J. A. Schouten and E. R. van Kampen, “Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde,” Math. Ann. **103** (1930), 752-783.

On the theory of embedding and curvature of non-holonomic structures.

By

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An m-direction field in an X_n ($m \leq n$) that that does not consist of the m-directions of ∞^{n-m} m-dimensional X_m is a non-holonomic structure that possesses properties relative to embedding and curvature that are analogous to those of an embedded X_m . Structures of that type have been investigated from various angles ([1]). In the present work, after a preparatory paragraph, we will next discuss what types of quantities arise in a complemented (*eingespannte*) X_n^m and what sort of identifications can be made for the sake of easing the calculations. That will yield a treatment of the X_n^m and the complements that is completely dual and which leads, on the one hand, to the identification of contravariant quantities of the X_n^m with quantities of the X_n, and on the other, to the identification of covariant ones. A general linear displacement will then be established in the X_n, and the various covariant differentials that arise in that way will be discussed. However, the formulas remain endowed with many factors B and C that only serve as component structures. That inconvenience will then be eliminated by employing the D-symbolism, which is the extension of a method that goes back to van der Waerden and Bortolotti and which allows one to avoid the use of the factors B and C almost completely by the clever employment of the indices that belong to the local reference system. The treatment of the theory of curvature will become especially simple with the help of that D-symbolism, as will be shown for the V_n^m in V_n (viz., X_n with the Riemannian metric), in particular. In that way, four curvature quantities that belong to V_n^m will appear, and that will imply identities for them that are analogous to the usual four identities. In

([1]) G. Vranceanu, “Sur les espaces non-holonomes,” Comptes Rendus **183** (1926), 825-854. “Sur le calcul différentiel absolu pour les variétés non-holonomes,” Comptes Rendus **183** (1926), 1083-1085. Horak (Czech), “On a generalization of the notion of manifold,” Publ. de l’Univ. Masaryk, Brno (1927). J. A. Schouten, “Über nicht-holonome Übertragungen in einer L_n ,” Math. Zeit. **30** (1929), 149-172. G. Vranceaunu, “Studio geometrico dei sistemi anholonomi,” Ann. di Mat. **6** (1929), 9-43. “Les trois points de vue dans l’étude des espaces non-holonomes,” Comptes Rendus **188** (1929), 973-976. One can also confer the literature that was given in these papers. – D. Sintzow, “Zur Krümmungstheorie der Integralkurven der Pfaffschen Gleichung,” Math. Ann. **101** (1929), 261-272, examined the curvature theory of a V_3^2 in R_3 independently of the author.

conclusion, the V_n^m that are complemented in X_n will be treated, which will make it possible to establish a displacement using the curious metric and complement, even though no displacement exists in X_n .

§ 1. – Preliminaries.

Local reference system that depends upon the ξ^ν .

We next understand an X_n to mean the totality of all values that the *n*-variables (viz., the *Ur-variables*) ξ^ν , $\nu = 1, \ldots, n$ can assume, but in such a way that as long as functions of the ξ^ν appear, it is only a domain in which those functions are differentiable sufficiently often. The *running* indices $\alpha, \ldots, \omega$ can be replaced by each symbol in a series of *fixed* indices. We have chosen to write *italic* numbers *1*, …, *n* for those fixed indices. The *kernel symbol* ξ remains unchanged under transformations of the Ur-variables:

(1) $$\xi^{\mathrm{N}} = \xi^{\mathrm{N}}(\xi^\nu),$$

while the *running indices* will take on a different series of symbols that is associated with a definite series of *fixed indices.* In what follows, e.g., the running indices Α, …, Ω shall always be assigned the fixed indices $\overline{1}$, …, $\overline{n}$ such that we can write ξ^{N} for the new Ur-variables with the use of that series of symbols. By contrast, if the X_n were subjected to a *transformation* then we would denote the *new* points with *the same index*, but with a *different* kernel symbol:

(2) $$\eta^{\mathrm{N}} = \eta^{\mathrm{N}}(\xi^\nu).$$

As is known, the equation:

(3) $$d\eta^{\mathrm{N}} = (\partial_\lambda \xi^{\mathrm{N}})\, d\xi^\lambda, \quad \partial_\lambda = \partial / \partial \xi^\lambda,$$

which is derivable from (1), serves as the starting point for the definition of *quantities* (which we understand to mean first-degree quantities) that are contravariant and covariant vectors:

(4a) $$v^{\mathrm{N}} = (\partial_\lambda \xi^{\mathrm{N}})\, v^\lambda,$$

(4b) $$w_\Lambda = (\partial_\Lambda \xi^\nu)\, w_\nu,$$

secondly, the *higher-degree quantities* that are derivable from them in a known way, and thirdly, the *scalars* (or *zero*-degree quantities), which are characterized by the invariance of the numbers that they determine under (1). *The kernel symbol also remains unchanged under transformations, and a change in the kernel symbol will always mean a change in the quantities themselves.*

The unit affinor A_λ^ν and the basis vectors $\underset{\lambda}{e}^{\nu}$ that belong to the $\overset{\nu}{e}_{\lambda}$, are defined by the equations:

(5) $$\underset{\lambda}{e}^{\nu} \overset{*}{=} \overset{\nu}{e}_{\lambda} \overset{*}{=} \delta_\lambda^\nu = \begin{cases} 0 & \nu \neq \lambda, \\ 1 & \nu = \lambda, \end{cases}$$

(6) $$A_\lambda^\nu = \overset{\mu}{e}_{\lambda}\, \underset{\mu}{e}^{\nu} \overset{*}{=} \delta_\lambda^\nu ,$$

in which the symbol $\overset{*}{=}$ means that the equation is true in only the reference system being employed and is not invariant under the transition to another system. δ_λ^ν is the well-known Kronecker symbol, which can be used in all sequence of symbols that happen to be employed. We shall suggest the system of basis vectors $\underset{\lambda}{e}^{\nu}$, $\overset{\nu}{e}_{\lambda}$ by (ν) in the text. Similarly, the defining numbers A_λ^{N} of the unit affinor and a system of basis vectors $\underset{\Lambda}{e}^{\mathrm{N}}$ and $\overset{\mathrm{N}}{e}_{\Lambda}$ belong to the ξ^{N}:

(7) $$A_\Lambda^{\mathrm{N}} \overset{*}{=} \overset{\mathrm{N}}{e}_{\Lambda} \overset{*}{=} \underset{\Lambda}{e}^{\mathrm{N}} \overset{*}{=} \delta_\Lambda^{\mathrm{N}} ,$$

which will be suggested in the text by (N), and according to (3), one will have:

$$A_\lambda^{\mathrm{N}} = (\partial_\nu \xi^{\mathrm{N}})(\partial_\Lambda \xi^\lambda) A_\lambda^\nu = (\partial_\mu \xi^{\mathrm{N}})\, \partial_\Lambda \xi^\mu ,$$

(8) $$\underset{\lambda}{e}^{\mathrm{N}} = (\partial_\mu \xi^{\mathrm{N}})\, \underset{\lambda}{e}^{\mu} \overset{*}{=} \partial_\lambda \xi^{\mathrm{N}} , \qquad \underset{\Lambda}{e}^{\nu} = (\partial_{\mathrm{N}} \xi^\nu)\, \underset{\Lambda}{e}^{\mathrm{N}} \overset{*}{=} \partial_\Lambda \xi^\nu ,$$

$$\overset{\nu}{e}_{\Lambda} = (\partial_\Lambda \xi^\mu)\, \overset{\nu}{e}_{\mu} \overset{*}{=} \partial_\Lambda \xi^\nu , \qquad \overset{\mathrm{N}}{e}_{\lambda} = (\partial_\lambda \xi^{\mathrm{M}})\, \overset{\mathrm{N}}{e}_{\mathrm{M}} \overset{*}{=} \partial_\lambda \xi^{\mathrm{N}} .$$

The indices that appear above and below in the middle in (5) and (8) are called *distinguishing* indices, as opposed to the transformed indices. Naturally, there are also *running* and *fixed* distinguishing indices, just as there are for the transformed indices. By convention, distinguishing indices are always considered to belong to the kernel symbols under a change of kernel symbols. The distinguishing indices, *as well as the running ones*, *do not* transform, and will never be written above or below to the right, which will remain reserved for exclusively the transformed indices. On historical grounds, an exception will be made for only the two distinguishing indices of the Kronecker symbol δ_λ^ν.

Not all indices of a quantity need to be referred to the same local reference system. Defining numbers with different types of indices are called *linking*. If one goes over to the (N) for only the upper index in the unit affinor then that will yield:

(9) $$A_\lambda^{\mathrm{N}} = \overset{\mu}{e}_\lambda \underset{\mu}{e}^{\mathrm{N}} = \overset{\mu}{e}_\lambda \underset{\mu}{e}^{\nu} \partial_\nu \xi^{\mathrm{N}} = \partial_\lambda \xi^{\mathrm{N}} ,$$

and that underscores the fact that the $\partial_\lambda \xi^{\mathrm{N}}$ are nothing but the linking defining numbers of the unit affinor. One now obtains the A_Λ^ν in the same way, such that the defining equations (4) can now be written:

(10) $$v^{\mathrm{N}} = A_\lambda^{\mathrm{N}} v^\lambda , \qquad w_\Lambda = A_\Lambda^\nu w_\nu .$$

The linking defining numbers of all quantities can be derived with the help of A_λ^{N} and A_Λ^ν; e.g.:

(11) $$v_{\lambda\mathrm{M}}^{\cdots\mathrm{N}} = v_{\lambda\mu}^{\cdots\nu} A_{\nu\mathrm{M}}^{\mathrm{N}\mu} = v_{\Lambda\mathrm{M}}^{\cdots\mathrm{N}} A_\lambda^\Lambda .$$

The different ways that the A_λ^ν, $\underset{\lambda}{e}^{\nu}$, $\overset{\nu}{e}_\lambda$, and δ_λ^ν transform under the transition from (ν) to (N) are clearly represented in the following table:

(12)

	Trans. of the contra. index	Trans. of the cov. index	Combined
$\underset{\lambda}{e}^{\nu}$	$\underset{\lambda}{e}^{\mathrm{N}}$	$\underset{\lambda}{e}^{\nu}$	$\underset{\lambda}{e}^{\mathrm{N}}$
$\overset{\nu}{e}_\lambda$	$\overset{\nu}{e}_\lambda$	$\overset{\nu}{e}_\Lambda$	$\overset{\nu}{e}_\Lambda$
A_λ^ν	A_λ^{N}	A_Λ^ν	A_Λ^{N}
δ_λ^ν	δ_λ^ν	δ_λ^ν	δ_λ^ν

The necessary and sufficient condition for (N) and (ν) to coincide reads:

(13) $$A_\lambda^{\mathrm{N}} \overset{*}{=} \delta_\lambda^{\mathrm{N}} ,$$

where $\delta_\lambda^{\mathrm{N}}$ is an *extension* of the Kronecker symbol that means 1 or 0 according to whether the running indices N and λ are replaced with fixed indices from the series of symbols that they are associated with whose locations do or do not correspond, respectively. We will also employ this symbol for all sequences of symbols that will occur.

Any point of X_n will be assigned a *local manifold* by (3) with a homogeneous linear group that is defined in it, or, what amounts to the same thing, an E_n (X_n with ordinary affine geometry). The vectors and higher-degree quantities are systems of defining numbers that transform under just those *local* groups in ways that depend upon the transformation of the ξ^{ν}. If the defining numbers of a quantity – e.g., v^{ν} – are defined over X_n (i.e., they are as functions of the ξ^{ν}) then one speaks of a *field*. One considers the v^{N} that are given by (4) to be *new* defining numbers of *the same* field under the transformation (1). Hence, the transformation from v^{ν} to v^{N} has nothing to do with how the v^{ν} depend upon the ξ^{ν}. By contrast, with the transformation (3), along with the v^{ν}, one can consider a new field whose defining numbers $'v^{\nu}$ are expressed in terms of the ξ^{ν} in the same way that the v^{ν} are expressed in terms of ξ^{ν}. One easily proves that the $'v^{\nu}$ transform under the transition from (ν) to (N) like the defining numbers of a vector, and that the same thing is true for quantities of arbitrary degree. That process, namely, which is called the *dragging of the field* under a transformation of type (2) and is used in variational problems especially, is basically different from the process that takes the v^{ν} to the v^{N}.

More general local reference systems.

The conceptual structures up to now allow a generalization that is completed in three steps:

1. *Separating the local transformation from the transformation of the* ξ^{ν}.

The transformations of the local group that were uniquely associated with the transformations of the ξ^{ν} in the example above can be made completely free of them. That happens when one introduces a system (k) of n arbitrary linearly-independent contravariant vectors $\underset{i}{e}^{\nu}$ in place of the system (n) that belongs to the ξ^{ν}, along with the vectors $\overset{k}{e}_{\lambda}$ that are reciprocal to them, which transform into a system (K) that consists of the vectors $\underset{I}{e}^{\nu}$ and $\overset{K}{e}_{\lambda}$ arbitrarily in a manner that is independent of the ξ^{ν}. If the defining numbers relative to (k) [(K), resp.] are provided with running indices $h, \ldots, m$ ($H, \ldots, M$, resp.) then one will obviously have:

$$\underset{i}{e}^{k} \overset{*}{=} \overset{k}{e}_{i} \overset{*}{=} \delta_i^k ,$$

(14)

$$\underset{I}{e}^{K} \overset{*}{=} \overset{K}{e}_{I} \overset{*}{=} \delta_I^K .$$

We associate the *running* indices $h, \ldots, m$ with the *fixed* indices from the *vertically*-printed sequence of numbers 1, …, n, while the running indices $H, \ldots, M$ are associated with the fixed indices from the sequence $\bar{1}, \ldots, \bar{n}$. The relationships between the defining numbers relative to (ν) and relative to (k) are obtained from the equations:

$$v^k = v^\nu \overset{j}{e}_\nu \underset{j}{e}^k \overset{*}{=} v^\nu \overset{k}{e}_\nu ,$$

(15)

$$w_i = w_\lambda \underset{j}{e}^\lambda \overset{j}{e}_i \overset{*}{=} w_\lambda \overset{\lambda}{e}_i .$$

The system (k) cannot always be coupled with a system of Ur-variables ξ^k. The necessary and sufficient condition for that is known to be:

(16) $$\partial_{[\mu} \overset{k}{e}_{\lambda]} = 0.$$

In the other case, which we refer to as *non-holonomic*, the defining numbers of the $d\xi^\nu$ relative to the (k) shall be described by the $(d\xi)^k$, since the ξ^k have no intrinsic meaning and play the same role as the non-holonomic parameters in mechanics.

2. *Introduction of several local manifolds.* – Each point of X_n will be assigned several local manifolds, each of which have their own affine group.

The simplest case of that kind appears when we do not replace (ν) with (k), but introduce (k), *along with* (ν). The two local manifolds then coincide in a single E_n, but two groups are now defined in E_n that belong to (ν), which depends upon the transformations of the ξ^ν, and to (k), which are independent of those transformations. Defining numbers that are *linked with* a quantity of higher degree also appear now; e.g., the unit affinor:

$$A_\lambda^k = \overset{j}{e}_\lambda \underset{j}{e}^k \overset{*}{=} \overset{k}{e}_\lambda ,$$

(15)

$$A_i^\nu = \overset{j}{e}_i \underset{j}{e}^\nu \overset{*}{=} \underset{i}{e}^\nu ,$$

with the aid of which, all other quantities can be derived, such as:

(18) $$v_{\lambda j}^{\cdot\cdot\cdot k} = v_{\lambda\mu}^{\cdot\cdot\cdot\nu} A_{j\nu}^{\mu k} = v_{ij}^{\cdot\cdot\cdot k} A_\lambda^i .$$

A more general case that will appear continually in this paper arises when each point of X_n is associated with not only the E_n with the group that is associated with (n), but also an E_m ($m \neq n$) whose group is independent of the transformations of the ξ^ν. There will then exist three types of quantities, namely, the ones that belong purely to E_n (E_m, resp.) and then *linking quantities*, whose indices refer partly to (ν) and partly to the reference system that lies in E_m.

3. *Transition to an arbitrary group in the local manifold.* – Naturally, with the latter extension, no further homogeneous linear transformations can be gained from the systems of defining numbers considered. The simplest (indeed, almost trivial) example is defined by the system of the ξ^ν themselves, such that each point of X_n is now associated with X_n itself, and the local group is the group of all transformations (1). We would like to call such systems that do not transform in a homogeneous linear way "geometric objects" (2). By the way, they also already appear in the local E_n, and the best-known example is probably that of the parameters $\Gamma^{\nu}_{\lambda\mu}$ of an affine displacement. Everywhere that such geometric objects appear in differential geometry, one notices the ambition to reduce the treatment to systems with homogeneous linear transformations (3). In affine geometry, that comes about by the introduction of covariant differentiation. In the more general projective and conformal geometries, in which each point of X_n is associated with a local X_n with a projective (conformal, resp.) group, one will achieve the same objective by introducing superfluous coordinates that will replace the local X_n with an E_N ($N > n$) with a group that is rigorously coupled with the transformation of the ξ^ν and the subsequent introduction of a covariant differentiation. For geometric objects, as well, we keep to the rule that the *kernel symbol* remains *fixed* under transformations, while the new *running indices* will assume a *different* sequence of symbols; e.g.:

$$\Gamma^{\mathrm{N}}_{\Lambda\mathrm{M}} = A^{\lambda\mu\mathrm{N}}_{\Lambda\mathrm{M}\nu}\,\Gamma^{\mathrm{N}}_{\Lambda\mathrm{M}} + A^{\mathrm{N}}_{\mu}\,\partial_{\mathrm{M}} A^{\mu}_{\Lambda} . \tag{19}$$

Throttling.

n scalar fields $\overset{\nu}{\xi}$ can be constructed from the Ur-variables ξ^ν that are numerically equal to the ξ^ν. Under the transition to new Ur-variables, the $\overset{\nu}{\xi}$ then remain invariant (as scalars), while the ξ^ν go to the ξ^{N}. We express that by the equation:

$$\overset{\nu}{\xi} \overset{*}{=} \xi^\nu . \tag{20}$$

The ξ^ν are not be confused with the $\xi^\mu \overset{\nu}{e}_\mu$, which are not scalars.

(2) Veblen, who was probably the first to refer to the meaning of those systems expressly, coined the expression "invariant," which we prefer to replace with "geometric object," primarily in view of the fact that the systems of that kind that appear in a differential-geometric examination always have a geometric meaning that is independent of any reference system, but also in order to avoid any false associations that might arise from the usual meaning of the word "invariant."

(3) Along with quantities, quantity densities and pseudo-quantities also possess such a transformation. Cf., J. A. Schouten and V. Hlavaty, "Zur Theorie der allgemeinen linearen Übertragung," Math. Zeit. **30** (1929), 414-432.

It is clear that the covariant basis vectors $\overset{\nu}{e}_\lambda$ arise from the $\overset{\nu}{\xi}$ by the covariant operation of taking the gradient:

$$\overset{\nu}{e}_\lambda = \partial_\lambda \overset{\nu}{\xi}, \tag{21}$$

and the fact that one gets the contravariant basis vectors $\underset{\lambda}{e}^\nu$ by dividing the vector $d\xi^\nu$ by the scalars $d\overset{\lambda}{\xi}$:

$$\underset{\lambda}{e}^\nu = \frac{\partial \xi^\nu}{\partial \overset{\lambda}{\xi}}. \tag{22}$$

Finally, if one divides the scalars $d\overset{\nu}{\xi}$ by the $d\overset{\lambda}{\xi}$ then what will arise are the n^2 scalars of the Kronecker symbol:

$$\delta^\nu_\lambda = \frac{\partial \overset{\nu}{\xi}}{\partial \overset{\lambda}{\xi}}. \tag{23}$$

Equations (21) to (23), along with:

$$A^\nu_\lambda = \frac{\partial \xi^\nu}{\partial \xi^\lambda}, \tag{24}$$

show the difference between the four symbols that appear in equations (5), (6), whose manners of transformations were clearly represented in Table (12).

We call the transition from ξ^ν to $\overset{\nu}{\xi}$ the *throttling (Abdrosseln*) of the index ν. In the same way, we call the transition from a quantity or a geometric object with p indices to the n^p scalars that are equal to the defining numbers relative to the reference system that belongs to those indices the *throttling* of the indices relative to that system and indicate that throttling by placing the indices in question above and below the kernel symbols; e.g.:

$$\begin{aligned} &\underset{\lambda\mu}{\overset{\nu}{\Gamma}} \overset{*}{=} \Gamma^\nu_{\lambda\mu}, \\ &\underset{\lambda}{\overset{\nu}{A}} = \delta^\nu_\lambda \overset{*}{=} A^\nu_\lambda, \\ &\underset{\lambda}{\overset{\nu}{e}} = \delta^\nu_\lambda \overset{*}{=} \overset{\nu}{e}_\lambda \overset{*}{=} \underset{\lambda}{e}^\nu. \end{aligned} \tag{25}$$

It is clear that throttling the indices *on a quantity* (but *not* on a general geometric object) can be achieved by contracting over all possible combinations of suitably-chosen basis vectors, e.g.:

(26) $$\underset{\lambda\mu}{\overset{\nu}{v}} = v_{\alpha\beta}^{\cdot\cdot\cdot\gamma}\, \underset{\lambda}{e}^{\alpha}\, \underset{\mu}{e}^{\beta}\, \overset{\nu}{e}_{\gamma} \ ,$$

and that can be reversed by multiplying the scalars obtained by suitable basis vectors and adding, e.g.:

(27) $$v_{\alpha\beta}^{\cdot\cdot\cdot\gamma} = \underset{\lambda\mu}{\overset{\nu}{v}}\, \overset{\lambda}{e}_{\alpha}\, \overset{\mu}{e}_{\beta}\, \underset{\nu}{e}^{\gamma} \ .$$

One can make use of that fact to define the throttling of one or more indices of a *quantity.* That will then be understood to mean moving each throttled contravariant (covariant, resp.) index by means of the n covariant (contravariant, resp.) basis vectors that belong to the Ur-variable of the index; e.g.:

(28) $$\underset{\lambda\mathrm{M}}{v}{}^{\nu}_{\cdot\cdot} = v_{\alpha\mathrm{B}}^{\cdot\cdot\cdot\nu}\, \underset{\lambda}{e}^{\alpha}\, \underset{\mathrm{M}}{e}^{\mathrm{B}} \ .$$

That throttling can also be reversed; e.g.:

(29) $$v_{\alpha\mathrm{B}}^{\cdot\cdot\cdot\nu} = \underset{\lambda\mathrm{M}}{v}{}^{\nu}_{\cdot\cdot}\, \overset{\lambda}{e}_{\alpha}\, \overset{\mathrm{M}}{e}_{\mathrm{B}} \ .$$

It is clear that a covariant equation will keep the property of covariance under throttling of one or more indices.

Indices employed.

Finally, we shall give an overview of the sequences of symbols and numerals that will be employed for the running (fixed, resp.) indices:

(30)

Running indices	Fixed indices
$\alpha, \ldots, \omega$	$1, \ldots, n$
$\mathrm{A}, \ldots, \Omega$	$\overline{1}, \ldots, \overline{n}$
$h, \ldots, m$	$\mathsf{1}, \ldots, \mathsf{n}$
$H, \ldots, M$	$\overline{\mathsf{1}}, \ldots, \overline{\mathsf{n}}$
$a, \ldots, g$	$\mathsf{1}, \ldots, \mathsf{m}$
$A, \ldots, G$	$\overline{\mathsf{1}}, \ldots, \overline{\mathsf{m}}$
$p, \ldots, w$	$m+1, \ldots, n$
$P, \ldots, W$	$\overline{m+1}, \ldots, \overline{n}$

Naturally, in theory, one is completely free to choose the running indices from within the assumed sequence of symbols. However, the readability of the formulas and the likelihood of avoiding errors (discovering them early on, resp.) will both be greatly increased when one restricts that freedom somewhat (which must naturally be maintained in each special case) for the cases that occur very often. Therefore, in the Greek alphabet, it is advisable *in general* to prefer to use ν for the contravariant index, λ for the covariant one, μ for the first covariant differentiation, ω for the second, and ξ for the third. One then writes, e.g.:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^{\nu}_{\lambda\mu} v^\lambda , \tag{31}$$

and not, say:

$$\nabla_\nu v^\lambda = \partial_\nu v^\lambda + \Gamma^{\lambda}_{\mu\nu} v^\mu , \tag{32}$$

which would give rise to much confusion and printing errors. The corresponding indices in the other series of symbols are implied by the following table:

(33)

	Contra.	Cov.	1. Diff.	2. Diff.	3. Diff.
(ν)	ν	λ	μ	ω	ξ
(N)	N	Λ	M	Ω	Ξ
(k)	k	i	j	l	h
(K)	K	I	J	L	H
(c)	c	a	b	d	e
(C)	C	A	B	D	E

§ 2. – X_n^m that are embedded and complemented in X_n .

Let an X_m with the Ur-variables η^c, $a, \dots, g = 1, \dots, m$ be "embedded" in X_n by means of the equation:

$$\xi^\nu = \xi^\nu(\eta^c) , \tag{34}$$

in which we recall the continuity conditions that were stated in the beginning of § **1**. Let some possibly-new Ur-variables in X_n (X_m, resp.) be ξ^{N} (η^C, resp.). Each point of X_m is then assigned a local E_n with the reference system (ν) with basis vectors $\underset{\lambda}{e}{}^\nu$, $\overset{\nu}{e}_\lambda$, and a local E_m with the reference system (c) with basis vectors $\underset{a}{e}{}^c$, $\overset{c}{e}_a$. Correspondingly, there is a unit affinor A_λ^ν for X_n at that point, the unit affinor B_a^c for the X_m, and the coupling quantities:

(35) $$\overline{B}_a^{\nu} = \frac{\partial \xi^{\nu}}{\partial \eta^a},$$

with the transformation law:

(36) $$\overline{B}_A^{\mathrm{N}} = A_{\nu}^{\mathrm{N}} B_A^a \overline{B}_a^{\nu}.$$

The m contravariant vectors in X_m :

(37) $$\underset{a}{\overline{e}}^{\nu} = \underset{a}{\overline{B}}^{\nu} = \overline{B}_b^{\nu} \underset{a}{e}^b = \frac{\partial \xi^{\nu}}{\underset{a}{\partial \eta}}$$

then arise from those quantities by throttling, and at each point of X_n , they determine, on the one hand, an m-direction that we shall call the *m-direction of the* X_m , and on the other hand, an E_m that lies in the local E_n that we shall denote by $\overline{B}_a^{\nu}$, in order to distinguish it from the local E_m of the X_m . It follows from (35) and (37) that $\overline{B}_a^{\nu}$ can be expressed in terms of the basis vectors $\overset{c}{e}$ and $\underset{a}{\overline{e}}$ as follows:

(39) $$\boxed{\overline{B}_a^{\nu} = \overset{b}{e}_a \underset{b}{\overline{e}}^{\nu}.}$$

Each line element $d\eta^c$ in the X_m of the local E_m of X_m is assigned the line element:

(40) $$d\xi^{\nu} = \overline{B}_a^{\nu} \, d\eta^a$$

of the X_n in the local $\overline{E}_m$. Now one can also choose the system (c) arbitrarily at each point and transform arbitrarily in a homogeneous linear way without referring to any sort of Ur-variables for the X_m . One needs only to replace $\partial\eta^c$ with $(\partial\eta)^c$ and $d\eta^c$ with $(d\eta)^c$ in formulas (35) to (38), since the η^c will be non-holonomic parameters, in general. However, one can go a step further and forgo an equation of the form (34) entirely, and instead of starting from it, one can start from equation (40) with $(d\eta)^a$ in place of $d\eta^a$, where the quantities $\overline{B}_a^{\nu}$, which shall have rank m, are now given over X_n , and the $(d\eta)^c$ do not represent exact differentials, in general. The arbitrary homogeneously-linear transforming $(d\eta)^c$ can be regarded as the defining numbers of a vector in an E_m , and the $\overline{B}_a^{\nu}$ will determine an E_m in the local E_n , which might be denoted by $\overline{E}_m$. That will change nothing in equations (35) to (40), except that the $d\eta^c$ and $\partial\eta^c$ will be replaced with $(d\eta)^c$ and $(\partial\eta)^c$, resp., and it should be observed that the domain of definition of the $\overline{B}_a^{\nu}$ will now be n-dimensional, in general. In that way, the X_m would be replaced at each point with an X_n that is provided with an m-direction and which shall be called the *X_n^m that is embedded in X_n* . If the m-directions of X_m were defined then the X_n^m will reduce to a system of ∞^{n-m} X_m , and if the domain of definition of the m-directions were sufficiently restricted then the case of the individual X_m would then arise.

We consider the most general case of X_n^m in X_n. The correspondence between the local E_m and the local $\overline{E}_m$ that was already spoken of in the context of line elements can also be expressed by saying that every contravariant vector v^c in X_n^m is always associated with a unique contravariant $\overline{v}^\nu$ in X_n :

$$\overline{v}^\nu = \overline{B}_b^\nu \, v^b . \tag{41}$$

From (37), the vectors $\underset{a}{e}^{c}$ will then correspond to the vectors $\underset{a}{\overline{e}}^{\nu}$. Conversely, a single vector in X_n^m will correspond to a contravariant vector $\overline{v}^\nu$ in X_n if and only if there is a vector v^c that satisfies equation (41). Since that equation can also be written as:

$$\overline{v}^\nu = v^b \underset{b}{\overline{e}}^{\nu} \tag{42}$$

as a result of (37), that will be the case if and only if $\overline{v}^\nu$ belongs to $\overline{E}^m$. The correspondence between contravariant vectors in E_m and $\overline{E}_m$ is then one-to-one. We would like to say that a vector $\overline{v}^\nu$ that belongs to $\overline{E}_m$ *lies in* X_n^m.

By contrast, any covariant vector w_λ in X_n is always associated with a covariant vector $'w_a$ in X_n^m in a single-valued way:

$$'w_a = \overline{B}_a^\mu \, w_\mu . \tag{43}$$

That vector will be zero if and only if the $(n-1)$-direction of w_λ contains the m-direction of X_n^m. It follows from the equation:

$$\overset{\nu}{\overline{B}}_a = \overline{B}_a^\mu \overset{\nu}{e}_\mu \tag{44}$$

that the $\overset{\nu}{\overline{B}}_a$ correspond to the covariant basis vectors $\overset{\nu}{e}_\mu$. Conversely, a covariant vector $'w_a$ in X_n^m is *not* associated with a covariant in X_n .

We shall now make use of the one-to-one correspondence between the E_m and the $\overline{E}_m$ by identifying the corresponding contravariant vectors and thus regarding the $\overline{v}^\nu$ and v^c as the defining numbers of one and the same quantity, which can be regarded as a vector in X_n, as well as a vector in X_n^m. Correspondingly, we ignore the difference between the kernel symbols and write v^ν instead of $\overline{v}^\nu$ from now on. That identification is geometrically related to the fact that $d\xi^\nu$ and $d\eta^a$ in (40) can be regarded as the defining numbers of the same line element that lies in X_n as well as in X_n^m. *After* E_m and $\overline{E}_m$ have been identified, (43) will now take on the following geometric interpretation: The vector $'w_a$, which can be represented by two parallel E_{m-1} in E_m, arises from the vector w_λ, which can be represented by two parallel E_{n-1} in E_n, by intersecting with E_m.

With an application of B^c_a, it will follow from the *former identification* that:

(45) $$B^\nu_a = \overline{B}^\nu_b B^b_a = \overline{B}^\nu_a ,$$

which will identify $\overline{B}$ and B, and with an application to $\underset{a}{e}{}^c$:

(46) $$\underset{a}{e}{}^\nu = \underset{a}{\overline{e}}{}^\nu ,$$

which will imply the identification of $\underset{a}{e}$ and $\underset{a}{\overline{e}}$. The kernel symbols $\overline{B}$ and $\underset{a}{\overline{e}}$ vanish from the calculations from here on out, so (41) will go to:

(47) $$\boxed{v^\nu = B^\nu_b\, v^b}$$

and (43) will go to:

(48) $$\boxed{'w_a = B^\mu_a\, w_\mu .}$$

Further simplifications can first be made when X^m_n is *complemented;* i.e., each point of X_n is associated with an $(n - m)$-direction that has no direction in common with the m-direction of X^m_n. That can come about by establishing m independent covariant vectors $\overset{c}{\underline{e}}_\lambda$ at each point in X_n whose $(n - 1)$-directions do not include the m-direction of X^m_n. Those $\overset{c}{\underline{e}}_\lambda$ can be transformed arbitrarily in a homogeneous linear way in their own right. The $\overset{c}{\underline{e}}_\lambda$ can then arise from a coupling quantity $\underline{B}^c_\lambda$ of rank m by throttling, whose upper index lies in X_n, while the lower one lies in the E_m that belongs to those transformations:

(49) $$\overset{c}{\underline{e}}_\lambda = \underline{B}^b_\lambda \overset{c}{e}_b \qquad \text{[cf., (37)]}$$

(50) $$\boxed{\underline{B}^c_\lambda = \overset{b}{\underline{e}}_\lambda \underset{b}{e}{}^c} \qquad \text{[cf., (39)].}$$

Naturally, the basis vectors $\overset{c}{e}_b$ and $\underset{b}{e}{}^c$ of this E_m are not generally identical to the basis vectors that appear in (37) and (39) for the E_m that is introduced into them, but they will become identical, and the two E_m will then coincide, as soon as one couples the transformations of the $\overset{c}{\underline{e}}_\lambda$ with those of the $\underset{a}{\bar{e}}^\nu$ by choosing the $\overset{c}{\underline{e}}_\lambda$ such that:

$$\overset{c}{\underline{e}}_\mu \underset{a}{\bar{e}}^\mu = \delta^c_a . \tag{51}$$

Geometrically, that means that the intersection of the $\overset{c}{\underline{e}}_\lambda$ with the local E_m of the $\underset{a}{e}^\nu$ is the double E_{m-1} of the parallelepiped of the $\underset{a}{e}^\nu$.

However, for the time being, we shall not make use of that coupling, but we will start from (50), and in that way completely overlook the fact that an m-direction is defined at each point in X_n, along with the $(n - m)$-direction. That will then lead to a line of reasoning that is entirely dual to the one that starts from (39). Along with the E_m of the $\underset{a}{e}^c$ in (50), which now has nothing to do with the E_m of the $(d\eta)^c$, there exists the m-dimensional set of vectors that are linearly derivable from the $\overset{c}{\underline{e}}_\lambda$, which are mapped in a one-to-one way to the covariant vectors of the E_m that arise from the E_m by laying them together along the $(n - m)$-direction of the complement. The latter E_m shall be denoted by $\underline{E}_m$ in order to distinguish it. Any covariant vector w_a in E_m is always assigned a covariant vector $\underline{w}_\lambda$ in X_n in a single-valued way:

$$\underline{w}_\lambda = \underline{B}^b_\lambda w_b \qquad \text{[cf., (41)]}. \tag{52}$$

From (49), the vectors $\overset{c}{e}_a$ then correspond to the vectors $\overset{c}{\underline{e}}_\lambda$. Conversely, a single vector in E_m will correspond to a covariant vector $\underline{w}_\lambda$ in X_n if and only if there is a vector w_c that satisfies equation (52). Since that equation can also be written:

$$\underline{w}_\lambda = \overset{b}{\underline{e}}_\lambda \underset{b}{w} \qquad \text{[cf., (42)]}, \tag{53}$$

that will be the case if and only if $\underline{w}_\lambda$ belongs to $\underline{E}_m$. The correspondence between the covariant vectors in E_m and the ones in $\underline{E}_m$ is then one-to-one. We would like to say that a covariant vector $\underline{w}_\lambda$ that belongs to $\underline{E}_m$ *lies in the* X_n^m.

By contrast, any contravariant vector v^ν in X_n is always associated with a contravariant vector in E_m in a one-to-one way:

$$'v^c = \underline{B}^c_\mu v^\mu \qquad \text{[cf., (43)]}. \tag{54}$$

That vector will be zero if and only if v^ν lies in the $(n - m)$-direction of the complement. It follows from the equation:

$$\underline{B}^c_\mu = \underline{B}^c_\mu \underset{\lambda}{e}^\mu \qquad \text{(55)} \qquad \text{[cf., (44)]}$$

that the $\underset{\lambda}{\underline{B}}^c$ correspond to the contravariant basis vectors $\underset{\lambda}{e}^\nu$. Conversely, a contravariant vector $'v^c$ in E_m is not associated with any contravariant vector in X_n. We shall now make use of the one-to-one correspondence between the E_m and the $\underline{E}_m$ by identifying the corresponding covariant vectors, and thus regarding $\underline{w}_\lambda$, as well as w_a, as the defining numbers of one and the same quantity that can be regarded as a vector in X_n, as well as a vector in E_m. Correspondingly, we shall now write w_λ instead of $\underline{w}_\lambda$. Once E_m and $\underline{E}_m$ have been identified, (54) will now take on the following meaning: The vector $'v^c$ that can be represented by two points in E_m arises from the vector v^ν that can be represented by two points in E_n because the latter two points, along with the $(n - m)$-direction of the complement, determine two E_{n-m} that will become points of $\overline{E}_m$ when they are laid together.

It follows from this *second identification* by an application of B^c_a that:

$$B^c_\lambda = \underline{B}^b_\lambda B^c_b = \underline{B}^c_\lambda \qquad \text{(56)} \qquad \text{[cf., (45)],}$$

which will identify $\underline{B}$ and B, and when that is applied to $\overset{c}{e}_a$:

$$\overset{c}{e}_\lambda = \overset{c}{\underline{e}}_\lambda \qquad \text{(57)} \qquad \text{[cf., (46)],}$$

which will imply the identification of $\overset{c}{e}$ and $\overset{c}{\underline{e}}$. The kernel symbols $\underline{B}$ and $\overset{c}{\underline{e}}$ will vanish from the calculations from now on. (52) will then go to:

$$\boxed{w_\lambda = B^b_\lambda w_b} \qquad \text{(58)} \qquad \text{[cf., (47)]}$$

and (54) will go to:

$$\boxed{'v^c = B^c_\mu v^\mu} \qquad \text{(59)} \qquad \text{[cf., (48)].}$$

Only now do we introduce the equation (51), which couples the two lines of reasoning that start from (39) [(50), resp.]. (51), (45), and (56) will then yield that:

$$\boxed{B^{\mu e}_{a\mu} = B^e_a.} \qquad \text{(60)}$$

(47) and (58) yield:

(61) $$\boxed{B^{\nu}_{\lambda} = B^{\nu}_{b} B^{b}_{\lambda}.}$$

It follows from (61) and (47) [(58), resp] that:

(62) $$\boxed{v^{c} = B^{c}_{\mu}\, v^{\mu},}$$

(63) $$\boxed{w_{a} = B^{\mu}_{a}\, w_{\mu}}$$ [cf., (62)].

Finally, it follows from (60) and (61) that:

(64) $$\begin{aligned} a)\quad B^{\nu}_{\lambda} &= B^{\nu}_{\mu} B^{\mu}_{\lambda}, \\ b)\quad B^{c}_{\lambda} &= B^{c}_{\mu} B^{\mu}_{\lambda}, \\ c)\quad B^{\nu}_{a} &= B^{\nu}_{\mu} B^{\mu}_{a}. \end{aligned}$$

The following diagram (*) gives an overview of how the equations relate to each other and shows how the various formulas follow from the three assumptions (39), (50), (51), and the two identifications I_1 and I_2 .

For the case in which v^{ν} (w_{λ}, resp.) lies in X^{m}_{n} (cf., pp. 12 and 14), and there must then exist a vector v^{c} (w_{a}, resp.) in X^{m}_{n} that satisfies equation (47) [(58), resp.], (62) [(63), resp.] will now give the means for expressing that vector in terms of v^{ν} (w_{λ}, resp.). If v^{ν} (w_{λ}, resp.) does not lie in X^{m}_{n} then a vector $'v^{c}$ ($'w_{a}$, resp.) can be constructed with the help of (59) [(48), resp.] whose defining numbers $'v^{\nu}$ ($'w_{\lambda}$, resp.) will follow from (47) [(58), resp.]. An application of (62) [(63), resp.] will then yield:

(65) $$'v^{c} = B^{c}_{\nu}\, v^{\nu} = B^{c}_{\nu}\, 'v^{\nu}$$

or

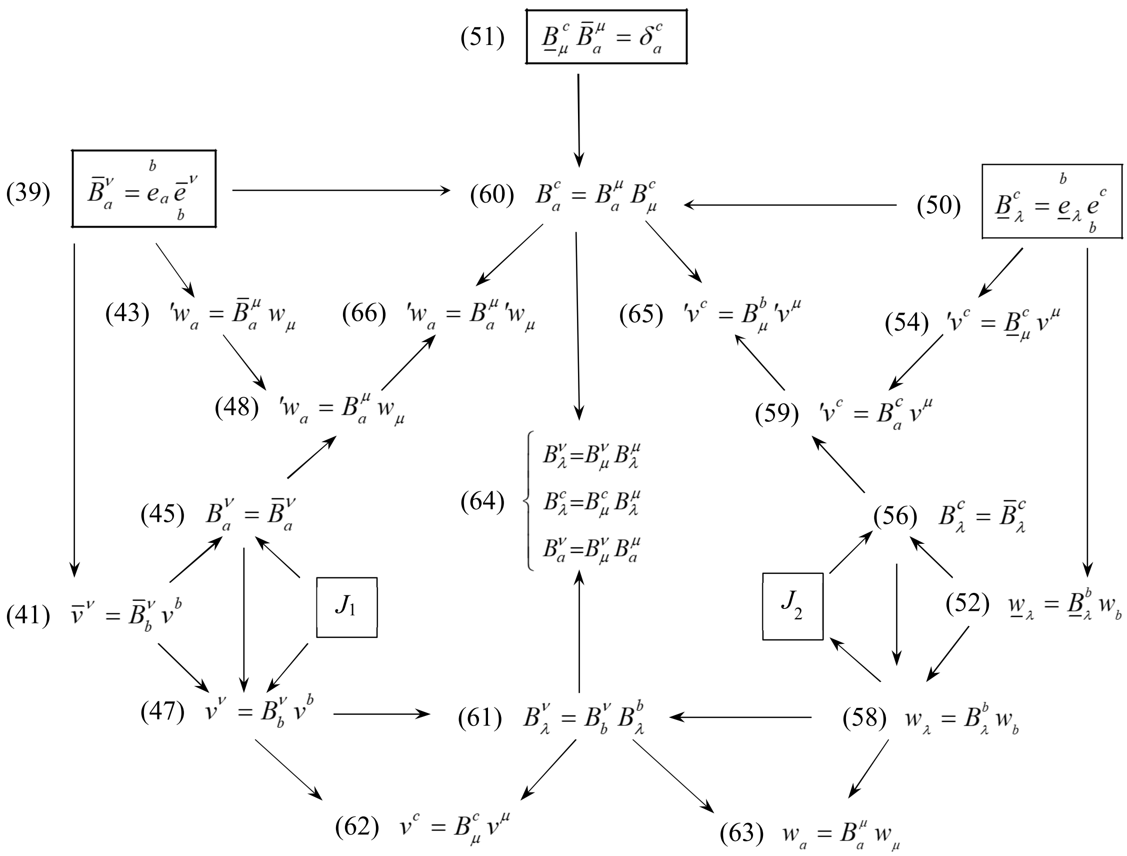
(66) $$'w_{a} = B^{\lambda}_{a}\, w_{\lambda} = B^{\lambda}_{a}\, 'w_{\lambda},$$

resp., from which it will emerge that v^{ν} (w_{λ}, resp.) cannot be decomposed into a component $'v^{\nu}$ ($'w_{\lambda}$, resp.) that lies in X^{m}_{n} and a component that lies in the $(n - m)$-direction of the complement (is contained in the m-direction of X^{m}_{n}). From now on, we shall correspondingly call $'v^{\nu} = B^{\nu}_{\mu}\, v^{\mu}$ ($w_{\lambda} = B^{\mu}_{\lambda}\, w_{\mu}$, resp.) the X^{m}_{n}*-component* of v^{ν} (w_{λ}, resp.). In summary, we have now achieved the following: There exist arbitrary contravariant (covariant, resp.) vectors in X_n and ones that *lie in*

(*) In the diagram, the indices b in formulas (39) and (50) are plotted from left to far right, along with the indices a and c in (51).

X_n^m. Each vector that lies in X_n^m possesses defining numbers with indices $\alpha, \ldots, \omega$, as well as ones with the indices $a, \ldots, g$. We shall say that a quantity in X_n *lies at the location of a certain index in* X_n^m when it does not change when contracted with B_λ^ν at that location, so one will have the rule:

A quantity on X_n *can have defining numbers that carry an index in the sequence* $a, \ldots, h$ *at a certain place when it lies in that place in the* X_n^m.



An example is given by the quantity B, which possesses four defining numbers: B_λ^ν, B_a^ν, B_λ^c, B_a^c, whose relationships are expressed by (60), (61), and (64).

After complementation, each point will be assigned an m-direction and an $(n-m)$-direction, and the space that the latter spans can also be regarded as a complementary $X_n^{m'}$ that is embedded in X_n then, where m' is written instead of $n-m$, for the sake of simplicity. If one applies precisely the same considerations to that $X_n^{m'}$ and one denotes the unit affinor of $X_n^{m'}$ by C_p^r, $p, \ldots, w = m + 1, \ldots, n$ then that will yield the following equations for C_p^ν and C_λ^r:

$$a) \qquad C_p^\nu = \overset{q}{e}_p \underset{q}{e}^\nu, \qquad\qquad C_\lambda^r = \underset{q}{e}^r \overset{q}{e}_\lambda,$$

(67) b) $C_p^{r} = C_{\mu p}^{r\mu} = \underset{q}{e}^{r} \overset{q}{e}_p$,

c) $C_\lambda^{\nu} = C_{q\lambda}^{\nu q} = \underset{q}{e}^{\nu} \overset{q}{e}_\lambda$,

in which the vectors $\underset{q}{e}^{\nu}$ and $\overset{q}{e}_\lambda$, together with $\underset{a}{e}^{\nu}$ and $\overset{c}{e}_\lambda$, define two reciprocal systems. Vectors that lie in $X_n^{m'}$ have two types of defining numbers whose relationships read as follows:

(68)

a) $v^\nu = C_r^{\nu}\, v^r$, $\qquad v^r = C_\nu^{r}\, v^\nu$,

b) $w_\lambda = C_\lambda^{p}\, w_p$, $\qquad w_p = C_p^{\lambda}\, w_\lambda$,

and the connection between A, B, and C is given by the formula:

(69) $$A_\lambda^{\nu} = B_\lambda^{\nu} + C_\lambda^{\nu} .$$

Furthermore, one has the rule:

A quantity can have defining numbers that carry an index in the sequence p, …, w in a certain position if and only if they lie in that position in $X_n^{m'}$.

Every point in X_n is now associated with an E_n , an E_m , and an $E_{m'}$ then, and there are:

1. Quantities in X_n with indices from the sequence $\alpha, \ldots, \omega$.

2. Quantities in X_n^m with indices from the sequences $\alpha, \ldots, \omega$ and $a, \ldots, g$.

3. Quantities in $X_n^{m'}$ with indices from the sequences $\alpha, \ldots, \omega$ and $p, \ldots, w$.

4. Coupling quantities that can carry indices from the sequence $\alpha, \ldots, \omega$ anywhere, and they can also carry indices from the sequences $a, \ldots, g$ ($p, \ldots, w$, resp.) in the positions where they lie in X_n^m ($X_n^{m'}$, resp.).

We now consider the special case in which the complement comes about by way of a fundamental tensor $a_{\lambda\mu}$ that is given in X_n . The X_n will be orthogonal to V_n, and the $(n - m)$-direction of the complement will be orthogonal to the m-direction in X_n^m relative to the fundamental tensor that was introduced. It must then be possible to construct the B_λ^{c} and B_λ^{ν} from B_a^{ν} and $a_{\lambda\mu}$ alone. We next define the fundamental tensor on V_n^m :

(70) $$\boxed{b_{ab} = B_{ab}^{\lambda\mu}\, a_{\lambda\mu}.}$$

It follows from the equation:

(71) $$\boxed{B_a^\lambda\, C_p^\mu a_{\lambda\mu} = 0,}$$

which express the orthogonality of the complement, along with (69) and (70):

(72) $$B_a^\lambda\, A_\omega^\mu a_{\lambda\mu} = B_a^\lambda\, B_\omega^\mu a_{\lambda\mu} = B_\omega^b\, b_{ab}\,,$$

from which it follows that:

(73) $$\boxed{B_\lambda^c = b^{cb} B_b^\mu a_{\lambda\mu}}$$

and

(74) $$\boxed{B_\lambda^\nu = B_{ab}^{\nu\mu}\, b^{ab} a_{\lambda\mu}.}$$

In the same way, one has:

(75) $$\boxed{c_{pq} = C_{pq}^{\lambda\mu}\, a_{\lambda\mu}}$$

for the fundamental tensor c_{pq} in $V_n^{m'}$, as well as the equations:

(76) $$C_\mu^q\, c_{pq} = C_p^\lambda\, a_{\lambda\mu}\,,$$

(77) $$\boxed{C_\lambda^r = c^{rq}\, C_q^\mu\, a_{\lambda\mu},}$$

(78) $$\boxed{C_\lambda^\nu = C_{pq}^{\nu\mu}\, c^{rq}\, a_{\lambda\mu},}$$

which follow from (71) and (75).

§ 3. – Induced displacement in a X_n^m that is complemented in L_n .

X_n will become L_n with the introduction of a linear displacement with the parameters $\Gamma_{\lambda\mu}^\nu$:

(79) $$\delta v^\nu = dv^\nu + \Gamma_{\lambda\mu}^\nu v^\lambda\, d\xi^\mu\,.$$

The equation of translation reads:

(80) $$\delta v^k = dv^k + \Gamma_{ij}^k\, v^i\,(d\xi)^j$$

relative to the non-holonomic reference system (k) of the previous paragraphs, which will have the transformation equation:

$$\Gamma_{ij}^{k} = A_{ij\nu}^{\lambda\mu k}\,\Gamma_{\lambda\mu}^{\nu} + A_{\mu}^{k}\,\partial_{j} A_{i}^{\mu}\,. \tag{81}$$

If v^ν and $d\xi^\nu$ lie in X_n^m then a translation in X_n^m will be given by the X_n^m-components of δv^ν, namely, the so-called *induced* translation. An X_n^m with a linear translation that is given in it is called an L_n^m; the same thing is true for the $X_n^{m'}$. We shall now go a step further by putting v^ν in X_n^m and w^ν in $X_n^{m'}$ and always considering only the components of the differential in the manifold that belongs to the field. Two covariant differentials will then arise:

$$\begin{aligned}
&\alpha) \quad \overset{m}{\delta}\, v^c = dv^c + \Gamma_{ab}^{c}\, v^a\, (d\xi)^b\,,\\
&\beta) \quad \overset{m}{\delta}\, w^r = dw^r + \Gamma_{pb}^{r}\, w^p\, (d\xi)^b\,,\\
&\gamma) \quad \overset{m'}{\delta}\, v^c = dv^c + \Gamma_{aq}^{c}\, v^a\, (d\xi)^q\,,\\
&\delta) \quad \overset{m'}{\delta}\, w^r = dw^r + \Gamma_{pq}^{r}\, w^p\, (d\xi)^q\,,
\end{aligned} \tag{82}$$

and each of them will define a covariant differential quotient:

$$\begin{aligned}
&\alpha) \quad \overset{m}{\nabla}_b\, v^c = B_{b\nu}^{\mu c}\,\nabla_\mu v^\nu = \partial_b v^c + \Gamma_{ab}^{c}\, v^a\,,\\
&\beta) \quad \overset{m}{\nabla}_b\, w^r = B_b^\mu\, C_\nu^r\,\nabla_\mu w^\nu = \partial_b w^r + \Gamma_{pb}^{r}\, w^p\,,\\
&\gamma) \quad \overset{m'}{\nabla}_q\, v^c = C_q^\mu\, B_\nu^c\,\nabla_\mu v^\nu = \partial_q v^c + \Gamma_{aq}^{c}\, v^a\,,\\
&\delta) \quad \overset{m'}{\nabla}_q\, w^r = C_{q\nu}^{\mu r}\,\nabla_\mu w^\nu = \partial_q w^r + \Gamma_{pq}^{r}\, w^p\,.
\end{aligned} \tag{83}$$

Both sets of equations can also be written with Greek indices ([4]):

$$\begin{aligned}
&\alpha) \quad \overset{m}{\nabla}_\mu\, v^\nu = B_{\mu c}^{b\nu}\,\overset{m}{\nabla}_b\, v^c\,,\\
&\beta) \quad \overset{m}{\nabla}_\mu\, w^\nu = B_\mu^b\, C_r^\nu\,\overset{m}{\nabla}_b\, w^r\,,\\
&\gamma) \quad \overset{m'}{\nabla}_\mu\, v^\nu = C_\mu^q\, B_c^\nu\,\overset{m'}{\nabla}_q\, v^c\,,
\end{aligned} \tag{84}$$

([4]) It should be observed that with our assumptions on the use of indices, expressions such as δv^c or $\nabla_b\, v^c$ are meaningless, since δv^ν and $\nabla_\mu\, v^\nu$ do not lie in X_n^m. It is on precisely those grounds that new differentiation symbols must be introduced here.

$$\delta) \qquad \overset{m'}{\nabla}_\mu w^\nu = C^{q\nu}_{\mu r} \overset{m'}{\nabla}_q w^r .$$

(84α) implies that:

(85) $$B^\beta_\mu \nabla_\beta v^\nu = \overset{m}{\nabla}_\mu v^\nu + v^\lambda \overset{m}{H}{}^{\cdot\cdot\cdot\nu}_{\mu\lambda} ,$$

and likewise:

(86) $$B^\beta_\mu \nabla_\beta v_\lambda = \overset{m}{\nabla}_\mu v_\lambda + v_\nu \overset{m}{L}{}^{\cdot\nu}_{\mu\cdot\lambda} ,$$

where

(87)
$$\alpha) \qquad \overset{m}{H}{}^{\cdot\cdot\cdot\nu}_{\mu\lambda} = -B^{\beta\alpha}_{\mu\lambda} \nabla_\beta C^\nu_\alpha = B^{\beta\alpha}_{\mu\lambda} \nabla_\beta B^\nu_\alpha = -B^{\beta\alpha}_{\mu\lambda} (\nabla_\beta \overset{q}{e}_\alpha) \underset{q}{e}^\nu ,$$
$$\beta) \qquad \overset{m}{L}{}^{\cdot\nu}_{\mu\cdot\lambda} = -B^{\beta\nu}_{\mu\gamma} \nabla_\beta C^\gamma_\lambda = B^{\beta\nu}_{\mu\gamma} \nabla_\beta B^\gamma_\lambda = -B^{\beta\nu}_{\mu\gamma} (\nabla_\beta \underset{q}{e}^\gamma) \overset{q}{e}_\lambda$$

are the first and second *curvature affinors* of L^m_n, for which ν (λ, resp.) belong to $L^{m'}_n$ and $\mu\lambda$ ($\mu\nu$, resp.) belong to L^m_n. In the same way, there are two curvature affinors $\overset{m'}{H}{}^{\cdot\cdot\cdot\nu}_{\mu\lambda}$ and $\overset{m'}{L}{}^{\cdot\nu}_{\mu\cdot\lambda}$ in $L^{m'}_n$.

Since *the new operators satisfy the usual rules in regard to sums and products*, they can be applied to higher-degree quantities that have some indices that belong to L^m_n, while the remaining ones belong to $L^{m'}_n$; e.g.:

(88) $$\overset{m}{\nabla}_b T_a^{\cdot r} = B^{\mu\lambda}_{ba} C^r_\nu \nabla_\mu T_\lambda^{\cdot r} = \partial_b T_a^{\cdot r} + \Gamma^{\,r}_{pb} T_a^{\cdot p} - \Gamma^{\,c}_{ab} T_c^{\cdot r} .$$

However, each of the operators $\overset{m}{\delta}$, $\overset{m'}{\delta}$, $\overset{m}{\nabla}$, $\overset{m''}{\nabla}$ can generate only quantities that have indices that belong to L^m_n, while the remaining ones belong to $L^{m'}_n$, so it follows from this that any arbitrary sequence of those operators will always makes sense when it is applied to quantities of the given kind.

§ 4. – The *D*-symbolism.

Let $v^{\cdot\cdot\cdot\nu}_{\lambda\mu}$ be a field whose index λ belong to L^m_n and whose index μ belong to $L^{m'}_n$, while it can be regarded as a quantity in L_n relative to ν. The usual covariant differential $\delta v^{\cdot\cdot\cdot\nu}_{\lambda\mu}$ will then exist, for which all indices of $v^{\cdot\cdot\cdot\nu}_{\lambda\mu}$ are regarded as belonging to L_n. Now, the expression:

(89) $$d\xi^\omega B^{\delta\alpha}_{\omega\lambda} C^\beta_\mu \nabla_\delta v^{\cdot\cdot\cdot\nu}_{\alpha\beta}$$

is likewise a covariant differential of $v_{\lambda\mu}^{\cdot\cdot\cdot\nu}$, and indeed it is one whose indices are in the same positions relative to L_n^m, $L_n^{m''}$, and L_n as the those of $v_{\lambda\mu}^{\cdot\cdot\cdot\nu}$. *However, that differential cannot be constructed with the symbols δ,* $\overset{m}{\delta}$, $\overset{m''}{\delta}$ *that were introduced up to now without employing the factors B and C.* One must then introduce new differential symbols here and a corresponding differential operator, and that must be done anew for every field that can be regarded as having some indices that belongs to L_n^m and others that belongs to $L_n^{m''}$, while the remaining ones are regarded as belonging to L_n. Now the various operators obviously differ only by the number of factors *B* and *C* that one contracts with in formulas such as (89), and the places of application where the contraction takes place. Different authors have almost simultaneously come to the conclusion that this troublesome introduction of new operators can be avoided in such a way that the number of factors *B* and *C* and the positions where they are applied can be given by the choice of indices from the sequences *a*, …, *g*, *p*, …, *w*, or *α*, …, *ω*, instead of by any index that the operator is endowed with. One could then get by with a single operator – say *D,* for example:

(90) $$D_b\, v_{ap}^{\cdot\cdot\cdot\nu} = B_{ba}^{\delta\alpha}\, C_p^{\beta}\, \nabla_\delta v_{\alpha\beta}^{\ \ \ \nu}.$$

However, there is one thing that must be observed: Up to now, the rule was true for all expressions that the indices *a*, …, *g* can appear *only* where the expression is contracted with *B*, but that indices *α*, …, *ω* can also appear in *all* positions, and correspondingly for the indices *p*, …, *w*, such that, e.g., $\nabla_\omega v_{\lambda\mu}^{\cdot\cdot\cdot\nu}$ has meaning, but not $\nabla_b v_{ap}^{\cdot\cdot\cdot\nu}$. *The second part of this rule will no longer be true for D-formulas*; e.g., if *a* were replaced with *λ* in (90) then that would mean that $v_{\lambda\mu}^{\cdot\cdot\cdot\nu}$ would have to be regarded as a quantity in L_n under differentiation, and also as far as the index *λ* is concerned, and a completely new quantity would then arise:

(91) $$D_b\, v_{\lambda p}^{\cdot\cdot\cdot\nu} = B_{b\lambda}^{\delta\alpha}\, C_p^{\beta}\, \nabla_\delta v_{\alpha\beta}^{\ \ \ \nu}.$$

D-formulas and ∇-formulas or *δ*-formulas *must never be confused or employed when mixed together, since indices in them have basically different meanings.* It is characteristic of ∇-formulas and formulas without differentiation that their meaning depends upon only the *skeleton* (= totality of kernel symbols, positions of the indices, and positions of the contractions performed), but not upon which of the allowed types of indices are employed. The *D*-formulas do not possess that property.

We now define the *D*-operators as follows: u^ν is a field in L_n, v^c lines in L_n^m, and w^r lies in $L_n^{m'}$.

(92)
$$\begin{aligned}
&\alpha) \quad D_\mu\, p = \nabla_\mu\, p,\\
&\beta) \quad D_\mu\, u^\nu = \nabla_\mu\, u^\nu,\\
&\gamma) \quad D_\mu\, v^c = B_\nu^c\, \nabla_\mu v^\nu,\\
&\delta) \quad D_\mu\, w^r = C_\nu^r\, \nabla_\mu w^\nu,
\end{aligned}$$

$$(93)\quad \begin{aligned} &\alpha) && D_b\, p &&= B_b^{\mu} \nabla_{\mu} p\,, \\ &\beta) && D_b\, u^{\nu} &&= B_b^{\mu} \nabla_{\mu} u^{\nu}\,, \\ &\gamma) && D_b\, v^{c} &&= B_{b\nu}^{\mu c} \nabla_{\mu} v^{\nu}\,, \\ &d) && D_b\, w^{r} &&= B_b^{\mu}\, C_{\nu}^{r} \nabla_{\mu} w^{\nu}\,, \end{aligned}$$

$$(94)\quad \begin{aligned} &\alpha) && D_q\, p &&= C_q^{\mu} \nabla_{\mu} p\,, \\ &\beta) && D_q\, u^{\nu} &&= C_q^{\mu} \nabla_{\mu} u^{\nu}\,, \\ &\gamma) && D_q\, v^{c} &&= C_b^{\mu} B_{\nu}^{c} \nabla_{\mu} v^{\nu}\,, \\ &d) && D_q\, w^{r} &&= C_{q\nu}^{\mu r} \nabla_{\mu} w^{\nu}\,. \end{aligned}$$

The operations D_μ, D_b, D_q shall be called *differentiation in* L_n, L_n^m, $L_n^{m'}$, resp.; corresponding formulas are true for covariant vectors. Therefore, B and C can be expressed as follows:

$$(95)\qquad B_a^{\nu} = D_a\, \xi^{\nu}, \quad C_p^{\nu} = D_p\, \xi^{\nu}.$$

The operators D *satisfy the formal rules of differentiation of sums and products*, and that will imply the rules for the differentiation of a higher-degree quantity with different types of indices, such as e.g., (90), (91). It is very important that the formal rules for contraction are also true, and indeed for the three possible types of contractions that correspond to the three types of indices, e.g.:

$$(96)\qquad D_b\, v_a^{\cdot\nu} w^a = (D_b\, v_a^{\cdot\nu})\, w^a + v_a^{\cdot\nu} (D_b\, w^a).$$

Formulas (92), (93), (94) are lacking precisely the structures that arise by contracting the v that belongs to differentiation in L_n^m with C and contracting the w that belongs to differentiation in $L_n^{m'}$ with B. However, those structures are not actual differential concomitants, since they depend upon only the local values of v (w, resp.) and the curvature affinors that were defined in the previous paragraphs. However, if:

$$(97)\quad \begin{aligned} &\alpha) && C_{\nu}^{r} \nabla_{\mu} v^{\nu} = -\, C_{\gamma}^{r}\, v^{\nu} \nabla_{\mu} C_{\nu}^{\gamma} = -\left(\overset{m}{H}{}_{\mu\lambda}^{\cdots r} + \overset{m'}{L}{}_{\mu\cdot\lambda}^{\cdot r} \right) v^{\lambda}\,, \\ &\beta) && B_{\nu}^{c} \nabla_{\mu} w^{\nu} = -\, B_{\gamma}^{c}\, w^{\nu} \nabla_{\mu} B_{\nu}^{\gamma} = -\left(\overset{m'}{H}{}_{\mu\lambda}^{\cdots c} + \overset{m}{L}{}_{\mu\cdot\lambda}^{\cdot c} \right) v^{\lambda}\,, \end{aligned}$$

then contracting with B (C, resp.) will yield:

$$\alpha)\qquad B_b^{\mu}\, C_{\nu}^{r} \nabla_{\mu} v^{\nu} = -B_b^{\mu}\, C_{\gamma}^{r}\, v^{\nu} \nabla_{\mu} C_{\nu}^{\gamma} = -\overset{m}{H}{}_{ba}^{\cdots r}\, v^{a}\,,$$

$$
(98)\qquad
\begin{aligned}
&\beta) && B^{\mu c}_{b\nu}\,\nabla_\mu w^\nu && = -B^{\mu c}_{b\gamma}\, w^\nu\,\nabla_\mu B^\gamma_\nu && = -\overset{m}{L}{}_{b\cdot p}^{\cdot r}\, w^p, \\
&\gamma) && C^{\mu r}_{q\nu}\,\nabla_\mu v^\nu && = -C^{\mu r}_{q\gamma}\, v^\nu\,\nabla_\mu C^\gamma_\nu && = -\overset{m'}{L}{}_{q\cdot a}^{\cdot r}\, v^a, \\
&\delta) && C^\mu_q\, B^c_\nu\,\nabla_\mu w^\nu && = -C^\mu_q\, B^r_\gamma\, w^\nu\,\nabla_\mu B^\gamma_\nu && = -\overset{m'}{H}{}_{qp}^{\cdots c}\, w^p.
\end{aligned}
$$

For the curvature affinors, it follows from (92), (94) that:

$$
(99)\qquad
\begin{aligned}
&\alpha) && \overset{m}{H}{}_{ba}^{\cdots\nu} = D_b B^\nu_a = D_b\, D_a\, \xi^\nu, \\
&\beta) && \overset{m}{L}{}_{b\cdot p}^{\cdot r} = D_b B^c_\lambda,
\end{aligned}
$$

and corresponding formulas will be true for $\overset{m'}{H}$ and $\overset{m'}{L}$. Of the differential quotients that were defined in (92), (93), (94) for a complement L^m_n in L_n, (92γ), (92δ), and (94) will lose their meanings when one goes to a complement L_m in L_n. The same thing will be true of (92δ), (94), (93γ), and (93δ) when one drops the complement L^m_n. However, if a displacement exists in L^m_n that is based on any grounds then the defining equation $D_b\, v^c = \overset{m}{\nabla}_b\, v^c$ will remain, but the relationship to ∇_μ will be lacking.

The D-symbolism was found by various authors independently. A first attempt can be found in R. Lagrange ([5]), who wrote down the covariant differential of a quantity in L_n for the coupling defining numbers relative to the two systems of Ur-variables (ν) and (N):

$$
\delta v_\lambda^{\cdot \mathrm{N}} = dv_\lambda^{\cdot \mathrm{N}} + \Gamma^{\mathrm{N}}_{\Lambda \mathrm{M}}\, v_\lambda^{\cdot \Lambda} d\xi^{\mathrm{M}} - \Gamma^{\nu}_{\lambda\mu}\, v_\nu^{\cdot \mathrm{N}} d\xi^\mu .
$$

Naturally, there are no grounds for distinguishing between δ and D here, since one is still dealing with two different systems of n coordinates in the same manifold ([6]). It was Van der Waerden ([7]) who first considered quantities with some indices that belong to V_n, while the other belong to a V_m that is embedded in V_n, and the operators $d\xi^\mu D_\mu$ are defined to act upon the quantities in V_n, while $d\eta^b D_b$ acts upon quantities in V_n and V_m (with the notation ***d***). Those operators were found by E. Bortolotti ([8]) in the same sphere of action independently of Van der Waerden. Unfortunately, Boltolotti did not introduce any new differentiation symbols, but kept the symbol ∇, which could lead to confusion and would unnecessarily make it impossible to use the old formulas in the course

([5]) *Calcul différentiel absolu*, Mém. des Sciences Math. Fasc. **19** (1926), pp. 10.

([6]) The index p on pp. 10, line 9, *cf. supra*, which suggests just the opposite, is obviously only a printing error.

([7]) "Differentialkovarianten von n-dimensionalen Mannigfaltigkeiten in Riemannschen m-dimensional Räumen," Abh. Math. Sem. Hamburg **5** (1927), 153-160.

([8]) "Spazi subordinate: equazioni di Gauß e Codazzi," Boll. Un. Matem. **6** (1927), 134-137. "Sulla varietà subordinate negli spazi a connessione affine e su di una espressione dei simboli di Riemann," Boll. Un. Matem. **7**, 2 (1928), pp. 8.

of calculations, since only their skeleton is defined. Finally, one also finds the same operator $d\eta^b D_b$ (with the notation D) in Duschek-Mayer ([9]). The operator D_b, as it is applied to quantities in V_m, can be found in a later work of Bortolotti ([10]), and is denoted by ∇^* there (in his less-consistent system). Furthermore, the decomposition of the differentials of v and w that is expressed in (82) was already found much earlier by Weyl ([11]) and Cartan ([12]).

§ 5. – Curvature quantities for the V_n^m in V_n.

We consider the case of a V_n^m in V_n, whose metric is given by the fundamental tensor $a_{\lambda\nu}$ in V_n (cf., pp. 18). As a result of (93) and (94), the following is then true for the fundamental tensors b_{ab} and c_{pq}:

(100) $\alpha) \quad D_b\, b_{ac} = 0, \qquad \beta) \quad D_q\, b_{ac} = 0,$

(101) $\alpha) \quad D_b\, c_{pr} = 0, \qquad \beta) \quad D_q\, c_{pr} = 0,$

(102) $D_\mu\, a_{\lambda\nu} = 0, \qquad D_b\, a_{\lambda\nu} = 0, \qquad D_q\, a_{\lambda\nu} = 0,$

and with consideration given to (99), it will follow from this that:

(103) $$\overset{m}{L}{}_{ba}^{\cdot\cdot\cdot\nu} = D_b\, b_{ac} B_\lambda^c\, a^{\lambda\nu} = D_b\, B_a^\nu = \overset{m}{H}{}_{ba}^{\cdot\cdot\cdot\nu}.$$

The difference between $\overset{m}{L}$ and $\overset{m}{H}$ will then vanish, and the same thing will be true for $\overset{m'}{L}$ and $\overset{m'}{H}$, such that instead of (97), one will have:

(104)
$$\alpha) \quad c_{p\nu}\,\nabla_\mu\, v^\nu = -\left(\overset{m}{H}{}_{\mu\lambda p} + \overset{m'}{H}{}_{\mu\lambda p}\right) v^\lambda,$$
$$\beta) \quad b_{p\nu}\,\nabla_\mu\, w^\nu = -\left(\overset{m}{H}{}_{\mu a\lambda} + \overset{m'}{H}{}_{\mu\lambda a}\right) w^\lambda.$$

Differentiating a scalar p twice with ∇_n^m and alternating leads to:

([9]) *Lehrbuch der Differentialgeometrie*, Teubner, 1930, pp. 156. Mayer informed me in a letter on 21-1-1930 that the relevant work of the aforementioned authors was unknown to him and that he had already lectured about the operator D in the Winter semester of 1926/27 at the University of Vienna.

([10]) "Scostamento geodetico e sue generalizzazioni," Giorn. di Matem. di Battaglini **66** (1928), 153-191.

([11]) "Zur Infinitesimalgeometrie: *p*-dimensionale Flächen im *n*-dimensional Raum," Math. Zeit. **12** (1922), 154-160, in particular, pp. 155.

([12]) "La géometrie des espaces de Riemann," Mém. des Sci. Math. **9** (1925), pp. 47.

(105) $$D_{[b}\, D_{a]}\, p = D_{[b} B^{\lambda}_{a]}\, D_{\lambda}\, p = \overset{m}{H}{}_{[ba]}^{\cdots r}\, D_r\, p\,.$$

However, since, on the other hand:

(106) $$D_{[b}\, D_{a]}\, p = \partial_{[b}\, \partial_{a]}\, p + \Gamma^{c}_{[ab]}\, \partial_c\, p\,,$$

it will follow that:

(107) $$\partial_{[b}\, \partial_{a]}\, p = -\Gamma^{c}_{[ab]}\, \partial_c\, p + \overset{m}{H}{}_{[ba]}^{\cdots r}\, D_r\, p\,.$$

The left-hand sides (107) vanishes identically if and only if $\overset{m}{H}{}_{[ba]}^{\cdots r}$, *as well as* $\Gamma^{c}_{[ab]}$, vanishes. Now, according to (87):

(108) $$\overset{m}{H}{}_{[ba]}^{\cdots r} = -\, B^{\beta\alpha}_{ba} \left(\delta_{[\beta}\, \overset{q}{e}{}_{\alpha]} \right) \underset{q}{e}{}^{r}\,,$$

and $\overset{m}{H}{}_{[ba]}^{\cdots r}$ vanishes if and only if the V_n^m degenerates into a system of $\infty^{m'}$ V_n's . In that case, in addition, the $\Gamma^{c}_{[ab]}$ will vanish if and only if the η^{c} are holonomic parameters in that V_m . It follows from (108) for $p = \overset{\nu}{\xi}$ when one contracts with $\underset{\nu}{e}{}^{c}$ that:

(109) $$\Gamma^{c}_{[ab]} = \underset{\nu}{e}{}^{c}\, \partial_{[b}\, \overset{\nu}{e}{}_{a]} = B^{c}_{\nu}\, \partial_{[b}\, B^{\nu}_{a]}\,,$$

and the $\Gamma^{c}_{[ab]}$ *then depend upon only the choice complement, and not the displacement in* V_n . However, the $\Gamma^{c}_{[ab]}$ can be calculated in a known way from (109) and formula (100*a*), which is written out as:

(110) $$\partial_b\, b_{ac} - \Gamma^{d}_{ab}\, b_{dc} - \Gamma^{d}_{cb}\, b_{ad} = 0,$$

and it will then follow that the displacement that is induced in V_n^m depends upon only b_{ab} and the complement, and as a result, *it is invariant under changes of the fundamental tensor a that leave b and the complement unchanged.*

Differentiating a vector u^{ν} in V_n twice in V_n then yields the known equation:

(111) $$D_{[\omega}\, D_{\mu]}\, u^{\nu} = -\,\tfrac{1}{2} K_{\omega\mu\lambda}^{\cdots\nu}\, u^{\lambda}\,,$$

in which $K_{\omega\mu\lambda}^{\cdots\nu}$ is the curvature quantity in V_n .

Differentiating a vector v^c in V_n^m twice in V_n then yields:

(112) $$D_{[\omega} D_{\mu]}\, v^c = D_{[\omega} B^c_{|\nu|}\, D_{\mu]}\, v^c = \left\{ \overset{m'}{H}{}_{[\omega|\lambda|}^{\cdots\; c} + \overset{m}{H}{}_{[\omega\cdot|\lambda|}^{\;\cdot\; c} \right\} D_{\mu]} v^\lambda - \tfrac{1}{2} B^c_\nu\, K_{\omega\mu\lambda}^{\cdots\;\nu} v^\lambda ,$$

and a corresponding formula exists for a vector w^r in $V_n^{m'}$. As a result of (103) and (111), differentiating a vector u^ν in V_n in V_n^m and alternating will yield:

(113) $$D_{[d}\, D_{b]}\, u^\nu = D_{[\omega} B^\mu_{b]}\, D_{\mu]}\, u^\nu = \overset{m}{H}{}_{[db]}^{\cdots\; q} D_q\, u^\nu - \tfrac{1}{2} B^{\omega\mu}_{d\,b}\, K_{\omega\mu\lambda}^{\cdots\;\nu} u^\lambda .$$

As a result of (106), differentiating a vector v^c in V_n^m in V_n^m and alternating yields:

(114) $$\begin{aligned} D_{[d}\, D_{b]}\, v^c = D_{[d}\, D_{b]}\, \overset{a}{v}\, \underset{a}{e}{}^c &= \overset{m}{H}{}_{[db]}^{\cdots\; r} D_r\, v^c - \overset{m}{H}{}_{[db]}^{\cdots\; r}\, \Gamma^c_{ar}\, v^a + \overset{a}{v}\, D_{[d}\, D_{b]}\, \underset{a}{e}{}^c \\ &= \overset{m}{H}{}_{[db]}^{\cdots\; r} D_r\, v^c - \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\cdots\; c} v^a , \end{aligned}$$

in which:

(115) $$\begin{aligned} \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\cdots\; c} &= - \overset{e}{e}{}_a\, D_{[d}\, D_{b]}\, \underset{e}{e}{}^c + \overset{m}{H}{}_{[db]}^{\cdots\; r}\, \Gamma^c_{ar} \\ &= \overset{m}{H}{}_{[db]}^{\cdots\; r}\, \Gamma^c_{ar} - \partial_{[d}\, \Gamma^c_{|a|b]} - \Gamma^c_{e[d}\, \Gamma^c_{|a|b]} - \Gamma^e_{[db]}\, \Gamma^c_{|ae} . \end{aligned}$$

It likewise follows that for a covariant vector v_a in V_n^m:

(116) $$D_{[d}\, D_{b]}\, v_a = \overset{m}{H}{}_{[db]}^{\cdots\; r} D_r\, v_a - \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\cdots\; c} v_c .$$

Unlike the left-hand sides of (114) and (116), the quantity $\overset{m}{K}$ does *not* depend upon only b and the choice of complement and is therefore not a true curvature quantity in V_n^m. That is based upon the fact that the first terms on the right-hand sides of (114) and (116) depends upon the displacement in V_n^m, as well as the choice of complement. However, one has:

(117) $$D_r\, v^c + \overset{m}{H}{}_{a\cdot r}^{\;c}\, v^a = \partial_r\, v^c + 2 v^a\, B^\omega_a\, C^\mu_r\, \partial_{[\omega} B^c_{\mu]} ,$$

and the expression on the left-hand side of this equation will then *depend upon only the choice of complement, and not on the translation in* V_n. It is a type of derivative that we shall denote by the symbol D'_q. Since corresponding things are true for covariant vectors, the defining equations of D'_q will read:

$$(118)\quad \begin{aligned} &\alpha) \qquad D'_q v^c = D_q v^c + \overset{m}{H}{}_{a\cdot q}^{\ \ c} v^a = \partial_q v^c + 2v^a B_a^\omega C_q^\mu \partial_{[\omega} B_{\mu]}^c , \\ &\beta) \qquad D'_q v_a = D_q v_a - \overset{m}{H}{}_{a\cdot q}^{\ \ c} v_c = \partial_q v_a - 2v_c B_a^\omega C_q^\mu \partial_{[\omega} B_{\mu]}^c \\ &\qquad\qquad\qquad = 2B_a^\omega C_q^\mu \partial_{[\omega} v_{\mu]} . \end{aligned}$$

(114) and (116) can then be written:

$$(119)\qquad D_{[d} D_{b]} v^c = \overset{m}{H}{}_{[db]}^{\ \cdots q} D'_q v^c - \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\ *\cdots c} v^a ,$$

$$(120)\qquad D_{[d} D_{b]} v_a = \overset{m}{H}{}_{[db]}^{\ \cdots q} D'_q v_a + \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\ *\cdots c} v_c ,$$

in which

$$(121)\qquad \overset{m}{K}{}_{dba}^{\ *\cdots c} = \overset{m}{K}{}_{dba}^{\ \cdots c} - 2 \overset{m}{H}{}_{[db]}^{\ \cdots r} \overset{m}{H}{}_{a\cdot r}^{\ \cdot c}$$

depends upon only b_{ab} and the choice of complement and is thus a proper curvature quantity in V_n^m.

(118β) gives one information about the geometric meaning of D'_q, since $D'_q v_a$ is a component of 2 $\partial_{[\omega} v_{\mu]}$ in $V_n^{m'}$ in its first index and a component in V_n^m in its second index. *Therefore,* D'_q *does not depend upon either the metric or any translation*, and as a result, it will also exist for a X_n^m that is a complement in X_n. One easily convinces oneself of the fact that the component in $V_n^{m'}$ or V_n^m does *not* lead to a linear translation in *either* index.

Differentiating a vector w^r in $V_n^{m'}$ in V_n^m twice and alternating will give:

$$(122)\qquad \begin{aligned} D_{[d} D_{b]} w^r = D_{[d} D_{b]} \overset{p}{w} \underset{p}{e}{}^r &= \overset{m}{H}{}_{[db]}^{\ \cdots q} D'_q w^r - \overset{m}{H}{}_{[db]}^{\ \cdots q} w^p + \overset{p}{w} D_{[d} D_{b]} \underset{p}{e}{}^r \\ &= \overset{m}{H}{}_{[db]}^{\ \cdots q} D_q w^r - \tfrac{1}{2} \overset{mm'}{K}{}_{dbp}^{\ \cdots r} w^p , \end{aligned}$$

where

$$(123)\qquad \tfrac{1}{2} \overset{mm'}{K}{}_{dbp}^{\ \cdots r} = \overset{m}{H}{}_{[db]}^{\ \cdots q} \Gamma_{pq}^r - \partial_{[d} \Gamma_{|p|q]}^r - \Gamma_{q[d}^r \Gamma_{|p|b]}^r - \Gamma_{[db]}^e \Gamma_{pe}^r .$$

In the same way, when one starts with $V_n^{m'}$, one can derive the curvature quantities $\overset{m'}{K}{}_{sqp}^{\ \cdots r}$ and $\overset{mm'}{K}{}_{sqp}^{\ \cdots r}$, along with a differential operator D'_b that acts upon quantities in $V_n^{m'}$. The translation that belongs to D has the property that there is a parallelogram with two sides in V_n^m and two in $V_n^{m'}$, and as one easily sees by calculation, it is determined uniquely by that property.

It emerges easily from (113), (114), and (122) that upon differentiating a quantity with three different types of indices in V_n^m, a term with the corresponding curvature quantity will appear for each index, along with a single term with $\overset{m}{H}{}_{[db]}^{\cdots q}$; e.g.:

(124) $$D_{[d} D_{b]} T_{\lambda a}^{\cdots r} = \overset{m}{H}{}_{[db]}^{\cdots q} D_q T_{\lambda a}^{\cdots r} + \tfrac{1}{2} B_{db}^{\omega\mu} K_{\omega\mu\lambda}^{\cdots\nu} T_{\nu a}^{\cdots r} + \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\cdots c} T_{\lambda c}^{\cdots r} - \tfrac{1}{2} \overset{mm'}{K}{}_{dbp}^{\cdots r} T_{\lambda a}^{\cdots p}.$$

§ 6. – The generalized equations of Gauss, Codazzi, and Ricci for V_n^m in V_n, derived with the help of the *D*-symbolism.

Equation (124) leads us to the generalized equations of Gauss, Ricci, and Codazzi in the simplest way. Upon applying B_a^ν, that will imply:

(125) $$D_{[d} \overset{m}{H}{}_{b]a}^{\cdots\nu} = \overset{m}{H}{}_{db}^{\cdots q} D_q B_a^\nu + \tfrac{1}{2} \overset{m}{K}{}_{dba}^{\cdots c} B_c^\nu - \tfrac{1}{2} B_{dba}^{\omega\mu\lambda} K_{\omega\mu\lambda}^{\cdots\nu}$$

or

(126) $$B_{dba}^{\omega\mu\lambda} K_{\omega\mu\lambda}^{\cdots\nu} = \overset{m}{K}{}_{dba}^{\cdots\nu} - 2\overset{m}{H}{}_{[db]}^{\cdots c} \overset{m'}{H}{}_{q\cdot a}^{\cdot\nu} - 2D_{[d} H_{b]a}^{\cdots\nu}$$

$$= \overset{m}{K}{}_{dba}^{*\cdots\nu} + 2\overset{m}{H}{}_{[db]}^{\cdots q} \left(\overset{m}{H}{}_{a\cdot q}^{\cdot\nu} - \overset{m'}{H}{}_{q\cdot a}^{\cdot\nu} \right) - 2D_{[d} H_{b]a}^{\cdots\nu},$$

and that equation will yield the generalized Gauss equation when one contracts with B_ν^c [cf., *Der Ricci-Kalkül*, Berlin, Julius Springer, 1923, which will be cited as R.K. from now on, pp. 198, formula (157)]:

(127) $$\boxed{B_{dba\nu}^{\omega\mu\lambda c} K_{\omega\mu\lambda}^{\cdots\nu} = \overset{m}{K}{}_{dba}^{*\cdots c} + 2\overset{m}{H}{}_{[db]}^{\cdots q} \overset{m}{H}{}_{a\cdot q}^{\cdot c} + 2\overset{m}{H}{}_{[d\cdot|q|}^{\cdot c} \overset{m}{H}{}_{b]a}^{\cdots q},}$$

which will assume the form:

(128) $$\boxed{B_{dba\nu}^{\omega\mu\lambda c} K_{\omega\mu\lambda}^{\cdots\nu} = \overset{m}{K}{}_{dba}^{*\cdots c} + 2\overset{q}{h}{}_{[db]}^{\cdots} \underset{q}{l}{}_{a\cdot}^{\cdot c} + 2\underset{q}{l}{}_{[d}^{\cdot c} \overset{q}{h}{}_{b]c}}$$

when one uses the indices q [cf., R.K. pp. 198, formula (158)], where:

(129) $$\overset{r}{h}{}_{ba} = -\overset{m}{H}{}_{ba}^{\cdots q} \overset{r}{e}{}_q = (D_b \overset{r}{e}{}_\lambda) B_a^\lambda,$$

$$\underset{p}{l}{}_{b\cdot}^{\cdot c} = -H_{b\cdot a}^{\cdot c} \underset{p}{e}{}^q = (D_b \underset{p}{e}{}^\nu) B_\nu^c,$$

and contracting with C^r_ν will give the generalized Codazzi equation:

$$(130)\qquad \boxed{B^{\omega\mu\lambda}_{dba} C^r_\nu K_{\omega\mu\lambda}^{\cdots\nu} = -2 \overset{m}{H}{}_{[db]}^{\cdots q} \overset{m'}{H}{}_{q\cdot a}^{\cdot r} + 2 D_{[d} \overset{m}{H}{}_{b]a}^{\cdots r},}$$

or, in another form, when one introduces the indices *q* and *r* [cf., R.K., pp. 200, formula (168*b*)]:

$$(131)\qquad \boxed{2D_{[d}\overset{r}{h}_{b]a} = +B^{\omega\mu\lambda}_{dba}\overset{r}{e}_\nu K^{\cdots\nu}_{\omega\mu\lambda} - 2\overset{q}{h}_{[db]}\underset{q}{\overset{r}{u}}_a + 2\underset{q}{\overset{r}{v}}_{[d}\overset{q}{h}_{b]a},}$$

where

$$(132)\qquad \begin{aligned} \underset{p}{\overset{r}{v}}_d &= -\underset{p}{e}^q D_d \overset{r}{e}_q = \overset{r}{e}_q D_d \underset{p}{e}^q \overset{*}{=} \Gamma^r_{pd}, \\ \underset{p}{\overset{r}{u}}_a &= \overset{m'}{H}{}^{\cdot s}_{q\cdot a}\underset{p}{e}^q\overset{r}{e}_s. \end{aligned}$$

Applying (124) to C^ν_p will give:

$$(133)\qquad \begin{aligned} D_{[d}\overset{m}{H}{}^{\cdot\nu}_{b]\cdot r} &= -D_{[d}D_{b]}C^\nu_p \\ &= \overset{m}{H}{}^{\cdots q}_{[db]} D_{[d} C^\nu_p + \tfrac12 \overset{mm'}{K}{}^{\cdots r}_{dbp} C^\nu_r - \tfrac12 B^{\omega\mu}_{db}K^{\cdots\nu}_{\omega\mu\lambda}C^\lambda_p \end{aligned}$$

or

$$(134)\qquad B^{\omega\mu}_{db}C^\lambda_p K^{\cdots\nu}_{\omega\mu\lambda} = \overset{mm'}{K}{}^{\cdots\nu}_{dbp} + 2\overset{m}{H}{}^{\cdots q}_{[db]}\overset{m'}{H}{}^{\cdots\nu}_{qp} - 2D_{[d}\overset{m}{H}{}^{\cdot\nu}_{b]\cdot p}.$$

Contracting this with C^ν_r will imply the generalized Ricci equation:

$$(135)\qquad \boxed{B^{\omega\mu}_{db}C^{\lambda r}_{p\nu}K^{\cdots\nu}_{\omega\mu\lambda} = \overset{mm'}{K}{}^{\cdots\nu}_{dbp} + 2\overset{m}{H}{}^{\cdots r}_{[d|e}\overset{m'}{H}{}^{\cdot e}_{b]\cdot p},}$$

while contracting with B^c_ν will lead back to (130). Throttling the indices *p* and *r* will yield:

$$(136)\qquad B^{\omega\mu}_{db}\underset{p}{e}^\lambda\overset{r}{e}_\nu K^{\cdots\nu}_{\omega\mu\lambda} = \overset{mm'}{K}{}^{\cdots\nu}_{db\lambda}\underset{p}{e}^\lambda\overset{r}{e}_\nu + 2\overset{r}{h}_{[d|e|}\underset{p}{l}{}^{\cdot e}_{b]}.$$

However, since:

$$(137)\qquad \begin{aligned} \overset{mm'}{K}{}^{\cdots\nu}_{db\lambda}\overset{r}{e}_\nu\underset{p}{e}^\lambda &= 2D_{[d}\left\{\underset{|p|}{e}^q D_{b]}\overset{r}{e}_q\right\} - 2\left(D_{[d}\underset{p}{e}^q\right)\left(D_{b]}\overset{r}{e}_q\right) - 2\overset{s}{h}_{[db]}\underset{s}{e}^t\left(D_t\overset{r}{e}_q\right)\underset{p}{e}^q \\ &= 2D_{[d}\underset{p}{\overset{r}{v}}_{b]} - 2\underset{p}{\overset{q}{v}}_{[d}\underset{q}{\overset{r}{v}}_{b]} - 2\overset{s}{h}_{[db]}\underset{s}{\overset{r}{u}}_p, \end{aligned}$$

where:

(138) $$\overset{r}{\underset{sp}{u}} = \underset{s}{e}^{t}\underset{p}{e}^{q} D_t \overset{r}{\underset{q}{e}} \overset{*}{=} -\Gamma^{r}_{ps},$$

the generalized Ricci equation can be written in the form [cf., R.K., pp. 200, formula (170b)]:

(139) $$\boxed{B^{m\mu}_{db} K_{\omega\mu\lambda}^{\cdots\nu} \underset{p}{e}^{\lambda} \overset{r}{e}_{\nu} = 2D_{[d} \overset{r}{\underset{p}{v}}_{b]} - 2\overset{q}{\underset{p}{v}}_{[d} \overset{r}{\underset{q}{v}}_{b]} + 2\overset{s}{h}_{[db]} \overset{r}{\underset{sp}{u}} + 2\overset{s}{h}_{[d|e|} \underset{p}{l}^{\cdot e}_{b]}.}$$

Equations (130), (131), and (139) differ from the corresponding equations for V_m in V_n by the appearance of additional terms that contain $\overset{m}{H}{}_{db}^{\cdots r}$ ($\overset{s}{h}_{[db]}$, resp.). By their definitions, the curvature quantities $\overset{m}{K}$, $\overset{mm'}{K}$, $\overset{m'm}{K}$, and $\overset{m'}{K}$ satisfy the *first identity* (cf., R.K., pp. 87):

(140) $$\overset{m}{K}{}_{(db)a}^{\cdots\cdots c} = 0, \qquad \overset{mm'}{K}{}_{(db)p}^{\cdots\cdots r} = 0,$$
$$\overset{m'm}{K}{}_{(sq)a}^{\cdots\cdots c} = 0, \qquad \overset{m'}{K}{}_{(sq)p}^{\cdots\cdots r} = 0.$$

It follows from (101), (102), in a known way, that they also satisfy the *third identity* (cf., R.K., pp. 88):

(141) $$D_{[d} D_{b]} b_{ac} = \overset{m}{K}{}_{db(ac)} = 0, \qquad \overset{mm'}{K}{}_{db(pr)} = 0,$$
$$\overset{m'm}{K}{}_{sq(ac)} = 0, \qquad \overset{m'}{K}{}_{sq(pr)} = 0.$$

The curvature quantities $\overset{mm'}{K}$ and $\overset{m'm}{K}$ collectively satisfy a type of *fourth identity* (cf., R.K., pp. 89), which is obtained by comparing (135) with an analogous formula for $\overset{m'm}{K}$:

(142) $$\overset{mm'}{K}{}_{acpr} + 2\overset{m}{H}{}_{[a\cdot|r|}^{\cdot e}\overset{m}{H}{}_{c|ep} = \overset{m'm}{K}{}_{dbpac} + 2\overset{m'}{H}{}_{[p\cdot|c|}^{\cdot e}\overset{m'}{H}{}_{r|sa}.$$

If we compute $D_{[d} D_{b]} v_a$ in two different ways:

$$D_{[d} D_b v_{a]} = \overset{m}{H}{}_{[db}^{\cdots q} D_{|q|} v_{a]} + \tfrac{1}{2}\overset{m}{K}{}_{[dba]}^{\cdots\cdots c} v_c,$$

$$D_{[d} D_b v_{a]} = D_{[b} B^{\lambda\mu}_{ba]} D_{\lambda|} v_{\mu} = \overset{m}{H}{}_{[db}^{\cdots q} D_{[q|} v_{a]} - v_c \overset{m}{H}{}_{[db}^{\cdots q} \overset{m}{H}{}_{a]\cdot q}^{\cdot c}$$

then a *second identity* will arise (cf., R.K., pp. 88) for $\overset{m}{K}$:

(143) $$\overset{m}{K}{}_{[dba]}^{\cdots\cdots c} = -\overset{m}{H}{}_{[db}^{\cdots q}\overset{m}{H}{}_{a]\cdot q}^{\cdot c},$$

and one will also have one for $\overset{m}{K}{}^{*}$ then:

(144) $$\overset{m}{K}{}_{[dba]}^{*\cdots c} = -4\overset{m}{H}{}_{[db}^{\cdots q}H_{a]\cdot q}^{\cdot c}$$

That formula can also be derived from (127) directly.

One will obtain a relation between $\overset{m}{H}$ and $\overset{m'}{H}$ from (130) by alternating *d*, *b*, and *a* :

(145) $$\overset{m}{H}{}_{[db}^{\cdots q}\overset{m'}{H}{}_{|q|\cdot a]}^{\cdot r} + D_{[d}\overset{m}{H}{}_{ba]}^{\cdots r} = 0 .$$

The first, second, and third identity for $\overset{m}{K}$ imply the *fourth identity:*

(146) $$\overset{m}{K}{}_{dbac} - \overset{m}{K}{}_{acdb} = 12\overset{m}{H}{}_{[ac}^{\cdots q}\overset{m}{H}{}_{db]q},$$

and in another form:

(147) $$\overset{m}{K}{}_{dbac}^{*} - \overset{m}{K}{}_{acdb}^{*} = 12\overset{m}{H}{}_{[ac}^{\cdots q}\overset{m}{H}{}_{db]q} - 2\overset{m}{H}{}_{[db]}^{\cdots q}\overset{m}{H}{}_{acq} + 2\overset{m}{H}{}_{[ac]q}\overset{m}{H}{}_{db}^{\cdots q}$$

§ 7. – Curvature theory of a V_n^m in X_n .

We shall now consider an X_n^m that is a complement X_n and couple it with a V_n^m by introducing a fundamental tensor b_{ab} ([13]). The metric in V_n^m, *by itself, is not in a position to generate a translation*, but if *the choice of complement is also given* then the Γ_{ab}^c can be calculated from equations (109) and (110), and in that way a metric translation will be established in V_n^m. In general, the operator D_b will take on meaning only by applying it quantities in V_n^m:

(148) $$D_b\, v^c = \partial_b\, v^c + \Gamma_{ab}^c\, v^a ,$$

([13]) Cf., Vranceanu, C. R. **188** (1929), 973-975.

while the operators D_q and D_μ do not exist at all. Since there is no translation in X_n, $\overset{m}{H}{}_{ba}^{\cdot\cdot\cdot\nu}$ will not exist, but (107) says that the quantities $\overset{m}{H}{}_{[ba]}^{\cdot\cdot\cdot\nu}$ do exist, which we would rather write as $M_{ba}^{\cdot\cdot\cdot\nu}$ here, since $\overset{m}{H}{}_{bc}^{\cdot\cdot\cdot\nu}$ do not exist [cf., (108)], and which will depend upon the choice of complement, and not upon the b_{ab} :

(149) $$M_{ba}^{\cdot\cdot\cdot\nu} = \partial_{[b} B_{a]}^{\nu} - \Gamma_{[ab]}^{c} B_c^{\nu} = C_\lambda^{\nu} \partial_{[b} B_{a]}^{\nu} .$$

The operator D'_q, which depends upon only the choice of complement, will become meaningful:

(150) $$D'_q v^c = \partial_q v^c + 2v^a B_a^{\omega} C_q^{\mu} \partial_{[\omega} B_{\mu]}^{c} ,$$

but will lose the relationship to the $\overset{m}{H}{}_{ba}^{\cdot\cdot\cdot r}$ that is expressed in (118). Along with that operator, there exists D'_b :

(153) $$D'_b w^r = \partial_b w^r + 2w^p C_p^{\omega} B_b^{\mu} \partial_{[\omega} C_{\mu]}^{r},$$

and, as before, one has:

(152) $$D'_{[q} D'_{b]} p = 0 .$$

As in (119), differentiating a vector v^c in V_n^m twice in V_n^m and alternating will give:

(153) $$D_{[d} D_{b]} v^c = M_{db}^{\cdot\cdot\cdot q} D'_q v^c - \tfrac{1}{2} \overset{m}{K}{}_{dba}^{*\cdot\cdot\cdot c} v^a ,$$

in which $\overset{m}{K}{}_{dba}^{*\cdot\cdot\cdot c}$ is no longer given by (121), but by:

(154) $$\overset{m}{K}{}_{dba}^{*\cdot\cdot\cdot c} = 4M_{db}^{\cdot\cdot\cdot\mu} B_a^{\omega} \partial_{[\omega} B_{\mu]}^{c} - 2\partial_{[d} \Gamma_{|a|b]}^{c} - \Gamma_{e[d}^{c} \Gamma_{|a|b]}^{e} - \Gamma_{[db]}^{e} \Gamma_{ae}^{c} .$$

Naturally, a corresponding quantity $\overset{m'}{K}{}_{sqp}^{*\cdot\cdot\cdot r}$ does not exist here. By contrast, a curvature quantity can be defined by applying the operator $D_q^* D_b^* - D_b^* D_q^*$ to v^c, where the operator D^* has the following meaning:

(155) $$\begin{aligned} D_b^* v^c &= D_b v^c, \\ D_q^* v^c &= D'_q v^c , \end{aligned}$$

$$D^*_b w^r = D'_b w^r .$$

That will then yield:

(156) $$D^*_{[q} D^*_{b]} v^c = \overset{mm'}{M}{}_{qba}^{\cdots c} v^a ,$$

in which

(157) $$\overset{mm'}{M}{}_{qba}^{\cdots c} = \partial_q \Gamma^c_{ab} - 2\partial_b C^\mu_q B^\lambda_a \partial_{[\mu} B^c_{\lambda]}$$
$$- \partial_{[\mu} B^e_{\lambda]} \{2\Gamma^c_{ae} C^\mu_q B^\lambda_b + 2\Gamma^c_{eb} C^\mu_q B^\lambda_a - 2\Gamma^d_{ab} C^\mu_q B^{\lambda c}_{be} - 4B^\mu_b C^\lambda_q \partial_{[\mu} C^r_{\lambda]}\} .$$

From (118β), the quantities:

(158) $$D'_q b_{ab} = N_{abq}$$

will be equal to:

$$\overset{m}{H}{}_{(ab)q} ,$$

such that those quantities will also be independent of the translation in X_n. If $\partial_{[\omega} B^c_{\mu]} = 0$ then, first of all, the choice of complement will tell us about $X_{m'}$, and secondly, all vectors $\overset{c}{e}{}_\mu$ will tell us about X_{n-1}, from which it will follow that the basis vectors in V^m_n arise by intersecting V^m_n with the X_{n-1} of those vectors. According to (118), the operator D'_q will then go to the operator ∂_q, and the quantities $\overset{mm'}{M}{}_{qba}^{\cdots c}$ will go to $\partial_q \Gamma^c_{ab}$ ([14]).

§ 8. – Concluding remarks.

The range of applications of the *D*-symbolism is in no way exhausted by the exposition above. In fact, we have established that the *D*-symbolism can be of great use in the theory of deformation and the treatment of higher curvatures of a V_m in V_n .

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([14]) An analogous case comes about in, e.g., unitary geometry; cf., J. A. Schouten and D. v. Dantzig, "Unitäre Geometrie," Math. Ann. **103** (1930), 319-346.