"Sur l'intégration de l'équation différentielle de Hamilton," C. R. Acad. Sci. Paris **121** (1895), 489-492.

On the integration of Hamilton's differential equation

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In a note that was inserted in *Comptes rendus* on 9 March 1893, I indicated a class of problems in dynamics whose differential equations can be solved by quadratures. One will arrive at those problems by the following consideration, which admits a remarkable generalization:

If one is given a *Hamilton differential equation:*

$$\tag{H} \tfrac{1}{2}\sum_{k=1}^{n} A_k \left(\frac{\partial W}{\partial q_k}\right)^2 = \Pi + \alpha_1$$

then one must determine a *complete solution:*

$$W = V(q_1, \ldots, q_n ; \alpha_1, \ldots, \alpha_n) .$$

In order to find equations (H) for which one can perform the integration, make that hypothesis that the quantities:

$$\left(\frac{\partial W}{\partial q_k}\right)^2 \qquad (k = 1, 2, \ldots, n)$$

are linear functions of the arbitrary constants $\alpha_1, \ldots, \alpha_n$:

$$\tag{H_k} \tfrac{1}{2}\left(\frac{\partial W}{\partial q_k}\right)^2 = \varphi_{k0} + \varphi_{k1}\,\alpha_1 + \ldots + \varphi_{kn}\,\alpha_n .$$

In order for the integrability conditions to be fulfilled, it is necessary and sufficient that the functions:

$$\varphi_{k0} , \varphi_{k1} , \ldots, \varphi_{kn}$$

should depend upon only the variable q_k .

Let $\Phi_{k\lambda}$ be the determinant that is adjoint to $\varphi_{k\lambda}$ with respect to the system of n^2 quantities:

$$\begin{matrix} \varphi_{11}, & \varphi_{12}, & \cdots & \varphi_{1n}, \\ \varphi_{21}, & \varphi_{22}, & \cdots & \varphi_{2n}, \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}, & \varphi_{n2}, & \cdots & \varphi_{nn}, \end{matrix}$$

and

$$\sum_{k=1}^{n} \varphi_{k\lambda}\Phi_{k\lambda} = \Phi , \qquad \sum_{k=1}^{n} \varphi_{k0}\Phi_{k\lambda} = \Psi_\lambda .$$

The equation:

$$(H^*) \qquad \tfrac{1}{2} \sum_{k=1}^{n} \frac{\Phi_{k1}}{\Phi} \left(\frac{\partial W}{\partial q_k} \right)^2 = \frac{\Psi_1}{\Phi} + \alpha_1$$

admits the complete solution:

$$W = \sum_{k=1}^{n} \int \sqrt{2\,\varphi_{k0} + 2\,\varphi_{k1}\,\alpha_1 + \cdots + 2\,\varphi_{kn}\,\alpha_n}\, dq_k \; .$$

The problem in dynamics that corresponds to equation (H^*) is characterized by the *vis viva* equation:

$$T \equiv \tfrac{1}{2} \sum_{k=1}^{n} \frac{\Phi}{\Phi_{k1}} q_k'^{\,2} = \frac{\Psi_1}{\Phi} + \alpha_1 \; .$$

Upon calculating the constants $\alpha_2, \ldots, \alpha_n$ and appealing to the equations:

$$\frac{\partial W}{\partial q_k} = \frac{\partial T}{\partial q_k'} = \frac{\Phi}{\Phi_{k1}} q_k' ,$$

one will again get the other $n – 1$ integrals, which are quadratic in $q_1', q_2', \ldots, q_n'$:

$$\tfrac{1}{2} \sum_{k=1}^{n} \frac{\Phi\,\Phi_{k\lambda}}{\Phi_{k1}^2} q_k'^{\,2} = \frac{\Psi_\lambda}{\Phi} + \alpha_\lambda \qquad (\lambda = 2, 3, \ldots, n).$$

It was by reflecting on such things that I was led to generalize my theorem of 9 March 1893. I introduced the n systems of variables:

$$\begin{matrix} q_{11}, & q_{12}, & \cdots & q_{1h_1}, \\ q_{21}, & q_{22}, & \cdots & q_{2h_2}, \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}, & q_{n2}, & \cdots & q_{nh_n}, \end{matrix}$$

in place of the n variables $q_1, \ldots, q_n$ whose total number is:

$$r = h_1 + h_2 + \ldots + h_n .$$

I then suppose that one can integrate the n equations:

$$(G_k) \qquad \tfrac{1}{2}\sum_{\rho_k=1}^{h_k}\sum_{\sigma_k=1}^{h_k} A_{k\rho_k\sigma_k}\frac{\partial W}{\partial q_{k\rho_k}}\frac{\partial W}{\partial q_{k\sigma_k}} = \varphi_{k0} + \varphi_{k1}\,\alpha_1 + \cdots + \varphi_{kn}\,\alpha_n ,$$

in which all of the quantities whose first index is k depend upon only the variables whose first index is k.

Let:

$$V(q_{k1}, \ldots, q_{kh_k} ; \alpha_1, \ldots, \alpha_n ; \beta_{k1}, \ldots, \beta_{k,h_k-1})$$

be the complete solution to equation (G_k). If the integrability conditions have been fulfilled then the expression:

$$W = V_1 + V_2 + \ldots + V_n$$

will be a complete solution to the equation:

$$(G^*) \qquad \tfrac{1}{2}\sum_{k=1}^{n}\frac{\Phi_{k1}}{\Phi}\sum_{\rho_k=1}^{h_k}\sum_{\sigma_k=1}^{h_k} A_{k\rho_k\sigma_k}\frac{\partial W}{\partial q_{k\rho_k}}\frac{\partial W}{\partial q_{k\sigma_k}} = \frac{\Psi_1}{\Phi} + \alpha_1 .$$

The problem in dynamics that corresponds to equation (G^*) is characterized by the *vis viva* equation:

$$T \equiv \tfrac{1}{2}\sum_{k=1}^{n}\frac{\Phi}{\Phi_{k1}}\sum_{\rho_k=1}^{h_k}\sum_{\sigma_k=1}^{h_k} B_{k\rho_k\sigma_k}\,q'_{k\rho_k}\,q'_{k\sigma_k} = \frac{\Psi_1}{\Phi} + \alpha_1 .$$

The system of h_k^2 quantities $B_{k\rho_k\sigma_k} = B_{k\sigma_k\rho_k}$ ($\rho_k, \sigma_k = 1, 2, \ldots, h_k$) is reciprocal to the system of h_k^2 quantities $A_{k\rho_k\sigma_k} = A_{k\sigma_k\rho_k}$. Upon calculating the constants $\alpha_1, \ldots, \alpha_n$ and appealing to the equations:

$$\frac{\partial W}{\partial q_{k\rho_k}} = \frac{\partial W}{\partial q'_{k\rho_k}} = \frac{\Phi}{\Phi_{k1}}\sum_{\sigma_k=1}^{h_k} B_{k\rho_k\sigma_k}\,q'_{k\sigma_k} ,$$

one will again get $n - 1$ other quadratic integrals:

$$\tfrac{1}{2}\sum_{k=1}^{n}\frac{\Phi\,\Phi_{k\lambda}}{\Phi_{k1}^2}\sum_{\rho_k=1}^{h_k}\sum_{\sigma_k=1}^{h_k} B_{k\rho_k\sigma_k}\,q'_{k\rho_k}\,q'_{k\sigma_k} = \frac{\Psi_\lambda}{\Phi} + \alpha_\lambda .$$

That is the true generalization of the known theorem of Liouville, which is a generalization that will permit one to utilize all of the progress in the integration of Hamilton's equation in order to find some new types of integrable equations, or in other words, to form some new linear elements whose geodesic lines one can determine.