxodromes that satisfy the conditions and coincide only when the starting curve C defines the angle ε with the carrier t of the pencil of planes everywhere. C will then be a general helix and a geodetic on a cylinder whose generating lines are parallel to t. The loxodrome that is found will then be nothing but the usual helix, which does, in fact, lie on the cylinder of revolution as a loxodrome.

(*) **G. Scheffers**, "Über Loxodromen," Leipziger Ber. (1902), 363-370.

In that way, one will easily get all loxodromes that are simultaneously general helices. All such curves can, in fact, be related to the ordinary helices by parallel line-elements. One then exhausts that class of curves when one chooses the curve *C* to be the ordinary helix and then ascribes all possible positions to the line *t* and all possible values to the angle ε.

In order to put the line of reasoning that was just pursued into an analytical form, one denotes the direction cosines of the tangent to the given curve by a, b, c, those of the binormal by a', b', c', and those of the principal normal by a'', b'', c''.

Let the fixed line *t*, which is chosen to be isogonal to the pencil of planes that goes through it as the carrier of the loxodrome, be given by the equations:

$$\begin{aligned} x &= p + \alpha u , \\ y &= q + \beta u , \\ z &= r + \gamma u , \end{aligned} \tag{1}$$

in which p, q, r, α, β, γ are arbitrarily-chosen constants that might be restricted by only the condition:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 .$$

A plane through the line *t* is determined by the equations:

$$\begin{aligned} x &= p + \alpha u + \alpha_1 u , \\ y &= q + \beta u + \beta_1 u , \\ z &= r + \gamma u + \gamma_1 u . \end{aligned} \tag{2}$$

If the α_1, β_1, γ_1 in them do not mean constants, but functions of a parameter, then equations (2) will represent the pencil of planes:

$$\begin{vmatrix} x - p & \alpha & \alpha_1 \\ y - q & \beta & \beta_1 \\ z - r & \gamma & \gamma_1 \end{vmatrix} = 0 ,$$

whose axis is the line *t*.

In any of the planes that correspond to constant values of α_1, β_1, γ_1, the lines u = const. and v = const. will define a coordinate system that one can assume to be rectangular with no loss of generality. If that is the case then in addition to the condition:

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1 \tag{3}$$

that those quantities must fulfill as the direction cosines of a line, one must also add the equation:

$$\alpha\,\alpha_1 + \beta\,\beta_1 + \gamma\,\gamma_1 = 0 , \tag{4}$$

while the prescribed condition that the direction (a, b, c) defines the angle ε with the planes of the pencil will assume the form:

$$\text{(5)} \qquad \sin\varepsilon = \begin{vmatrix} a & \alpha & \alpha_1 \\ b & \beta & \beta_1 \\ c & \gamma & \gamma_1 \end{vmatrix}.$$

If one performs the elementary calculation that determines $\alpha_1, \beta_1, \gamma_1$ as functions of the given direction cosines a, b, c from equations (3), (4), (5) then one will get:

$$\text{(A)} \qquad \begin{aligned} \alpha_1 &= \frac{1}{1-\lambda^2}\left[(b\gamma - c\beta)\sin\varepsilon + (a - \alpha\lambda)\sqrt{\cos^2\varepsilon - \lambda^2}\right], \\ \beta_1 &= \frac{1}{1-\lambda^2}\left[(c\alpha - a\gamma)\sin\varepsilon + (b - \beta\lambda)\sqrt{\cos^2\varepsilon - \lambda^2}\right], \\ \gamma_1 &= \frac{1}{1-\lambda^2}\left[(a\beta - b\alpha)\sin\varepsilon + (c - \gamma\lambda)\sqrt{\cos^2\varepsilon - \lambda^2}\right]. \end{aligned}$$

The square root can be positive or negative in that. The quantities:

$$\begin{aligned} \lambda &= a\ \alpha + b\ \beta + c\gamma, \\ \lambda' &= a'\ \alpha + b'\beta + c'\gamma, \\ \lambda'' &= a''\alpha + b''\beta + c''\gamma \end{aligned}$$

mean the cosines of the angles that the fixed line makes with the tangent, binormal, and principal normal to the curve, resp.

Equations (2) will represent the desired loxodrome when one determines u and v in such a way that the direction of the tangent to the curve coincides with the direction a, b, c.

The condition for that, namely:

$$\begin{aligned} a\,ds - \alpha\,du - \alpha_1\,dv &= v\,d\alpha_1, \\ b\,ds - \beta\,du - \beta_1\,dv &= v\,d\beta_1, \\ c\,ds - \gamma\,du - \gamma_1\,dv &= v\,d\gamma_1, \end{aligned}$$

will be fulfilled by the expressions:

$$\text{(I)} \qquad \begin{aligned} \sin\varepsilon\frac{dv}{v} &= -\frac{d\omega}{1-\lambda^2}\left(\lambda\lambda''\sin\varepsilon + \lambda'\sqrt{\cos^2\varepsilon - \lambda^2}\right), \\ \sin\varepsilon\frac{du}{v} &= -\frac{\lambda\,d\omega}{1-\lambda^2}\left(\lambda' + \frac{\lambda\lambda''\sin\varepsilon}{\sqrt{\cos^2\varepsilon-\lambda^2}}\right), \end{aligned}$$

$$\sin\varepsilon \frac{ds}{v} = -\frac{d\omega}{1-\lambda^2}\left(\lambda' + \frac{\lambda\,\lambda''\sin\varepsilon}{\sqrt{\cos^2\varepsilon - \lambda^2}}\right),$$

after some reductions of the determinants to be solved. In those equations:

$$d\omega = \frac{da}{a''} = \frac{db}{b''} = \frac{dc}{c''} = \frac{d\lambda}{\lambda''}$$

means the contingency angle, u is the length of the axis of the pencil, as measured from an arbitrary starting point, and v is the distance from the fixed axis to the point (x, y, z).

In order to get the finite equations of the loxodrome, one must first integrate equation (I) in order to determine v and then, once the value that was found for v is substituted in the second equation, determine u from it by quadrature.

The determination of the finite equations of the loxodrome that can be associated with a given curve by parallel tangents requires two quadratures.

Of the natural equations for the curve, one of them gives the determination of v by merely eliminating it from the equations:

$$\frac{\tau}{\kappa} = \frac{d\omega}{d\omega'} = \frac{da}{da'}$$

and

$$\sin\varepsilon \frac{1}{\kappa} = \frac{v}{1-\lambda^2}\left(\lambda' + \frac{\lambda\,\lambda''\sin\varepsilon}{\sqrt{\cos^2\varepsilon - \lambda^2}}\right),$$

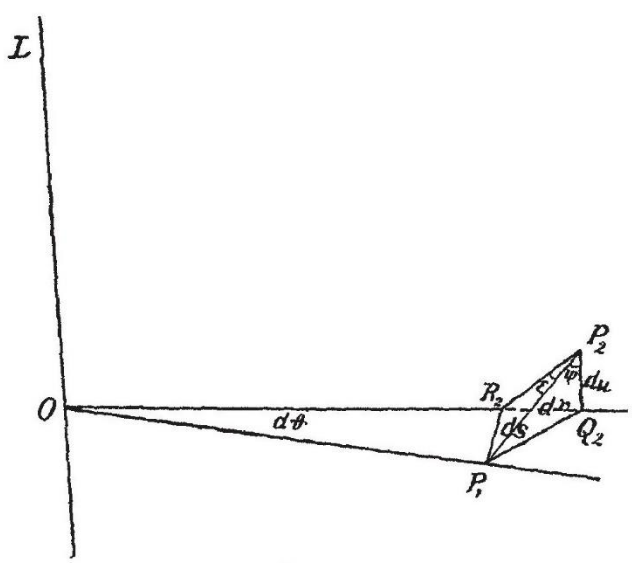

Fig. 3.

while the second one requires the integration of the third equation in (I). The natural equations of the loxodromes with a prescribed indicatrix for the tangents will then be likewise found by two quadratures, independently of their finite equations.

Relations exist between du, dv, and ds that are easy to give, as one can read them off from the formulas, but also as a simple geometric picture will show. Namely, if LOP_1 and LOQ_2 are two neighboring planes of the pencil and $P_1 P_2 = ds$ is the arc-length of the loxodrome then the altitudes OP_1 and OQ_2 to LO are equal to v and $v + dv$, resp., while the altitude $P_2 Q_2$ to OQ_2 is equal to du and the cosine of the angle $P_1 P_2 Q_2 = \psi$ is equal to λ. If one now drops the altitude $P_1 R_2$ from P_1 to $Q_2 O$ then $Q_2 R_2 = dv$, and the angle $R_2 P_2 P_1$ will equal ε. It will then follow immediately from the triangle $P_1 Q_2 P_2$ that:

(6) $$\frac{du}{ds} = \lambda ,$$

and from the triangle $R_2 P_2 Q_2$ that:

$$dv^2 = R_2 P_2^2 - du^2 = ds^2 \cos^2 \varepsilon - ds^2 \lambda^2 ,$$

such that one will have:

(7) $$\frac{dv}{ds} = \sqrt{\cos^2 \varepsilon - \lambda^2} .$$

If one denotes the angle $P_1 O Q_2$ between two neighboring planes of the pencil by $d\vartheta$ then that will give, analytically:

$$d\vartheta^2 = d\alpha_1^2 + d\beta_1^2 + d\gamma_1^2 = \frac{d\omega^2}{(1-\lambda^2)^2}\left(\lambda' + \frac{\lambda \lambda'' \sin \varepsilon}{\sqrt{\cos^2 \varepsilon - \lambda^2}}\right)^2 ,$$

which is identical to the geometrically-obvious equation:

$$v \, d\vartheta = ds \sin \varepsilon .$$

That equation, along with (6) and (7), mediates the transition from the path of investigation that was followed here to the usual representation, by which one achieves the general relation between u, v, and ϑ:

(II) $$v^2 \, d\vartheta^2 = \tan^2 \varepsilon \, (du^2 + dv^2)$$

as one does here, as well.

If one would like to find all *loxodromes that are simultaneously helices* then one must introduce the direction cosines of the tangent, binormal, and principal normal of the usual helix:

$$x = \cos t , \qquad y = \sin t , \qquad z = m\, t$$

for a, b, c ; a', b', c' ; a'', b'', c'' in the system of formulas (I), so one must set:

$$\lambda = \frac{1}{\sqrt{1+m^2}}(-\alpha \sin t + \beta \cos t + \gamma\, m) ,$$

$$\lambda' = \frac{1}{\sqrt{1+m^2}}\left(-\alpha \sin t + \beta \cos t + \frac{\gamma}{m}\right),$$

$$\lambda'' = -\alpha \cos t - \beta \sin t ,$$

$$d\omega = \frac{dt}{\sqrt{1+m^2}} .$$

That will give:

$$\sin \varepsilon \ln v = \sin \varepsilon \ln \sqrt{1-v^2} - \int \frac{m\,dt\left(-\alpha \sin t + \beta \cos t + \dfrac{\gamma}{m}\right)\sqrt{R}}{(1+m^2) - (-\alpha \sin t + \beta \cos t + \gamma m)^2}$$

and

$$u = \int \frac{-\alpha \sin t + \beta \cos t + \gamma m}{\sqrt{1+m^2} \cdot \sqrt{R}}\, dv ,$$

which are equations in which we have set:

$$R = \cos^2 \varepsilon - \frac{1}{1+m^2}(-\alpha \sin t + \beta \cos t + \gamma\, m)^2 ,$$

to abbreviate.

The generality of the formulas will not be restricted if one assumes that $\alpha = 0$ or $\beta = 0$. That assumption would require only a rotation of the coordinate system.

By contrast, if $\alpha = \beta = 0$ then the integrand of the quadrature that appears in the expression for log v will reduce to a constant, and in that case it will result that the cylinder axis is parallel to the axis of the pencil, as it must be, as the loxodromic line of the cylindro-conical helix.

If the axes of the helix and the loxodrome cross at right angles then $\gamma = 0$. In that way, log v can be expressed in terms of elementary functions of t, while in the general case, the integral that appears is an elliptic one:

The problem of the loxodromic helix is solved by elliptic integrals and their degenerations.

As Scheffers has proved (*), the *finite equations of all loxodromes* can be determined by performable operations. Their explicit presentation is easily connected with equation (2), which emerges from the equation of the minimal lines:

$$dx^2 + dy^2 + dz^2 = 0$$

by way of the transformation:

$$x = v \cos (i\, \vartheta \tan \varepsilon) ,$$
$$y = v \sin (i\, \vartheta \tan \varepsilon) , \tag{8}$$
$$z = u .$$

Now one knows the finite equations of the minimal lines:

(*) **G. Scheffers**, "Über Loxodromen," Leipziger Ber. (1902), 363-370.

$$x = T' \cos t - T'' \sin t\,,$$
$$(9) \qquad y = T' \sin t + T'' \cos t\,,$$
$$z = i\,(T + T'')\,,$$

which are equations in which T means an arbitrary function of the parameter t. The determining data u, v, ϑ of the general loxodrome are obtained immediately in finite form from equations (8) and (9). The assumption that $\alpha = \beta = 0$ is no loss of generality since it implies a simple rotation of the coordinate system. The equations:

$$x = v\,\alpha_1 = v \cos\,\vartheta\,,$$
$$y = v\,\beta_1 = v \sin\,\vartheta\,,$$
$$z = u\,,$$

in which u, v, ϑ are inferred from (8) and (9) in the given way, will then represent the general loxodrome whose axis is the z-axis (*).

For *loxodromes of constant curvature:*

$$\kappa = \frac{d\omega}{ds} = \frac{\sin\varepsilon}{v\,C} = \frac{\sin\varepsilon}{p}$$

must be constant, so:

$$v = \frac{p}{C}\,,$$

in which p means a constant, and one has set:

$$C = -\,\frac{1}{1-c^2}\left(c' + \frac{c\,c''\sin\varepsilon}{\sqrt{\cos^2\varepsilon - c^2}}\right)$$

to abbreviate.

The problem then leads to the following system of equations:

$$\sin\,\varepsilon\,dv = p\,\frac{dc}{c''}\,\sqrt{\cos^2\varepsilon - c^2}\,,$$

$$\frac{p}{v} = -\,\frac{1}{1-c^2}\left(c' + \frac{c\,c''\sin\varepsilon}{\sqrt{\cos^2\varepsilon - c^2}}\right),$$

$$1 = c^2 + c'^2 + c''^2,$$

(*) Cf., **E. Salkowski**, “Schraubenlinien und Loxodromen,” Sitz. Math. Ges. Berlin **7** (1908), 83-87.

which yields a first-order differential equation for c as a function of v. If one regards it as having been solved then one will find u by a quadrature.

The problem of loxodromes of constant curvature requires solving a first-order differential equation and performing a quadrature.

In order for *a curve to be regarded as a loxodrome in two ways*, it is necessary and sufficient that the system of equations (I) (pp. 11) that define the loxodrome as a curve in a pencil of planes must be fulfilled for two different pencils, so the equations:

$$\sin\varepsilon_1 \frac{dv_1}{v_1} = d\omega\, L_1 \sqrt{\cos^2\varepsilon_1 - \lambda_1^2}\,,$$

$$\sin\varepsilon_1 \frac{du_1}{v_1} = d\omega\, L_1\, \lambda_1\,,$$

$$\sin\varepsilon_1 \frac{ds_1}{v_1} = d\omega\, L_1\,,$$

as well as:

$$\sin\varepsilon_2 \frac{dv_2}{v_2} = d\omega\, L_2 \sqrt{\cos^2\varepsilon_2 - \lambda_2^2}\,,$$

$$\sin\varepsilon_2 \frac{du_2}{v_2} = d\omega\, L_2\, \lambda_2\,,$$

$$\sin\varepsilon_2 \frac{ds_2}{v_2} = d\omega\, L_2\,,$$

must be fulfilled simultaneously. In them, one has set:

$$L_1 = -\frac{1}{1-\lambda_1^2}\left(\lambda_1' + \frac{\lambda_1\,\lambda_1''\sin\varepsilon_1}{\sqrt{\cos^2\varepsilon_1 - \lambda_1^2}}\right),$$

and correspondingly:

$$L_2 = -\frac{1}{1-\lambda_2^2}\left(\lambda_2' + \frac{\lambda_2\,\lambda_1''\sin\varepsilon_2}{\sqrt{\cos^2\varepsilon_2 - \lambda_2^2}}\right).$$

Those equations will be compatible if and only if:

$$\frac{L_1\, v_1}{\sin\varepsilon_1} = \frac{L_2\, v_2}{\sin\varepsilon_2}\,,$$

since the same quantity – viz., the radius of curvature of the curve – appears on both sides of the equation.

Taking the logarithmic derivative of both sides of the equation and eliminating v_1 and v_2 will produce the differential equation of the problem in the form of:

$$\frac{dL_1}{L_1} + L_1 \frac{d\omega\sqrt{\cos^2\varepsilon_1 - \lambda_1^2}}{\sin\varepsilon_1} = \frac{dL_2}{L_2} + L_2 \frac{d\omega\sqrt{\cos^2\varepsilon_2 - \lambda_2^2}}{\sin\varepsilon_2} .$$

The following relations between λ_1, λ_1', λ_1'' and λ_2, λ_2', λ_2'' then enter in:

$$\lambda_1^2 + \lambda_1'^2 + \lambda_1''^2 = 1 ,$$

$$\lambda_2^2 + \lambda_2'^2 + \lambda_2''^2 = 1 ,$$

$$\frac{d\lambda_1}{\lambda_1''} = \frac{d\lambda_2}{\lambda_2''} = d\omega ,$$

$$\lambda_1\lambda_2 + \lambda_1'\lambda_2' + \lambda_1''\lambda_2'' = l ,$$

in which l denotes the cosine of the constant angle that the two axes of the pencils define with each other.

If one employs the three finite equations between the six quantities in order to eliminate three of them then what will remain is a system of two first-order differential equations:

The determination of the double loxodromes, i.e., the curves that can belong to two different surfaces of revolution as loxodromic lines, require that one must solve a system of two first-order ordinary differential equations.

III. – Algebraic curves of constant curvature.

The association of two space curves by parallel line-elements is especially useful in the theory of general helices that can only be related to each other in the given way. For other purposes, other transformations that differ from the ones that were examined in the first part generally prove to be of greater use.

In particular, the association of two space curves by orthogonal line elements shall now be called upon in order to investigate a certain class of space curves of constant curvature. Among the curves of that class, which all lie on second-order surfaces of revolution, there are infinitely-many algebraic ones. That result seems remarkable due to the fact that, up to now, despite the simple geometric way of generating curves of constant curvature by bending the plane of a circle in such a way that its tangents remain rectilinear, no one has succeeded in discovering algebraic curves of that kind. The fact that the curves of constant curvature that **Chassiotis** recently found (*) are not algebraic is implied by a consideration of the formulas for his solutions:

$$x = \quad a\int \sin u \cos u \sqrt{1+\cos^2 u}\, du\,,$$
$$y = -\, a\int \cos^2 u \sqrt{1+\cos^2 u}\, du\,,$$
$$z = \quad a\int \sin u \sqrt{1+\cos^2 u}\, du\,,$$

with no further analysis, since one sees that only the first of the quadratures will be solved by an algebraic function of cos u , while the second integral is elliptic and the third one is logarithmic.

A curve $(P) \equiv (x, y, z)$ has direction cosines (a, b, c) for its tangent, (a', b', c') for its binormal, and (a'', b'', c'') for its principal normal. Let its line element be ds, let its curvature be κ, and let its torsion be τ. If the curve $(P_1) \equiv (x_1, y_1, z_1)$ possesses orthogonal line-elements at points that correspond to (P) then one must have:

$$a\, a_1 + b\, b_1 + c\, c_1 = 0$$

or

$$\begin{aligned} a_1 &= a' \sin\vartheta + a'' \cos\vartheta, \\ b_1 &= b' \sin\vartheta + b'' \cos\vartheta, \\ c_1 &= c' \sin\vartheta + c'' \cos\vartheta. \end{aligned} \tag{1}$$

In those and all following formulas, the quantities with the index 1 will have the same meaning for (P_1) that the quantities without the index have for the curve (P). The angle ϑ between the tangent to (P_1) and the principal normal of (P) is subject to no restrictions.

If one now establishes a relation between the arc-length elements ds_1 and ds:

$$ds_1 = f(s)\, ds \tag{2}$$

arbitrarily then one will get:

$$f(s)\, \kappa_1 = \sqrt{\kappa^2 \cos^2 \vartheta + \left[\left(\frac{d\vartheta}{ds}\right)^2 - \tau\right]^2} \tag{3}$$

by means of an elementary calculation that is based upon the **Frenet** equations.

Since $\kappa_1 > 0$, the root must be endowed with the same sign that $f(s)$ possesses, so the positive one as long as one is dealing with real curves. Furthermore, one has:

(*) **S. Chassiotis**, "Notes sur les courbes gauches," Nouvelles Annales (4) **5** (1905), 394-399.

$$f(s)\ \tau_1 = -\frac{\kappa \sin\vartheta \dfrac{d\vartheta}{ds}\left(\dfrac{d\vartheta}{ds}-\tau\right)+\kappa^2\cos\vartheta\left[\dfrac{1}{\kappa}\left(\dfrac{d\vartheta}{ds}-\tau\right)\right]}{\kappa^2\cos\vartheta\left(\dfrac{d\vartheta}{ds}-\tau\right)^2} - \kappa\sin\vartheta. \tag{4}$$

Those equations differ only in form from the formulas that characterized the transformation that was derived geometrically (pp. 5), and they can be obtained from the latter by a easy conversion. They were first found in a recent paper by **G. Sannia** ([*]), who derived then with the help of the methods of natural geometry.

Any element of the curve (P_1) that is associated by orthogonality is obtained from equations (1) by quadratures:

$$\begin{aligned} x_1 &= \int f(s)(a'\sin\vartheta + a''\cos\vartheta)\,ds, \\ y_1 &= \int f(s)(b'\sin\vartheta + b''\cos\vartheta)\,ds, \\ z_1 &= \int f(s)(c'\sin\vartheta + c''\cos\vartheta)\,ds\ . \end{aligned} \tag{5}$$

f and ϑ are completely-arbitrary functions of s in those formulas. If the curve (P) is given then one can assume:

$$\kappa_1 = \kappa_1(s_1), \quad \tau_1 = \tau_1(s_1).$$

Namely, if one eliminates the function $f(s)$ from equations (2), (3), (4) then one will get a system of first and second order differential equations for s_1 and ϑ.

One can always bring two arbitrary curves (P) *and* (P_1) *into a position such that they are associated with each other by orthogonality of elements.*

Only the straight line $\kappa = 0$ defines an exception, since it obviously cannot be associated with anything but plane curves, which is a result that emerges immediately from (3) and (4).

In principle, one can employ that transformation in order to determine the finite equations of curve (κ_1, τ_1, s_1) that is given by its natural equations. Meanwhile, actually following through on one of the simple problems that will be solved by a Ricatti differential equation and quadratures would not require any corresponding analytical devices.

The curves of constant curvature $\kappa_1 = 1$ will be obtained when one introduces:

$$f = \sqrt{\kappa^2\cos^2\vartheta + \left(\frac{d\vartheta}{ds}-\tau\right)^2}$$

into equations (5), so it will be provided by the quadrature:

([*]) **G. Sannia**, "Deformazioni infinitesime delle curve inestendibili e correspondenza per orthogonalita di elementi," Rend. Circ. Mat. Palermo **21** (1906), pp. 236.

$$x_1 = \int (a' \sin \vartheta + a'' \cos \vartheta) \sqrt{\kappa^2 \cos^2 \vartheta + \left(\frac{d\vartheta}{ds} - \tau\right)^2} \, ds \ ,$$

$$y_1 = \int (b' \sin \vartheta + b'' \cos \vartheta) \sqrt{\kappa^2 \cos^2 \vartheta + \left(\frac{d\vartheta}{ds} - \tau\right)^2} \, ds \ ,$$

$$z_1 = \int (c' \sin \vartheta + c'' \cos \vartheta) \sqrt{\kappa^2 \cos^2 \vartheta + \left(\frac{d\vartheta}{ds} - \tau\right)^2} \, ds \ .$$

Those formulas, which represent any curve of constant curvature 1 in an infinitude of ways, since along with the determining data of the curve (P), ϑ will also appear as an arbitrary function of s, would offer no advantage over the numerous other formulas that are possible and would likewise produce curves of constant curvature if they did not allow one to arrive at special curves whose properties are geometrically interesting by a suitable choice of the associated curve and transformation.

Those formulas will become especially simple when one chooses:

$$\frac{d\vartheta}{ds} = \tau ,$$

so one will have, in fact:

(6)
$$f\,\kappa_1 = \kappa \cos \vartheta, \qquad f\,\tau_1 = -\kappa \sin \vartheta,$$

and it will follow from equations (1) that:

$$a_1'' = -a , \qquad b_1'' = -b , \qquad c_1'' = -c .$$

The chosen relation then *associates the tangent of the curve* (P) *with a parallel to the principal normal to the curve* (P_1), and from the results that were obtained previously, the polar surface of (P) will be associated with the rectifying surface of (P_1) by parallel generating lines.

One chooses the curve (P) to be the common helix:

$$x = \cos v , \qquad y = \sin v , \qquad z = m\,v ,$$

so the line (P_1) will be the general geodetic on the tangent surface to a general helix. Here, one will have:

$$\vartheta = \frac{m}{\sqrt{1+m^2}}\, v = n\,v .$$

If one looks for the geodetics for which an arbitrarily-prescribed relation exists between κ_1 and τ_1 then one will get f as a function of ϑ, and therefore the finite equations for the desired curves by quadratures. In particular, if one dealing with the curves of constant curvature $\kappa_1 = 1$ then one will have:

$$f = \frac{\cos nv}{1+m^2} ,$$

and when one consistently drops the index on the coordinate relation from now on:

$$x = -\frac{1}{\sqrt{1+m^2}}\int (n \sin v \sin nv + \cos v \cos nv) \cos nv \, dv ,$$

$$y = +\frac{1}{\sqrt{1+m^2}}\int (n \cos v \sin nv - \sin v \cos nv) \cos nv \, dv ,$$

$$z = -\frac{1}{1+m^2}\int \sin nv \cos nv \, dv ,$$

and after performing the quadratures:

$$x = -\frac{1}{\sqrt{1+m^2}}\left[\frac{1-n}{4(1+2n)}\sin(1+2n)v + \frac{1+n}{4(1-2n)}\sin(1-2n)v + \tfrac{1}{2}\sin v\right],$$

(A) $$y = \frac{1}{\sqrt{1+m^2}}\left[\frac{1-n}{4(1+2n)}\cos(1+2n)v + \frac{1+n}{4(1-2n)}\cos(1-2n)v + \tfrac{1}{2}\cos v\right],$$

$$z = \frac{1}{4m\sqrt{1+m^2}}\cos 2n\,v .$$

One can see from the right-hand sides that algebraic equations will always exist between the coordinates when n is a rational number. In that case, one will always get *algebraic curves of constant curvature* then. All of the curves that are represented by equations (A) lie on second-order surfaces of revolution with vertical axes of rotation:

$$A(x^2 + y^2) = B + C\,(z + D)^2,$$

in which A, B, C, D depend upon only the inclination of the helix:

$$A = 1+m^2 ,$$

$$B = \frac{27\,n^4}{4(1-n^2)(1-4n^2)^2} ,$$

$$C = \frac{4n^2}{(1-4n^2)(1-n^2)} ,$$

$$D = \frac{1}{4n}(1+2n^2) .$$

We combine all of those results into:

The geodetic lines on the tangent surface to the general helix can be found by quadratures, and the curves of constant curvatures are represented explicitly among them. They are always the intersection curves of the tangent surface with second-order surfaces of revolution. They will therefore be algebraic when the edge of regression of the developable surface on which they lie is algebraic.

Obviously, not just *any* tangent surface to a helix will have geodetic lines that possess constant curvature. One will get the helices on whose tangent surface such lines exist when one determines the edge of regression (ξ, η, ζ) of the rectifying surface to the curve (x, y, z) that is given by the system of equations (A). If one applies the general formulas (*):

$$\xi = x + \kappa \frac{\kappa a' - \tau a}{\kappa \tau' - \tau \kappa'} ,$$

$$\eta = y + \kappa \frac{\kappa b' - \tau b}{\kappa \tau' - \tau \kappa'} ,$$

$$\zeta = z + \kappa \frac{\kappa c' - \tau c}{\kappa \tau' - \tau \kappa'}$$

to our example then since the curvature is:

$$\kappa = 1 ,$$

the torsion is:

$$\tau = -\tan n v ,$$

and the arc-length element is:

$$ds = \frac{1}{\sqrt{1+m^2}} \cos n v \, dv ,$$

then that will give:

$$\xi = x + \frac{1}{\sqrt{1+m^2}} \sin v \cos^2 n v ,$$

$$\eta = y + \frac{1}{\sqrt{1+m^2}} \cos v \cos^2 n v ,$$

(*) Cf., say, **G. Scheffers**, *Einleitung in die Theorie der Kurven*, 1901, pp. 354-355.

$$\zeta = z + \frac{1}{m\sqrt{1+m^2}} \cos^2 n\,v\,,$$

or:

$$\xi = \frac{3n}{4\sqrt{1+m^2}}\left\{\frac{\sin\,(1+2n)\,v}{1+2n} - \frac{\sin\,(1-2n)\,v}{1-2n}\right\},$$

(B)
$$\eta = \frac{-3n}{4\sqrt{1+m^2}}\left\{\frac{\cos\,(1+2n)\,v}{1+2n} - \frac{\cos\,(1-2n)\,v}{1-2n}\right\},$$

$$\zeta = \frac{3}{4m\sqrt{1+m^2}}\cos 2n\,v + \frac{1}{2m\sqrt{1+m^2}}\,.$$

Those equations tell us that the helices on whose tangent surfaces the curves of constant curvature that we seek will lie also belong to a second-order surface of revolution with the z-axis as its axis of rotation. Namely, we will have:

$$A'(\xi^2 + \eta^2) = B' + C'(\zeta + D')^2,$$

in which we have set:

$$A' = \frac{16}{9}\frac{1+m^2}{n^2},$$

$$B' = \frac{4}{(1-4n^2)^2},$$

$$C' = -\frac{64}{9}\frac{m^2(1+m^2)}{1-4n^2},$$

$$D' = -\frac{1}{2m\sqrt{1+m^2}} = -\frac{1-n^2}{2n}\,.$$

If we compare that result to the previous one then we will see that A, A', B, B' are always positive, while C and C' always have opposite signs. Thus, if the helix lies on an ellipsoid of revolution, which happens for:

$$n < \frac{1}{2},$$

then the curve (A) will belong to a hyperboloid of revolution. By contrast, for:

$$n > \frac{1}{2},$$

the helix will lie on a hyperboloid and its geodetics of constant curvature will lie on an ellipsoid.

The elements of the helix are determined from equations (B), which then give its differential of arc-length:

$$ds = \frac{\pm 3}{\sqrt{1+m^2}} \cos n\,v \sin n\,v\,dv\,,$$

such that the arc-length will become:

$$\sigma = \pm \frac{3}{4m} \cos 2\,n\,v\,,$$

while its radius of curvature will be:

$$\rho = \frac{3}{4m} \sin 2n\,v\,.$$

Since the tangents to the curve are parallel to the binormals to the helix on the circular cylinder that the investigation started from, the constant ratio of the curvature to torsion of our curve will be equal to *m*. Its natural equations accordingly read:

$$\frac{\kappa}{\tau} = m$$

and

$$\frac{16\,m^2}{9}\sigma^2 + \frac{4\,m^2}{9}\rho^2 = 1\,.$$

With that, the following theorem has been proved:

The helices whose natural equations are:

$$\frac{\kappa}{\tau} = m\,, \qquad 4\sigma^2 + \rho^2 = \left(\frac{3}{2m}\right)^2$$

belong to second-order surfaces of revolution, and will indeed lie on an ellipsoid of revolution when:

$$n = \frac{m}{\sqrt{1+m^2}} < \frac{1}{2}$$

and a hyperboloid of revolution with one sheet when:

$$1 > n > \frac{1}{2}\,.$$

A geodetic line of constant curvature lies on its tangent surface, which likewise belongs to a second-order surface of revolution with the same axis of rotation. Both curves – viz., the helices, as well as the geodetics of constant curvature – will be algebraic when n is a rational number.

The natural equations of the helix imply that one will get them when one twists the astroid:

$$\xi^{2/3} + \eta^{2/3} = \left(\frac{1}{m}\right)^{2/3},$$

or

$$\xi = \frac{1}{m}\cos^3\varphi\,, \qquad \eta = \frac{1}{m}\sin^3\varphi\,,$$

in such a way that it goes to a helix with a curvature radius of m; in other words, in such a way that it defines a constant angle with the ξ-axis whose tangent is m. If one observes that the angle φ has the same meaning for a plane curve that the angle nv has for the twisted one, and that furthermore, the segment of the tangent to the astroid has a length:

$$e = \frac{1}{m}\cos^2\varphi$$

from the curve to the ξ-axis, which is a length that equals the distance from a point (x, y, z) on the curve (A) to the corresponding point (ξ, η, ζ) on the curve (B), then one will see that the curve of constant curvature emerges from the ξ-axis by the torsion of the astroid. In so doing, one should probably observe that the ξ-axis of the plane figure must be regarded in two different ways: On the one hand, it is the tangent to the astroid at a cusp, and then it is the shortest connecting line between two cusps, which are lines that take on an entirely-different meaning from the torsion in space. The former is also rectilinear on the tangent surface, while the latter is the geodetic line of constant curvature that we are concerned with.

In order to understand the form of the helix (B) and the curve (A) that is connected with it more precisely, we shall first consider the cylinder, on which the helix is geodetic. Its guiding curve in the xy-plane has the equations:

$$\xi_0 = p\left(\frac{\sin\alpha v}{\alpha} - \frac{\sin\beta v}{\beta}\right),$$

$$\eta_0 = -p\left(\frac{\cos\alpha v}{\alpha} - \frac{\cos\beta v}{\beta}\right),$$

in which we have set:

$$\frac{3n}{4\sqrt{1+m^2}} = p\,,$$

$$1 + 2n = \alpha\,,$$

$$1 - 2n = \beta,$$

to abbreviate. The line element of the curve is determined from the equation:

$$d\sigma_0 = 2p \sin\frac{\alpha - \beta}{2} v\, dv,$$

and its radius of curvature is determined from the formula:

$$\rho_0 = \frac{4p}{\alpha + \beta} \sin\frac{\alpha - \beta}{2} v,$$

such that the natural equation of the curve will read:

$$(\alpha - \beta)^2 \sigma_0^2 + (\alpha + \beta)^2 \rho_0^2 = 16p^2,$$

or when one considers the meaning of α and β:

$$4n^2 \sigma_0^2 + \rho_0^2 = 4p^2.$$

The curve is then (*) an epicycloid when $n < 1/2$ and a hypocycloid when $n > 1/2$. The fixed circle has the radius:

$$R_0 = \frac{4np}{1 - 4n^2},$$

and the rolling circle has the radius:

$$r_0 = \frac{p}{1 + 2n},$$

such that the modulus of the curve will be:

$$\frac{r_0}{R_0} = \frac{1 - 2n}{4n}.$$

The helix, like its projection onto the *xy*-plane and the astroid that arises by twisting it, has cusps everywhere at the points:

$$v = \frac{\lambda\pi}{2n} \qquad (\lambda = 1, 2, 3, \ldots).$$

(*) Cf., **Cesàro**, *Vorlesungen über natürliche Geometrie*, Leipzig, 1902, pp. 9 or **G. Loria**, *Spezielle ebene Kurven*, Leipzig, 1902, pp. 491. The helices on second-order surfaces of revolution, a special case of which is dealt with here, were systematically investigated by **W. Blaschke** in a paper “Bermerkungen über allgemeine Schraublinien” that appeared recently in Monatshefte für Math. u. Phys. **19**, pp. 188-204. One will also find references to the literature of the problem there.

According to whether λ is even or odd, they are distributed on the two parallel circles:

$$\zeta = \frac{5}{4m\sqrt{1+m^2}} \qquad \left(v = \frac{2k\pi}{2n}\right)$$

and

$$\zeta = \frac{-1}{4m\sqrt{1+m^2}} \qquad \left(v = \frac{(2k+1)\,\pi}{2n}\right),$$

resp., whose radii are equal to R_0 and which include the highest and lowest points of the helix.

The curve then lies on both sides of the equator:

$$\zeta = \frac{1}{2m\sqrt{1+m^2}},$$

which splits the curve into two overlapping halves. Its point of intersection with the equator corresponds to the values:

$$v = \frac{\lambda\pi}{4n}$$

of the parameter.

The curve of constant curvature (A) likewise attains its highest point for $v = k\,\pi / n$, and indeed it lies on the parallel circle:

$$\zeta = \frac{1}{4m\sqrt{1+m^2}}$$

of the second-order surface of revolution on which the entire curve lies. Its lowest point, which will be attained for:

$$v = \frac{2k+1}{2n}\pi,$$

coincides with the lowest point of the helix. The parallel circle:

$$\zeta = \frac{-1}{4m\sqrt{1+m^2}}$$

to which it belongs, is then the contact circle (*) of the two coaxial second-order surfaces of revolution that are dealt with here.

For that lowest point $v = \frac{2k+1}{2n}\pi$, one will have:

(*) Cf., pp. 43.

$$ds = 0 ,$$
$$\tau = \infty ,$$

while one has:

$$\kappa = 1 ,$$

as always.

Those relations mean that the curve also possesses cusps for those points. Namely, when the line element *ds* approaches zero, the contingency angle:

$$d\omega = \kappa\, ds$$

will become infinitely small of the same order. The tangent will then be stationary at the point. By contrast, the osculation angle:

$$d\omega' = \tau\, ds = \frac{-1}{\sqrt{1+m^2}} \sin n\, v\, dv$$

will remain of the same order of magnitude as in the remaining extent of the curve. The osculation angle then further rotates around the stationary tangent in the same sense.

By contrast, since:

$$v = \frac{\kappa\,\pi}{n} ,$$

at the highest point of the curve, one will have:

$$\tau = 0 ,$$

so since *ds* has the same order of magnitude as *dv* there:

$$d\omega' = 0 .$$

The curve will then possess a stationary osculation plane at that point.

Before we move on to examine in particular the simplest examples of the curves that are currently known in general, let us summarize the results that were obtained as follows:

If one twists an astroid without changing its curvature in such a way that it can be laid on a second-order surface of revolution whose axis is vertical as a curve of constant inclination then it will go to a helix whose projection onto the equatorial plane of the surface of revolution will be a cyclic curve, and indeed an epicycloid when the curve is laid on an ellipsoid and a hypocycloid when the curve is laid on a hyperboloid of one sheet. Under torsion, the axes of the astroid go to geodetic lines of the tangent surface that connect two cusps that are not successive. They are curves of constant curvature that likewise lie on a second-order surface of revolution and have cusps at the cusps of the edge of regression itself. They are algebraic when they are closed, and that will be the case when the astroid is twisted into an algebraic helix. The curves thus-

constructed are the only geodetics on the tangent surfaces of helices that possess constant curvature.

The assumption that:

$$n = \tfrac{1}{2}$$

played a special role in all of our investigations. In that case, the quadratures that determine the coordinates (x, y, z) of the curve (A) cannot be performed as usual. Namely, we will have:

$$x = -\frac{1}{\sqrt{1+m^2}}\int\left(\frac{1}{2}\sin v\sin\frac{v}{2}+\cos v\cos\frac{v}{2}\right)\cos\frac{v}{2}\,dv\;,$$

$$y = \frac{1}{\sqrt{1+m^2}}\int\left(\frac{1}{2}\cos v\sin\frac{v}{2}-\sin v\cos\frac{v}{2}\right)\cos\frac{v}{2}\,dv\;,$$

$$z = -\frac{1}{\sqrt{1+m^2}}\int\frac{1}{2}\sin v\cos\frac{v}{2}\,dv\;,$$

from which, the quadratures will yield the following:

$$x = -\frac{1}{\sqrt{1+m^2}}\left[\tfrac{3}{8}\,v+\tfrac{1}{2}\sin v+\tfrac{1}{16}\sin 2v\right],$$

$$y = \frac{1}{\sqrt{1+m^2}}\left[\tfrac{1}{2}\cos v+\tfrac{1}{16}\cos 2v\right],$$

$$z = -\frac{1}{\sqrt{1+m^2}}\cos v\,.$$

That curve no longer lies on a second-order surface of revolution but belongs to a parabolic cylinder whose generating lines are parallel to the x-axis and whose axis plane is parallel to the xy-plane. The curve is transcendental, so it is not worthy of the interest that the usual curves that correspond to rational n attract.

For the sake of better clarity, let us once more summarize the systems of formulas that one must consider in order to geometrically represent the curves.

I. – Curves of constant curvature.

1. Equations:

$$x = -\frac{1}{\sqrt{1+m^2}}\left[\frac{1-n}{4(1+2n)}\sin(1+2n)v + \frac{1+n}{4(1-2n)}\sin(1-2n)v + \tfrac{1}{2}\sin v\right],$$

$$y = \frac{1}{\sqrt{1+m^2}}\left[\frac{1-n}{4(1+2n)}\cos(1+2n)v + \frac{1+n}{4(1-2n)}\cos(1-2n)v + \tfrac{1}{2}\cos v\right],$$

$$z = \frac{1}{4m\sqrt{1+m^2}}\cos 2n\,v = \frac{1-n^2}{4n}\cos 2\,n\,v\,.$$

2. Arc-length element:

$$ds = \frac{1}{\sqrt{1+m^2}}\cos n\,v\,.$$

3. Torsion:

$$\tau = -\tan n\,v\,.$$

4. Cusps appear for:

$$v = \frac{(2k+1)\pi}{2n}\,.$$

5. The squares of the semi-axes of the second-order surfaces that they belong to are:

$$a^2 = \frac{27n^4}{4(1-4n^2)^2}\,,\qquad c^2 = \frac{-27n^2}{16(1-4n^2)}\,,$$

whose axis ratio is then:

$$\frac{a}{c} = \frac{2ni}{\sqrt{1-4n^2}}\,.$$

Their centers lie on the z-axis at a height of:

$$z = -\frac{1+2n^2}{4n}\,.$$

6. Natural equations:

$$\kappa = 1,\qquad \tau^2 = \frac{m^2 s^2}{1-m^2 s^2}\,.$$

II. – Helices.

$$\xi = p\left\{\frac{\sin(1+2n)v}{1+2n} - \frac{\sin(1-2n)v}{1-2n}\right\},$$

$$\eta = -p\left\{\frac{\cos(1+2n)v}{1+2n} - \frac{\cos(1-2n)v}{1-2n}\right\}, \qquad p = \frac{3n}{4\sqrt{1+m^2}},$$

$$\zeta = \frac{3}{4m\sqrt{1+m^2}}\cos 2nv + \frac{1}{2m\sqrt{1+m^2}}.$$

2. Arc-length element:

$$d\sigma_0 = \pm\frac{3}{2\sqrt{1+m^2}}\sin 2n\,v\,dv.$$

3. Radius of curvature:

$$\rho = \frac{3}{2m}\sin 2n\,v.$$

4. Cusps appear for:

$$v = \frac{k\pi}{2n}.$$

5. The squares of the semi-axes of the second-order surface of revolution that the curve belongs to are:

$$a'^2 = \frac{9}{4}\frac{n^2(1-n^2)}{(1-4n^2)^2}, \qquad b'^2 = \frac{9}{16}\frac{(1-n^2)^2}{n^2(1-4n^2)},$$

such that the ratio of the principal axes will be:

$$\frac{a'}{c'} = \frac{2n^2}{\sqrt{1-n^2}\sqrt{1-4n^2}}.$$

The coordinates of their centers are $x = 0$, $y = 0$, $\zeta = \frac{1-n^2}{2n}$.

III. – The guiding curve of the cylinder of a helix.

1. Equations:

$$\xi_0 = p\left\{\frac{\sin(1+2n)v}{1+2n} - \frac{\sin(1-2n)v}{1-2n}\right\},$$

$$\eta_0 = -p\left\{\frac{\cos(1+2n)v}{1+2n} - \frac{\cos(1-2n)v}{1-2n}\right\}.$$

2. Arc-length element:

$$d\sigma_0 = -\,2p \sin 2n\,v\;dv\,.$$

3. Radius of curvature:

$$\rho_0 = 2p \sin 2n\,v\,.$$

4. Radius of the fixed circle:

$$R_0 = \frac{4\,n\,p}{1-4n^2}\,.$$

5. Radius of the rolling circle:

$$r_0 = \frac{p}{1+2n}\,.$$

6. Modulus of the curve:

$$\frac{R\,v}{r_0} = \frac{4\,n}{1-2n}\,.$$

Moreover:

$$m = \frac{n}{\sqrt{1-n^2}}$$

means the trigonometric tangent of the angle that the tangent to the helix defines with the z-axis.

Examples.

1. Take:

$$n = \frac{1}{6}\,.$$

The helix is then the intersection of the cylinder that is constructed over the *cardioid:*

$$\zeta_0 = \tfrac{3}{4}\,p\left\{\sin\tfrac{4}{3}v - 2\sin\tfrac{2}{3}v\right\},$$

$$\eta_0 = -\tfrac{3}{4}\,p\left\{\cos\tfrac{4}{3}v - 2\cos\tfrac{2}{3}v\right\}$$

with the ellipsoid:

$$\frac{x^2+y^2}{a'^2} + \frac{z^2}{c'^2} = 1\,,$$

in which:

$$a' = \frac{3}{64}\sqrt{35}$$

and

$$c' = \frac{105}{32}\sqrt{2}\,,$$

such that one will have:

$$\frac{a'}{c'} = \frac{1}{2\sqrt{70}} < \frac{1}{16} .$$

The helix then belongs to an ellipsoid of rotation that has been stretched considerably so that its axis of rotation exceeds the diameter of the equator by more than sixteen-fold.

The corresponding curves of constant curvature:

$$x = \frac{\sqrt{35}}{6\cdot 32}\left\{5\sin\tfrac{4}{3}v + 14\sin\tfrac{2}{3}v + 13\sin v\right\} ,$$

$$y = \frac{\sqrt{35}}{6\cdot 32}\left\{5\cos\tfrac{4}{3}v + 14\cos\tfrac{2}{3}v + 16\cos v\right\} ,$$

$$z = \frac{35}{24} \cos 2n\, v$$

lie on the hyperboloid of revolution of one sheet whose real semi-axis is:

$$a = \frac{3}{64}\sqrt{3} = \left(\frac{\sqrt{3}}{4}\right)^3 ,$$

and whose imaginary semi-axis is:

$$c = \frac{3}{16}\frac{\sqrt{3}}{2} = \left(\sqrt{\frac{3}{8}}\right)^3 .$$

The distance between the centers of the two surfaces of revolution amounts to:

$$\zeta - z = \frac{3}{4n} = \frac{9}{2} .$$

The cusps of the helix correspond to the parameter values $v = \frac{k\pi}{2n} = 3k\pi$ (so they then reduce to two here) of the intersection points of the line:

$$\xi = 0 , \qquad \eta = \tfrac{3}{4}\, p = \frac{\sqrt{35}}{64}$$

with the ellipsoid at the points:

$$\zeta_1 = \frac{175}{24} \qquad \text{and} \qquad \zeta_2 = -\frac{35}{24} .$$

The curve of constant curvature possesses only one cusp for:

$$v = \frac{2k+1}{2n}\pi = 6\,k\,\pi + 3\pi\,.$$

Its coordinates are:

$$\xi = 0\,, \qquad \eta = \frac{\sqrt{35}}{64}\,, \qquad \zeta = -\frac{35}{24}\,.$$

It also attains its highest point in the $\eta\,\zeta$-plane, namely, for:

$$v = \frac{k\,\pi}{n} = 6\,k\,\pi\,.$$

Its coordinates have the values:

$$\xi = 0\,, \qquad \eta = \frac{35\sqrt{35}}{6\cdot 32}\,, \qquad \zeta = +\frac{35}{24}\,.$$

2. For:

$$n = \frac{1}{4}\,,$$

one will have:

$$\frac{R_0}{r_0} = 2\,.$$

The helix will then lie on a cylinder whose guiding line is the *epicycloid with two cusps:*

$$\xi_0 = \tfrac{2}{3}\,p\left\{\sin\tfrac{3}{2}v - 3\sin\tfrac{1}{2}v\right\},$$

$$\eta_0 = -\tfrac{2}{3}\,p\left\{\cos\tfrac{3}{2}v - 3\cos\tfrac{1}{2}v\right\}.$$

The radius of the fixed circle is:

$$R_0 = \tfrac{4}{3}\,p$$

and that of the moving circle is:

$$r_0 = \tfrac{2}{3}\,p\,,$$

in which one has set:

$$p = \frac{3}{64}\sqrt{15}\,.$$

The ellipsoid that the helix belongs to has the equation:

$$\frac{x^2+y^2}{a'^2} + \frac{z^2}{c'^2} = 1\,,$$

and indeed, one has:

$$a' = \frac{1}{8}\sqrt{15}, \quad c' = \frac{15}{8}\sqrt{3}, \quad \frac{a'}{c'} = \frac{1}{3\sqrt{5}}.$$

Here as well, the axis of rotation is more than six times as long as the equatorial diameter. The curve of constant curvature, which is obtained for $n = 1/4$, has the equations:

$$x = \frac{-\sqrt{15}}{32}\left\{\sin\tfrac{3}{2}v + 5\sin\tfrac{1}{2}v + 4\sin v\right\},$$

$$y = \frac{\sqrt{15}}{32}\left\{\cos\tfrac{3}{2}v + 5\cos\tfrac{1}{2}v + 4\cos v\right\},$$

$$z = \frac{15}{16}\cos\frac{v}{2}.$$

It has degree six, since it is the intersection of the hyperboloid of revolution:

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1,$$

in which one sets:

$$a = \tfrac{1}{8}\sqrt{3}, \qquad c = \frac{3}{8},$$

with the parabolic cylinder of degree three:

$$\sqrt{15}\,y = z\left[\frac{512}{225}z^2 + \frac{64}{15}z + 2\right] - \frac{\sqrt{15}}{8}$$

that will be obtained by eliminating $v / 2$ from the expressions for y and z.

3. The value:

$$n = \frac{1}{3}$$

corresponds to the ratio:

$$\frac{R_0}{r_0} = 4,$$

so to the *four-cusped epicycloid*, as the guiding curve of the cylinder on which the helix lies:

$$\xi = \tfrac{3}{5}p[\sin\tfrac{5}{3}v - \sin\tfrac{1}{3}v],$$

$$\eta = -\tfrac{3}{5}\, p\,[\cos\tfrac{5}{3}v - \cos\tfrac{1}{3}v]\,,$$

$$\zeta = 2\cos\tfrac{2}{3}v + \tfrac{4}{3}\,,$$

in which one has:

$$p = \tfrac{1}{6}\sqrt{2}\,.$$

The radius of the fixed circle is:

$$R_0 = \tfrac{12}{5}\,p\,,$$

and that of the moving one is:

$$r_0 = \tfrac{3}{5}\,p\,.$$

The helix belongs to the ellipsoid of revolution with the semi-axis:

$$a' = \tfrac{3}{5}\sqrt{2}\,, \quad c' = \tfrac{6}{5}\sqrt{2}\,,$$

which then have a ratio of 1 : $\sqrt{10}$.

The associated curve of constant curvature:

$$x = \frac{-\sqrt{2}}{15}(\sin\tfrac{5}{3}v + 10\sin\tfrac{1}{3}v + 5\sin v)\,,$$

$$y = \frac{\sqrt{2}}{15}(\cos\tfrac{5}{3}v + 10\cos\tfrac{1}{3}v + 5\cos v)\,,$$

$$z = \tfrac{2}{3}\cos\tfrac{2}{3}v$$

lies on the hyperboloid of revolution with the semi-axes:

$$a = \frac{3\sqrt{3}}{10}\,, \qquad c = \frac{3\sqrt{3}}{4\sqrt{5}}\,.$$

The highest and lowest points of the helix, which correspond to the cusps of the guiding epicycloid, lie on the parallel circles of the ellipsoid:

$$\zeta_1 = \frac{10}{3}\,,$$

$$\zeta_2 = -\frac{2}{3}\,.$$

The cusps of the curve of constant curvature coincide with the lowest points of the helix, as always, while their maxima, which occur for the parameter value $v = \frac{(2\kappa+1)\,\pi}{2n} = \frac{3}{2}(2\kappa+1)\,\pi$, possess the coordinates:

$$x = \mp\frac{2\sqrt{2}}{5}, \quad y = 0, \; z = +\frac{2}{3}.$$

4. Let the series of examples (which is easy to increase arbitrarily) be concluded with the simplest case, which can occur for a helix on a cylinder with a hypocycloidal guiding curve. In that case, the ratio of the radii of the generating circles:

$$\frac{R_0}{r_0} = \frac{4n}{2n-1}$$

will be restricted by the condition that one must have $n < 1$. As a result, one must have:

$$R_0 > 4r ,$$

such that the three-cusped hypocycloid and the astroid will not come under consideration as guiding curves for the cylinder.

For:

$$n = \frac{3}{4} ,$$

one will have:

$$R_0 = 6\, r_0 .$$

The helix:

$$\xi = \tfrac{2}{5}\, p\{\sin \tfrac{5}{2} v - \sin \tfrac{1}{2} v\}, \qquad p = \tfrac{9}{64}\sqrt{7},$$

$$\eta = \tfrac{2}{5}\, p\,\{\cos \tfrac{5}{2} v + \cos \tfrac{1}{2}\},$$

$$\zeta = \tfrac{7}{16} \cos \tfrac{3}{2} v + \tfrac{7}{24}$$

belongs to the hyperboloid of revolution:

$$\frac{\xi^2+\eta^2}{a'^2} - \frac{\zeta^2}{c'^2} = 1 ,$$

whose axes are:

$$a' = \tfrac{9}{40}\sqrt{7}, \quad c' = \tfrac{7}{40}\sqrt{5} .$$

The cusps that appear for $v = 2\kappa\,\pi / 3$ distribute themselves on the parallel circles:

$$\zeta_1 = \frac{35}{48} \qquad \text{and} \qquad \zeta_2 = -\frac{7}{48} .$$

The cusps of the corresponding curve of constant curvature:

$$x = -\frac{\sqrt{7}}{160}\left\{\sin\tfrac{5}{2}v + 35\sin\tfrac{1}{2}v + 20\sin v\right\} ,$$

$$y = \frac{\sqrt{7}}{160}\left\{\cos\tfrac{5}{2}v - 35\cos\tfrac{1}{2}v + 20\cos v\right\} ,$$

$$z = \frac{7}{48}\cos\frac{3}{2}v$$

also lie on the former circles, and the highest point of the latter curve is on the parallel circle:

$$z = \frac{7}{48} \qquad \text{for} \qquad v = \frac{\kappa\,\pi}{n} = \frac{4}{3}\kappa\,\pi$$

and possesses the coordinates:

$$x = +\frac{7\sqrt{7}}{80}\sin\frac{\kappa\,\pi}{3} ,$$
$$y = +\frac{7\sqrt{7}}{80}\cos\frac{\kappa\,\pi}{3} .$$

The ellipsoid that the curve belongs to is determined by the semi-axis:

$$a = \frac{27}{40}\sqrt{3} , \quad c = \frac{9}{40}\sqrt{15} .$$

IV. – Geodetic lines on the tangent surfaces to helices.

The curves of constant curvature that were the subject of investigation up to now represent only one special case of the larger class of space curves that take the form of geodetic lines on the tangent surface to general helices. If one would like to define them independently of the surfaces on which they appear then one must observe that their principal normals will be associated with parallels to the tangent to a helix in such a way that the spherical indicatrix of their principal normals will then be a circle (*).

The equations of those curves in Cartesian coordinates are derived from the results at the beginning of the third section:

$$
(1)\qquad
\begin{aligned}
x_1 &= \int f'(v)\,(a' \sin n v + a'' \cos n v)\, dv\,,\\
y_1 &= \int f'(v)\,(b' \sin n v + b'' \cos n v)\, dv\,,\\
z_1 &= \int f'(v)\,(c' \sin n v + c'' \cos n v)\, dv\,,
\end{aligned}
$$

which are formulas in which:

$$f'(v) = \frac{ds_1}{dv}$$

means an arbitrary function of v, and:

$$a' = -n \sin v\,, \qquad b' = n \cos v\,, \qquad c' = \frac{-1}{\sqrt{1+m^2}}\,,$$

$$a'' = -\cos v\,, \qquad b'' = -\sin v\,, \qquad c'' = 0$$

mean the direction cosines of the binormals and principal normals of the helix from which we started (**). The curvature and torsion of the curve (1) were connected with the corresponding quantities of the helix by the equations:

$$
\begin{aligned}
\kappa_1 \frac{ds_1}{ds} &= \tau \cos n v\,,\\
\kappa_1 \frac{ds_1}{ds} &= -\kappa \sin n v\,.
\end{aligned}
$$

It follows from this that:

$$\frac{\tau_1}{\kappa_1} = -\tan n v\,.$$

If one now sets:

$$\tan n v = -t$$

then one will have:

$$s_1 = f(v) = \varphi(t)\,,$$

$$\kappa_1 = \frac{\cos^3 n v}{m\varphi'(t)} = \frac{-1}{m\varphi'(t)\sqrt{(1+t^2)^3}}\,.$$

That will then give the theorem:

(*) The indicatrix of their tangents is a spherical helix.

(**) The quadratures can be solved explicitly, moreover.

One will get the natural equations of the geodetic lines on the tangent surfaces to helices when one eliminates the parameter from the equations:

$$\frac{\tau_1}{\kappa_1} = t ,$$
$$\frac{1}{\kappa_1} = - m\varphi'(t)\sqrt{(1+t^2)^3} ,$$
$$s_1 = \varphi(t) ,$$

in which φ *means an arbitrary function.*

The equations of the edge of regression of the rectifying surface, which is a helix for which the ratio of curvature to torsion is equal to m, are obtained from the general theory in the form:

$$\xi_1 = x_1 + \frac{ds_1}{\kappa_1}\frac{\kappa_1 a_1' - \tau_1 a_1}{d\frac{\tau_1}{\kappa_1}} ,$$
$$\eta_1 = y_1 + \frac{ds_1}{\kappa_1}\frac{\kappa_1 b_1' - \tau_1 b_1}{d\frac{\tau_1}{\kappa_1}} ,$$
$$\zeta_1 = z_1 + \frac{ds_1}{\kappa_1}\frac{\kappa_1 c_1' - \tau_1 c_1}{d\frac{\tau_1}{\kappa_1}} .$$

In the present case, those equations can be converted as follows:

$$\xi_1 = x_1 + \frac{1}{\sqrt{1+m^2}}\frac{1}{\kappa_1}\cos^2 nv \sin v ,$$
$$\eta_1 = y_1 + \frac{1}{\sqrt{1+m^2}}\frac{1}{\kappa_1}\cos^2 nv \cos v ,$$
$$\zeta_1 = z_1 + \frac{1}{m\sqrt{1+m^2}}\frac{1}{\kappa_1}\cos^2 nv .$$

On the other hand, if one determines the line of striction of the principal normal surface of the curve (x_1, y_1, z_1) then when the equations of that curve:

$$\xi_2 = x_1 + \frac{\kappa_1}{\kappa_1^2 + \tau_1^2}\, a_1'' ,$$

$$\eta_2 = y_1 + \frac{\kappa_1}{\kappa_1^2 + \tau_1^2}\, b_1'' \ ,$$

$$\zeta_2 = z_1 + \frac{\kappa_1}{\kappa_1^2 + \tau_1^2}\, c_1''$$

are converted by means of the relations that exist for the curve in question then they will become:

$$\xi_2 = x_1 + \frac{1}{\sqrt{1+m^2}} \frac{1}{\kappa_1} \cos^2 nv \sin v \ ,$$

$$\eta_2 = y_1 + \frac{1}{\sqrt{1+m^2}} \frac{1}{\kappa_1} \cos^2 nv \cos v \ ,$$

$$\zeta_2 = z_1 + \frac{1}{m\sqrt{1+m^2}} \frac{1}{\kappa_1} \cos^2 nv \ .$$

The line of striction of the principal normal surface of a geodetic line G on the tangent surface to a general helix lies on the cylinder on which the helix is a geodetic. All three curves meet at the cusps of the curve G.

Corresponding points of the line of striction and the helix lie on the same generating lines of the cylinder at a distance from each other of:

$$\zeta_1 - \zeta_2 = \frac{\cos^2 nv}{n\kappa_1} = \frac{\kappa_1}{n(\kappa_1^2 + \tau_1^2)} \ .$$

The plane of the rectifying lines and principal normals then contacts the cylinder:

The principal normals to the geodetic lines on the tangent surface to a general helix envelop the cylinder on which the helix is a geodetic line and contact it along the line of striction of the surface that they define.

The geometric proof of that theorem will encounter no difficulties. If Q_1, Q_2, Q_3, … are neighboring points on a helix of the cylinder C, and if c is the intersection curve of the cylinder with a plane that is laid perpendicular to the generating lines, moreover, and finally, if T_1, T_2, T_3, … are the intersection points of the tangents $Q_1\,Q_2$, $Q_2\,Q_3$, $Q_3\,Q_4$,

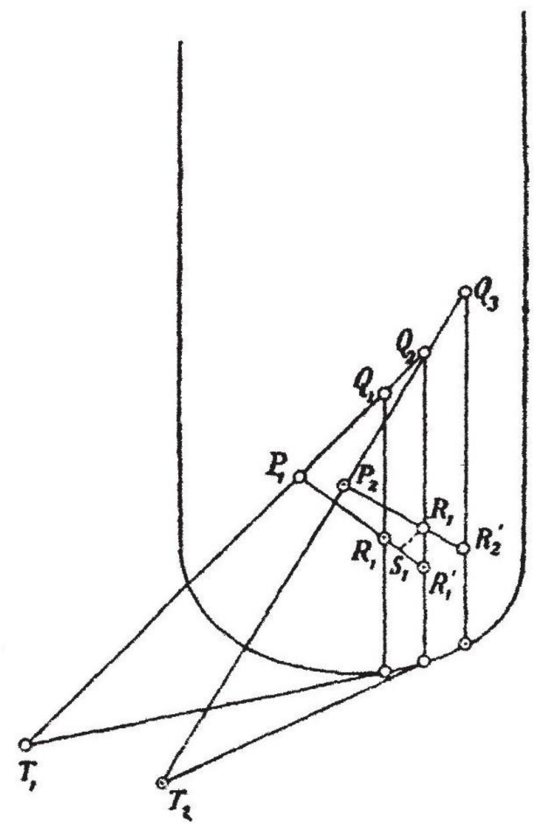

Fig. 4.

… with that plane then $T_1 T_2$ will be perpendicular to the plane $T_3 Q_3 q_2$, so conversely the plane $T_2 Q_3 q_3$ is perpendicular to the plane $T_1 Q_3 T_2$, and the altitude $P_1 R_1$ to $Q_1 T_1$ in the plane will then be a surface normal. It meets the cylinder C at the points R_1 and R_1' on the two neighboring generators $Q_1 q_1$ and $Q_2 q_2$. If one chooses a point P_2 in the vicinity of P_1 on the neighboring generator $Q_2 Q_3$ to the tangent surface then the surface normal to that point will lie in the plane $T_2 Q_3 q_3$, so it will cut the plane $T_1 Q_2 q_2$ at a point R_2 on the line $Q_2 q_2$. If one now regards $P_1 P_2$ as a line element of a geodetic line on the tangent surface then $P_1 R_1$ and $P_2 R_2$ will be two consecutive principal normals for it. The shortest distance between them is parallel to the rectifying line $T_1 Q_2$, so it lies in the plane $T_1 Q_2 q_2$ and will be obtained when one drops the altitude $R_2 S_1$ from the intersection point R_2 of the line $P_2 R_2$ with that plane to $P_1 R_1$. However, that says nothing but the theorem that was previously derived analytically.

Appendix.

The curves that emerge from the simplest cyclic curves are less suitable for representation since the resulting curves assume inconvenient forms that do not allow the characteristic forms of the curve classes to emerge clearly.

For that reason, cyclic curves with a large number of cusps are established in the accompanying figure, which represents the two main types in orthogonal projection.

The construction happens most simply in such a way that after calculating the numerical values of all necessary determining data, the cyclic curve that corresponds to the values R_0, r_0 is represented, and the cylinder that is perpendicular to it is erected, and its intersection curve with the second-order surface of revolution whose semi-axes are a' and b' is determined. That line of intersection is a helix whose tangents intersect the second-order surface of revolution with the semi-axes a, c at the points of the desired curve of constant curvature. The tangent surfaces are represented in the figure by isolating their generating lines in order to let the connection emerge more clearly.

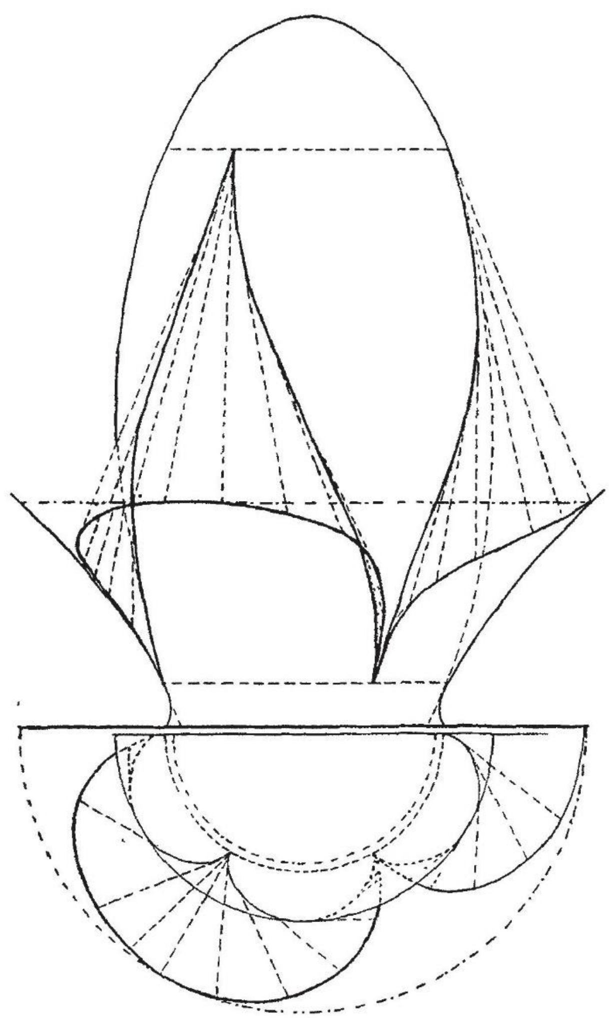

Fig. 5.

The two second-order surfaces of revolution on which the helix and the associated curve of constant curvature lie contact each other along the parallel circle on which the cusps of the curve of constant curvature lie.

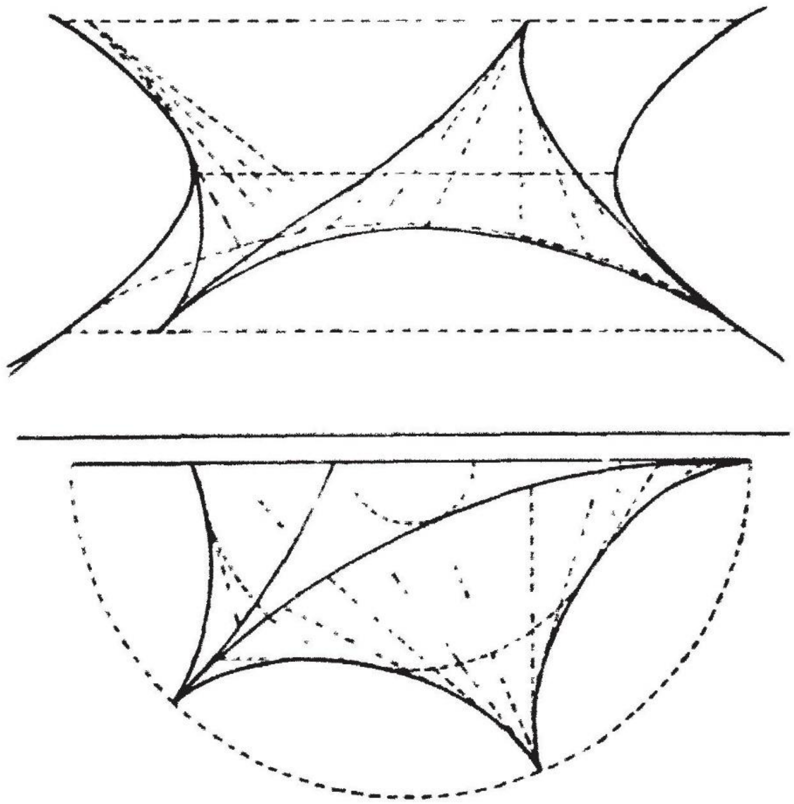

Fig. 6.

That already follows from the simple geometric argument that the helix and the curve of constant curvature must have common tangents at the cusps where they meet. However, one can also conclude that from the equations of the two surfaces of revolution in the usual way by calculation, which can be omitted here, due to its elementary character.