# Hertz's mechanics and Einstein's theory of gravitation 

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If one uses Jacobi's prescription for eliminating time from the principle of least action:

$$
\begin{equation*}
\delta \int_{t_{0}}^{t_{1}} 2 T d t=0 \tag{1}
\end{equation*}
$$

by using the energy principle before performing the variation then one will find that for the simplest case in which a material point of mass 1 moves in a conservative force field with a potential energy of $V$ that:

$$
\begin{equation*}
T=\frac{1}{2}\left(\frac{d s}{d t}\right)^{2}, \quad T+V=\text { const. }=E, \quad d t=\frac{d s}{\sqrt{2(E-V)}} \tag{2}
\end{equation*}
$$

so

$$
\begin{equation*}
\delta \int_{P_{0}}^{P_{1}} \sqrt{2(E-V)} d s=0 \tag{3}
\end{equation*}
$$

which implies that:

$$
\begin{equation*}
\delta \int_{P_{0}}^{P_{1}} d s=0 \tag{4}
\end{equation*}
$$

for a vanishing force field. The symbolic integration limits in that integral mean the initial (final, resp.) point of the motion, which are not varied.
(3) [(4), resp.] is a mathematical consequence of (1) and (2) and the prescription that Jacobi explained. Therefore, one can consider (3) to be the expression for the law of motion for an arbitrary conservative system when one takes care that the relation (2.1) should also remain justified with a suitable definition of $d s$. Let such a system with $n$ degrees of freedom be established by the $n$ Lagrange coordinates $\left({ }^{1}\right)$ :

$$
q_{1}, q_{2}, \ldots, q_{n},
$$

and let its energy be established by $\left({ }^{2}\right)$ :

[^0]\[

$$
\begin{equation*}
2 T=g_{i k}\left(q_{1}, \ldots, q_{n}\right) \dot{q}_{i} \dot{q}_{k}, \quad V=V\left(q_{1}, \ldots, q_{n}\right), \tag{5}
\end{equation*}
$$

\]

so one has to set $\left({ }^{1}\right)$ :

$$
\begin{equation*}
d s^{2}=2 T d t^{2}=g_{i k} d q_{i} d q_{k} \tag{6}
\end{equation*}
$$

That is, one introduces a general Riemannian metric into coordinate space by convention: Line elements that run through the system points with equal kinetic energy in equal time are regarded as equal. The time required to make $T=1 / 2$ will serve as the unit of mass. (Naturally, that unit of mass is independent of the choice of time unit and coordinate system.)

The geometry of motion is then determined by (3) quite generally. The curves of paths are the extremals of the action integral in the non-Euclidian $R_{n}$ under consideration. In that way, it is not the metric, but probably the action integral, that is independent of the parameter $E=$ total energy. Thus, there are $\infty^{1}$ paths from an arbitrary point $P_{0}$ to an arbitrary point $P_{1}$ then, so there will be $\infty^{1+2 n-2}=\infty^{2 n-1}$ in all, as it must be. - The elapsed time between given endpoints can be regarded as a function of $E$ alone, and (2) implies that it is given by the integral:

$$
\begin{equation*}
t=\int_{P_{0}}^{P_{1}} \frac{d s}{\sqrt{2(E-V)}}, \tag{7}
\end{equation*}
$$

which extends along the path, and which I would like to call the "chronometric integral."
In the time differential $d t$, which is regarded as invariant under the choice of coordinates, the velocities $\dot{q}_{k}$ appear as the contravariant components of a vector whose covariant components are the quantities that are ordinarily referred to as impulses:

$$
\begin{equation*}
p_{i}=\frac{\partial T}{\partial \dot{q}_{i}}=g_{i k} \dot{q}_{k} \tag{8}
\end{equation*}
$$

Its invariant $2 T$ takes the three known forms:

$$
\begin{equation*}
2 T=g_{i k} \dot{q}_{i} \dot{q}_{k}=p_{k} \dot{q}_{k}=g^{i k} p_{i} p_{k}, \tag{9}
\end{equation*}
$$

in which the contravariant components of the fundamental metric tensor are constructed from the minors of that metric, divided by $\sum \pm g_{i k}$, in the well-known way.

For purely geometric arguments in which the time is left out of consideration, in order to characterize the direction of the line element $d s$ with the components $d q_{i}$, it is preferable to introduce the direction vector $R$, whose contravariant components are:

[^1]\[

$$
\begin{equation*}
R^{i}=\frac{d q_{i}}{d s} \quad\left(=\frac{\dot{q}_{i}}{\sqrt{2 T}}\right) \tag{10}
\end{equation*}
$$

\]

and whose covariant ones are:

$$
\begin{equation*}
R_{i}=g_{i k} \frac{d q_{k}}{d s} \quad\left(=\frac{p_{i}}{\sqrt{2 T}}\right) \tag{11}
\end{equation*}
$$

In those equations, the expressions on the left are meaningful in all situations, while the ones that are enclosed in brackets assume that the line element actually moves with some velocity, which can, of course, be supposed for any line element (in holonomic systems!).

From (6), the invariant of the direction vector is equal to 1 :

$$
\begin{equation*}
g_{i k} R^{i} R^{k}=g^{i k} R_{i} R_{k}=R^{i} R_{i}=1, \tag{12}
\end{equation*}
$$

which corresponds to the known relationships between the direction coefficients in Euclidian $R_{3}$.
One regards the bilinear invariant of two direction vectors $R$ and $S$ that relate to the same point $P$ in $q$-space as the cosine of the angle between any line elements that include that point and agree with the directions in question. More briefly: the angle between those two directions or any vectors that relate to that point and have the relevant directions:

$$
\begin{equation*}
\cos (R, S)=g_{i k} R^{i} S^{k}=R^{i} S_{i}=R_{i} S^{i}=g^{i k} R_{i} S_{k} . \tag{13}
\end{equation*}
$$

Now, one must keep in mind that it is absurd to speak of the angle between directions that relate to different points in $q$-space. Thanks to the fact that our fundamental metric form is positive definite (which is geometrized by the essentially positive character of kinetic energy), the definition (13) always proves to be $\leq 1$, provided that $R$ and $S$ are actual direction vectors, and as such (12) must have the invariant 1 . The angle that was defined will then admit a real determination $\leq \pi$. Moreover, it has the all of the proposed elementary geometric meaning, and it is constructible in the Euclidian plane. One draws two line elements from $P$ with the directions in question (with arbitrary, but infinitely small lengths), denotes their lengths by $\delta^{\prime} s$ and $\delta^{\prime \prime} s$ according to (6), and constructs a planar triangle from them, along with the length of the line element that connects their endpoints $\delta^{\prime \prime \prime} s$ that was found before. The desired angle is opposite to $\delta^{\prime \prime \prime} s$. In fact, from (6), one has:

$$
\begin{gather*}
\delta^{\prime} s^{2}=g_{i k} \delta^{\prime} q^{i} \delta^{\prime} q^{k} \\
\delta^{\prime \prime} s^{2}=g_{i k} \delta^{\prime \prime} q^{i} \delta^{\prime \prime} q^{k} \tag{14}
\end{gather*}
$$

and up to higher-order infinitesimals, one has:

$$
\begin{equation*}
\delta^{\prime \prime \prime} s^{2}=g_{i k}\left(\delta^{\prime} q^{i}-\delta^{\prime \prime} q^{i}\right)\left(\delta^{\prime} q^{k}-\delta^{\prime \prime} q^{k}\right)=\delta^{\prime} s^{2}+\delta^{\prime \prime} s^{2}-2 \delta^{\prime} s \delta^{\prime \prime} s \cdot g_{i k} \frac{\delta^{\prime} q_{i}}{\delta^{\prime} s} \frac{\delta^{\prime \prime} q_{k}}{\delta^{\prime \prime} s}, \tag{15}
\end{equation*}
$$

which represents the cosine law in the Euclidian plane for the triangle that we speak of.
For the time being, let us call the contravariant direction components $R^{i}$ themselves (or even the covariant ones $R_{i}$ ) of the system of cosines that the direction $R$ defines the "coordinate $n$ hedron" at each point, which is distinguished by the choice of coordinates, and along whose edges one of the coordinates $q_{i}$ experiences an increase, or also in the other one, namely, the "impulse $n$ hedron," along whose edges the system impulse increases under motion when one assigns a nonzero mass to one of the impulses $p_{i}$. None of these apply. (?) However, let $K$ be the direction of the first edge of the coordinate $n$-hedron then, so only $K_{1}$ is non-zero (while all other $K^{i}=0$ ), and (12) [(13), resp.] will then imply that:

$$
\begin{equation*}
g_{11}\left(K^{1}\right)^{2}=1, \quad \cos (R, K)=R_{1} K^{1}=\frac{R_{1}}{\sqrt{g_{11}}} \tag{16}
\end{equation*}
$$

while similar things are true for the remaining $n-1$ edges. - Things are different when $J$ is the direction vector of the first edge of the impulse $n$-edge, so only $J_{1}$ will be non-zero (and all other $J_{i}=0$ ). One will then have:

$$
\begin{equation*}
g^{11}\left(J_{1}\right)^{2}=1, \quad \cos (R, J)=R^{1} J_{1}=\frac{R_{1}}{\sqrt{g^{11}}} \tag{17}
\end{equation*}
$$

as well as similar statements for the remaining edges of this $n$-hedron $\left({ }^{1}\right)$.
If $\cos (R, S)$ vanishes then the two directions will be mutually perpendicular, in the entirely elementary geometric sense, and the auxiliary triangle that was described above will be a right triangle. By the way, that relationship always exists between an arbitrary edge direction of one of the two aforementioned $n$-hedrons and each edge of the other $n$-hedron (with the single exception of the corresponding one). - In order to have perpendicularity, it obviously suffices that one of the expressions (13) should vanish for a system of quantities that are proportional to the system of components considered. In particular, we can always replace the contravariant components in it with the components $d q_{i}$ of a line element in the direction considered.

We now consider just one hypersurface in our $q$-space that is given by setting a function of the $n$ coordinates, which is thought of as an invariant, equal to a constant:

$$
\begin{equation*}
f\left(q_{1}, \ldots, q_{n}\right)=\text { const. } \tag{18}
\end{equation*}
$$

That is: We consider the totality of all system locations whose coordinates satisfy that condition, whereby in the case of a change of coordinates, the condition should be replaced with another one, in such a way that once more the same physical system positions will be selected (naturally, that will happen simply by "recalculating" $f$ in the new coordinates). Select a point $P$ (= a system location) on that surface and consider all line elements (i.e., infinitely-small displacements) that

[^2]start from it and lie in the surface (i.e., lead to system locations on the same distinguished set). The components $d q_{i}$ of each of them will then satisfy the relation:
\[

$$
\begin{equation*}
\frac{\partial f}{\partial q_{i}} d q_{i}=0 \tag{19}
\end{equation*}
$$

\]

so they will be perpendicular to a direction whose covariant components are proportional to $\partial f$ / $\partial q_{i}$. Conversely, it is known that the ratios of the $\partial f / \partial q_{i}$ are determined uniquely by the totality of these relations. One will refer to the direction thus-defined as the direction of the surface normal at $P$. The associated direction vector itself (viz., its covariant components) is obtained from the partial derivatives by multiplying them by a suitable factor:

$$
\begin{equation*}
N_{i}=\frac{\frac{\partial f}{\partial q_{i}}}{\sqrt{g^{k l} \frac{\partial f}{\partial q_{k}} \frac{\partial f}{\partial q_{l}}}} \tag{20}
\end{equation*}
$$

so that (12) will be satisfied:

$$
g^{i k} N_{i} N_{k}=1 .
$$

The $\partial f / \partial q_{i}$ themselves define the covariant components of a vector at $P$, namely, the gradient of $f$. It is not hard to show that the square root (as its invariant, by which we must divide, in order to obtain the direction components) yields the magnitude at $P$ of the steepest ascent of $f$ per unit of length of the line element that was found before, which admits its steepest increase in the direction $N$ [more precisely, in that one of the two opposite directions that fulfill (20) for a positive sign on the square root].

One now arrives at the Hamilton-Jacobi picture from the simple concept of the gradient in the most natural way (but in general only as it applies to conservative, holonomic systems) by the following convention:

If I regard the constant in (18) as a variable parameter then the equation will represent a family of surfaces:

$$
\begin{equation*}
f\left(q_{1}, \ldots, q_{n}\right)=C, \tag{18}
\end{equation*}
$$

so the direction of steepest ascent in $f$ at each point of $q$-space will then be given by an orthogonal trajectory of that family. Its magnitude will be a certain positive function - say $\Psi\left(q_{1}, \ldots, q_{n}\right)$ :

$$
\begin{equation*}
g^{i k} \frac{\partial f}{\partial q_{i}} \frac{\partial f}{\partial q_{k}}=\Psi^{2}\left(q_{1}, \ldots, q_{n}\right) . \tag{21}
\end{equation*}
$$

The increase in $f$ along the element $d s$ of such a trajectory is $\Psi d s$. That product has the same value for all trajectory elements between the neighboring surfaces:

$$
f=C, \quad f=C+d C,
$$

namely, $d C$, and that value is smaller than the product $\Psi d s^{\prime}$, when $d s^{\prime}$ is any line element that connects the aforementioned surface non-orthogonally, and thus obliquely. One simply has that $d s^{\prime}$ is greater than the perpendicular surface distance $d s$ at the same zero (that follows from the construction of the elementary triangle that was given above). Therefore, the spatial integral:

$$
\begin{equation*}
\int_{\left(C_{0}\right)}^{\left(C_{1}\right)} \varphi d s=C_{1}-C_{0} \tag{22}
\end{equation*}
$$

will have the same value on all trajectory arcs that connect the surfaces $C_{0}$ and $C_{1}$, and a smaller value than it would have on any other arc that extends between the two surfaces. In particular, it will have a smaller value on the trajectory arc $P_{0} P_{1}$ than it has on any other arc that extends between those two points.

However, that means that the orthogonal trajectories define an extremal field of the spatial integral that we speak of. From (3), there will then be mechanical paths when one says, in particular, that:

$$
\Psi=\sqrt{2(E-V)}
$$

i.e., from (21), when one demands of the function $f$ that:

$$
\begin{equation*}
g^{i k} \frac{\partial f}{\partial q_{i}} \frac{\partial f}{\partial q_{k}}=2(E-V) . \tag{sic}
\end{equation*}
$$

As one sees from (9), that is nothing but Hamilton's partial differential equation [multiplied by (2)] in the form that one cares to give it for conservative systems; i.e., when $V$ does not depend upon time explicitly.

One fulfills the differential equations of the paths that are found - qua trajectories - by setting their direction vectors equal to the altitudes, but in covariant form:

$$
g^{i k} \frac{d q_{k}}{d s}=\frac{\frac{\partial f}{\partial q_{i}}}{\sqrt{g^{l m} \frac{\partial f}{\partial q_{l}} \frac{\partial f}{\partial q_{m}}}}
$$

or from (24):

$$
\begin{equation*}
\sqrt{2(E-V)} g^{i k} \frac{d q_{k}}{d s}=\frac{\partial f}{\partial q_{i}} . \tag{25}
\end{equation*}
$$

That is the first law of Jacobi's relations. In fact, from (2.3) and (8), one will fulfill it simply with:

$$
\begin{equation*}
p_{i}=\frac{\partial f}{\partial q_{i}}, \tag{26}
\end{equation*}
$$

which is valid for motion along these trajectories. The second law of Jacobi's relations differs from the one that was obtained above only insofar as it represents the equations of the same trajectories in integrated form when (25) are the differential equations.

If we can do that then we would like to direct our attention to the geometrically intuitive way of generating surfaces of the type discussed. Any arbitrarily-given hypersurface:

$$
\begin{equation*}
\varphi=0 \tag{27}
\end{equation*}
$$

can be extended to such a family; at the same time, this implies that every mechanically possible path can be regarded as an intermediate term in such a field of trajectories (and indeed in an infinitude of ways).

Analytically that is not immediately clear, since an arbitrary function $f$ will not satisfy the partial differential equation at all, but it is geometrically obvious.

It is next clear that when the problem of "extension" is soluble at all, it will be geometrically unique. That is, when the given surface $f=0$ can be regarded as an intermediate term in a family:

$$
\begin{equation*}
f=C \tag{28}
\end{equation*}
$$

(where $f$ should satisfy the partial differential equation), that family can be geometrically established uniquely by the given surface and can be constructed in the following way:

The equation indeed requires that the gradient of $f$ should be a given function of position, namely, $\sqrt{2(E-V)}$. If we then proceed along all surface normals to $f=0$ by the infinitely-small increment:

$$
\begin{equation*}
\delta s=\frac{\varepsilon}{\sqrt{2(E-V)}} \tag{29}
\end{equation*}
$$

(where $\varepsilon$ is an infinitely-small constant) then the set of all points to which we will arrive will again define a surface of the family (28), namely, the surface:

$$
f=C_{0}+\varepsilon,
$$

in which:

$$
f=C_{0}
$$

represents the given surface $f=0$.
If one performs that process of constructing altitudes and connecting the endpoints to a new surface sufficiently often (and indeed on the two "edges" of the $f$-surface) then one can now derive a completely-determined family of surfaces from it in any case that must be represented in the form:

$$
\begin{equation*}
f^{*}=C^{*} \tag{30}
\end{equation*}
$$

and which must represent the unique solution when our problem admits a solution at all.
However, one easily sees that this is the case. One needs only to put (30) into a form such that the value of the parameter $C^{*}$ will be most characteristic of the system of "layers of an onion" that was constructed. On one of the orthogonal trajectories of (30), let:

$$
\sqrt{2(E-V)} \cdot \delta s=g\left(C^{*}\right) \delta C^{*}
$$

The same function $g\left(C^{*}\right)$ will then once more give the correct connection between the perpendicular distance $\delta s$ to the neighboring surface and the parameter variation $\delta C^{*}$, and on all other trajectories, as well. By construction [cf., (29)], the function that is given by the product on the left-hand side is, in fact, constant on each surface. Now, if the undetermined integral is:

$$
\int^{C^{*}} g\left(C^{*}\right) d C^{*}=h\left(C^{*}\right)
$$

then:

$$
h\left(f^{*}\right)=h\left(C^{*}\right)
$$

will represent the same family of surfaces that we defined in (30), etc., in the desired form, since now, by proceeding in the direction of the family of surfaces in all of space, one will have:

$$
\frac{\delta h\left(C^{*}\right)}{\delta s}=\sqrt{2(E-V)},
$$

and thus:

$$
\frac{\delta h\left(f^{*}\right)}{\delta s}=\sqrt{2(E-V)},
$$

and that is all that equation (24) requires.

We shall now turn to the intuitive proof of the second part of Jacobi's theorem.
Suppose that $f=C$ is a family of surfaces whose geometric character and manner of representation satisfy the conditions. Furthermore, let:

$$
\begin{equation*}
\eta\left(q_{1}, \ldots, q_{n}\right)=\text { const. }=\beta \tag{25}
\end{equation*}
$$

be a family of surfaces, each of which consists of nothing but orthogonal trajectories of the first family. For sufficiently small $\varepsilon$ :

$$
\begin{equation*}
f+\varepsilon \eta=\text { const. } \tag{26}
\end{equation*}
$$

will then be a family with the same character as the former one, namely, that $f+\varepsilon \eta$ satisfies the partial differential equation up to quantities of order $\varepsilon$, inclusive. In fact, when it is substituted in the equation (we now confine ourselves to a first glance), the terms that are linear in $\varepsilon$ will, in fact, differ as a result of the orthogonality of the two families $f$ and $\eta$. However, the opposite will also
take place then. Had one found a family (26) that likewise satisfied the equation for sufficiently small $\varepsilon$, then at each of their points each of the surfaces (25) would include the altitude to the $f$ surface through that point, which would then consist of enough orthogonal trajectories of the $f$ family.


[^0]:    $\left({ }^{1}\right)$ We therefore assume that it is holonomic!
    $\left({ }^{2}\right)$ General indices that appear twice are summed over (using Einstein's procedure)!

[^1]:    $\left({ }^{1}\right)$ See, however, H. von Helmholtz, "Zur Geschichte des Princips der kleinsten Action," Sitzber. d. Berl. Akad. d. Wiss. (1887), pp. 236.

[^2]:    $\left({ }^{1}\right)$ On the other hand, this representation should not mislead one to believe that one should not distinguish between the "covariant" and "contravariant" direction of a line element! $K$ and $J$ are totally different directions and are distinguished only by a fortuitous choice of coordinates, moreover!

