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On Hamilton's geometrical optics and its relationship to the theory of contact transformations.

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Herr **F. Klein** has already referred to the great significance of **Hamilton**'s papers on theoretical optics some time ago ([1]). The following exposition is intended to expand upon some of the things that he said, but naturally without exhausting the topic.

In **Hamilton**'s first paper, which was presented to the Irish Academy in the year 1824, but only appeared three years later in an extended version, we find a clean break being made between the actual theory of optical instruments and theory of everything that is common to maps that are mediated by such things. That requirement, which Herr **Bruns** has only recently introduced in his paper on the *Eikonal* (Leipzig, 1892), has already been imposed and realized ([2]), and indeed in a masterful way, by **Hamilton** ([3]). Therefore, an essential part of optics is a branch of line geometry, whose founders undoubtedly include **Hamilton** in a preeminent place.

From **Malus**'s theorem, for homogeneous, isotropic media, the relationship between the light paths in object space and image space, independently of the nature of the dioptric systems that are employed, is such that normal congruences of families of parallel surfaces will always be associated with such things again. That gave rise to **Hamilton**'s investigation of those normal congruences and their *focal surfaces,* so to arbitrary central surfaces. At the same time, he had extended that investigation to line congruences that were not normal and of the type that optics

([1]) Jahresbericht der Deutschen Mathematiker-Vereinigung **I** (1892), pp. 35.

([2]) Perhaps, in part, by **Malus** before, whose *Traité d'optique*, which was available for only a short time, was not unfortunately provided to the author. See the following remark.

([3]) The Preface is interesting, in which the nineteen-year-old researcher introduced his (unpublished) treatise "On Caustics" by saying:

"The Problems of Optics considered mathematically relate for the most part to the intersection of the rays of light proceeding from known surfaces according to known laws.

In the present paper, it is proposed to investigate some general properties common to all such Systems of Rays, and independent of the particular surface or particular law. It is intended in another paper to point out the application of these mathematical principles to the actual laws of Nature.

A fortnight ago, I believed that no writer had ever treated of Optics on a similar plan. But within that period, my tutor, the Rev. Mr. **Boynton**, …, has shown me … a beautiful memoir of **Malus** on the subject entitled "Traité d'Optique" and presented it to the Institute in 1807.

Those who may take the trouble to compare his memoir with mine will perceive a difference in method and extent.

With respect to those results which are common to both, it is proper to state that I arrived at them in my own researches before I was aware of the existence of his."

likewise presents under the assumption of anisotropic media, and indeed more generally ([1]). **E. Kummer** treated that same subject thirty-three years later in his celebrated treatise "Allgemeine Theorie der geradlinigen Strahlsysteme," Crelles Journal **57** (1860). However, while **Kummer** did full justice to his predecessor, later authors left **Hamilton**'s work unnoticed. Thus, adequate citations are absent in the splendid work of **L. Bianchi** (*Differentialgeometrie*, Leipzig, 1899), and even the names of **Malus** and **Hamilton** are missing from the index of literature used.

Another generalization that **Hamilton** had likewise implemented from the outset related to light motion in media of continuously-increasing optical density: viz., the problem of atmospheric ray refraction. With a different terminology, he treated a theory of geodetic lines in a manifold that is defined by a quadratic differential form and could be mapped conformally to ordinary space. One can consider the latter restriction to be inessential, just like the restriction to three dimensions in its own right since **Hamilton**'s method can be extended to the most general case of that genre.

It is even more recent that the general theory of geodetic lines has been treated from the standpoint that is related most closely to that of **Hamilton**. However, the name of **Hamilton** is missing from that treatment (**Bianchi**, *loc. cit.*, pp. 570), or one finds him cited in such a way that the historical facts cannot be inferred from it (**Darboux**, *Théorie de surfaces*, II, pp. 480), as a result of the obvious limited knowledge that the author had of the papers of that Irish mathematician. Similar remarks can even be made in regard to **Hamilton**'s treatises on mechanics. Thus, **P. Stäckel** (Bolyai-Festschrift, Klausenburg, 1902, pp. 66) referred back to **Jacobi** for information about the connection between problems in mechanics and the problem of geodetic lines. However, **Hamilton** had made just that remark and then later extended his investigations, which were originally restricted to the realm of optics, to the more-encompassing matters in mechanics.

We then come to the actual gist of **Hamilton**'s presentation and to perhaps his most original achievement. It would seem that it consists of the fact that a single function is placed at the center of attention in all of the problems that will be spoken of here: viz., **Hamilton**'s *characteristic function*. In problems of mechanics, that means the time that a material system (which can be represented by a point in a higher manifold) needs in order to go from one place to another. In problems of optics, it is likewise the time that emitted light needs in order to go from one point to another. That function is coupled by certain partial differential equations, so it cannot generally be found. However, if one assumes that it has been found then the more precise description of the course of motion or of dioptric devices will be achieved by only differentiations and eliminations. The profundity of that idea for theoretical mechanics is well-known. However, it does not generally seem to be known that **Hamilton** had anticipated an important component of a great modern theory with that fact: *Hamilton's characteristic function is not only the generating function of a family of* ∞^1 *contact transformations, but Hamilton himself also treated it in that way.*

In perhaps the example of ordinary dioptrics (i.e., for monochromatic light and refraction only in homogeneous, isotropic media with sufficiently-extended separation surfaces), one fixes the time, and thus the value of the characteristic function. If one then lets one of the two points considered in object space and image space vary then (except for some special cases) that will

([1]) Cf., **Levi-Civita**, "Complementi al teorema di Malus-Dupin," Rend. Acc. Linc. IX (1900), pp. 185, 327.

generate a wave surface each time ([1]). If one restricts the variability of one or the other point to a region whose extension one can allow to converge to zero then two surface elements will arise – *wave elements*, as we would like to say – which both belong to mutually-associated wave surfaces. However, the association of the element in object space with the element in image space is a contact transformation. Within certain limits, the optical system can then emerge from any wave element in object space that goes to a certain wave element in image space according to a certain time evolution and also from wave elements in united position. If one then varies the time interval then one will get ∞^1 contact transformations and a given wave element in object space will produce ∞^1 wave elements in image space with a common normal. The two wave elements that are associated with each other will be known when their normals are known, namely, the light path that associates them. However, the directions of those normals are obtained from the characteristic function by an elementary differentiation, which **Hamilton** naturally also performed ([2]).

The difference between **Hamilton**'s theory and the treatment that **S. Lie** would have lent to the same matters would then seem lie in their differing starting points (the variational problem for **Hamilton**, the concept of transformation for **Lie**), but otherwise it would only lie in the fact that **Hamilton**'s wave surface was still not described *explicitly* as a "union of elements." By contrast, the analytical apparatus was developed sufficiently, and he handled it in precisely the same way that one does in the modern theory of contact transformations ([3]). *Certainly, Hamilton then performed a not-inessential service in the name of exhausting the concept of a contact transformation.* However, the rich and multifaceted content of **Hamilton**'s papers to also able to offer many sources of inspiration today.

The fact that these completely obvious remarks have escaped the founders of the theory of contact transformations is clearly based upon the fact that **Hamilton**'s work was known to **Lie** only second-hand. However, we have also not been able to find any reference to that state of affairs, which is historically interesting and pedagogically worthwhile in all of its simplicity.

However, **S. Lie** made some suggestions about the presence of the further group-theoretic concepts into optics. Namely, he repeatedly referred to the fact that the concept of dilatation appeared in **Huygens**'s principle. Refractions belong to the contact transformations that leave the group of dilatations fixed, reflections, to the ones that commute with all dilatations ([4]). Naturally, that in itself does not represent a significant advance, but only a translation of long-known facts

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([1]) In all of the questions that are considered here, it is irrelevant whether one can actually implement the relationship between object space and image space with an optical system. If a physical realization of the map is not possible then one can nonetheless express it as if that were the case.

([2]) The state of affairs that is described in the text here was probably basically also referred to by Herr **Bruns**, when he called his eikonal the generating function of a contact transformation, which he equated with the association defined by the light path. (Pages 328 and 361 of the aforementioned paper.)

The association defined by the light path is not a contact transformation. However, one can also extend **Bruns**'s formulas to such a thing (in ∞^1 ways) by adding a new pair of variables. [Cf., **F. Haussdorff**, "Infinitesimale Abbildungen der Optik," Leipziger Berichte (1896), pp. 79, *et seq.*] The additional variables have geometric meaning that is similar to the variables Ω, Ω' that will be introduced later on, but less simple.

([3]) Namely, one should look at the formulas in the third supplement to "Systems of Rays," Trans. Jr. Ac. XVII (1837), esp., pp. 11. The presentation there also approximates the one that will be presented later on to some extent.

([4]) "Die infinitesimalen Berührungstransformationen der Optik," Leipziger Ber. (1896), pp. 130. Cf., also Math. Ann., Bd. 59, pp. 209. Moreover, the latter statement is based upon a definition (which was missing in **Lie**) that includes a certain arbitrariness. One would also have the same right to say that reflections invert all dilatations.

into a different language. Nonetheless, one can then say that the general methods of optics that are developed in the theory of contact transformations independently of optics can still be applied to optics.

One might guess how **Lie** himself had imagined that application from the fact that he referred to:

$$\sqrt{p_1^2 + p_2^2 + p_3^2}$$

as the symbol of an infinitesimal dilatation precisely in connection with optics. It emerged from this that there, as on other occasions, as well, he regarded the dilatations as (2-2)-valued contact transformations. Nonetheless, one can explain the concept of a dilatation in a simpler way. A correct implementation of an analytical theory that is based upon the foundations that **Lie** referred to would give rise to unnecessary complications.

Therefore, one would first have to prove that a free application of **Lie**'s methods ([1]) (but not also of the special conceptual and analytical apparatus that he preferred) would actually achieve a certain simplification of **Hamilton**'s theory, at least in the case of ordinary dioptrics. We would like to try to do that in the second part of our publication.

The ideas that are at the basis for those developments can be described briefly as follows: **Lie**'s concept of element will be replaced with another one, namely, the concept of an *oriented* element. It will be represented by suitable *coordinates*, and indeed in such a way that it will subsume the set of all oriented elements ([2]). Dilatations are then *single-valued* transformations of oriented elements. The so-called unions will be classified by the way that they distribute their elements on *planes*. Finally, the contact transformations of (ordinary) dioptrics will be generated, not by functions with two sequences of point-coordinates, but by ones with two sequences of plane-coordinates.

As we would like to emphasize immediately, those functions do *not* have a simple physical meaning, unlike **Hamilton**'s function. However, they do not have less significance than **Hamilton**'s function in regard to the treatment of general theoretical questions insofar as they also do not contain a higher degree of arbitrariness than one finds with the use of rectangular Cartesian coordinates. Analytically, they are easier to deal with: Namely, they are not coupled by partial differential equations. Their general form can then be given with no further analysis.

The direction of a light ray can be specified clearly only when one has assigned a certain so-called positive *direction* to the line that defines the light path beforehand. That will happen when

([1]) Cf., in particular, **Lie** and **Engel**, *Transformationsgruppen II* (Leipzig, 1890, pp. 125-156) and Math. Ann. **59** (1904), pp. 193, *et seq.*, where the analytical questions that come under consideration were treated with the use of variables that were not superfluous.

([2]) The exclusion of certain elements and rays is not a disadvantage in all cases, namely, it is not such a thing in the theory of optical instruments. However, the demands of mathematical clarity and precision do not allow one to exclude figures *tacitly*, and to then express whether one has likewise represented them (as is also customary in studies of geometrical optics, among other things).

one makes a certain choice of direction cosines, which can result for a given (actual) line in two ways. If those direction cosines are *X*, *Y*, *Z* then one has the equation:

$$X^2 + Y^2 + Z^2 = 1 . \tag{1}$$

If *x*, *y*, *z* is a point on the thus-*oriented* line then the line will be specified exhaustively by combining the quantities *X*, *Y*, *Z* with the quantities:

$$L = y\,A - z\,Y , \qquad M = z\,X - x\,Z , \qquad N = x\,Y - y\,X , \tag{2}$$

which are coupled with *X*, *Y*, *Z* by the **Plücker** relation:

$$X\,L + Y\,M + Z\,N = 0 . \tag{3}$$

An inversion of the sign in all six coordinates *X*, *Y*, *Z*, *L*, *M*, *N* then means an inversion of the ("positive") direction of the straight line.

However, one can also specify an oriented line, or as we would like to say from now on, an *oriented ray*, by combining the quantities *X*, *Y*, *Z* with the coordinates Ξ, Η, Ζ of the foot of the altitude that has been dropped from the point (0, 0, 0) to the ray. The quantities Ξ, Η, Ζ are then coupled with the quantities *L*, *M*, *N* by the equations:

$$\Xi = Y\,N - Z\,M , \qquad \mathrm{H} = Z\,L - X\,N , \qquad \mathrm{Z} = X\,M - Y\,L , \tag{4}$$

$$L = \mathrm{H}\,Z - \mathrm{Z}\,Y , \qquad M = \mathrm{Z}\,X - \Xi\,Z , \qquad N = \Xi\,Y - \mathrm{H}\,X , \tag{5}$$

and with the quantities *X*, *Y*, *Z* by the equation ([1]):

$$X\,\Xi + Y\,\mathrm{H} + Z\,\mathrm{Z} = 0 . \tag{6}$$

When one employs those ray coordinates, inverting the direction of an oriented ray will obviously be effected by just changing the signs of *X*, *Y*, *Z* .

Now, one calls the figure that takes the form of an oriented ray and an (real) plane that is perpendicular to it, whose equations can always be put into the form:

$$X\,\xi + Y\,\eta + Z\,\zeta = \Omega , \tag{7}$$

and in only one way, an *oriented element* or *wave element*. Ω then means the distance to the plane, thus-*oriented*, from the coordinate origin (so – Ω is the distance to that point from the plane), as

([1]) The concepts of oriented line and oriented ray are identical in the real domain. However, when they are extended to the complex domain with the use of homogeneous coordinates, they will lead to completely different conceptual constructions.

measured along the positive direction of the ray. The *point of the wave element* (i.e., the point of intersection of the ray with the plane) has the coordinates:

$$(8) \qquad x = \Xi + \Omega X, \qquad y = \mathrm{H} + \Omega Y, \qquad z = \mathrm{Z} + \Omega z .$$

However, we shall assign the coordinates X, Y, Z; Ξ, H, Z; Ω to the wave element itself, which are coupled by the equations (1) and (6).

Two so-called consecutive wave elements will now be called *united* when "the point of the one lies in the plane of the other," i.e., when the **Pfaff** expression:

$$(9) \qquad X\,dx + Y\,dy + Z\,dz = d\Omega - x\,dX - y\,dY - z\,dZ = d\Omega - \Xi\,dX - \mathrm{H}\,dY - \mathrm{Z}\,dZ$$

vanishes. An analytical manifold of ∞^r wave elements, any two consecutive members of which are united, is called a *union*. The number r can then have a value of at most 2, and that case has special significance. We classify the two-dimensional unions of oriented elements into three families by the following rules:

I. *Doubly-curved two-dimensional unions*, which are ones whose ∞^2 elements belong to ∞^2 oriented planes.

Such a union consists of either all oriented elements that belong to a so-called oriented non-developable surface ([1]) or of all oriented elements whose planes go through the tangents to a *curved* line and have their points at the associated contact points. Finally, it can consist of all oriented elements with a common point.

II. *Simply-curved two-dimensional unions* whose ∞^2 elements are distributed on ∞^1 oriented planes.

Such a union consists of either the oriented tangent planes to an oriented developable surface that are each coupled with their ∞^1 contact points that lie on a generator of the surface or it consists of all oriented planes through an (real) straight line such that each plane is coupled with the ∞^1 points on that line.

III. *Non-curved or planar two-dimensional unions*, namely, the ∞^3 unions whose ∞^3 elements each consist of a well-defined oriented plane and its ∞^3 points.

([1]) An analytical surface, as a locus of points, is oriented when one orients its normal at each location with a well-defined tangent plane, and then analytically continues the orientation. That can be done in two ways. The two unions that arise in that way differ from each other, but can also be identical (i.e., have an analytical connection).

Now, an invertible analytical association ($E \to E'$) of the ∞^5 oriented elements E with other ones E' that can again emerge from united consecutive elements is called an *oriented contact transformation* (in the region where the transformation behaves regularly) ([1]).

The simplest example of an oriented contact transformation is any *dilatation*, and that concept is explained by the fact that every element advances along its normal through a constant line segment. ($\Omega' = \Omega + \text{const.}$) The dilatations are then *single-valued* oriented contact transformations. They define a one-parameter continuous group. A dilatation can convert any oriented union into another such thing and indeed (and it is important to note this) it will convert any two-dimensional union of the types I, II, III into another union of *the same* type. By contrast, the dilatations will take the unions that are called special in I and II, which belong to curves or points, to oriented cylindrical surfaces or spheres. Those unions then define families that are not invariant under dilatations.

We call unions that can be taken to each other by dilatations *parallel*. ∞^1 parallel unions belong to each two-dimensional union, which are all oriented surfaces, with the exception of at most one of them, and indeed, they are the so-called parallel surfaces.

Upon generalizing the properties that one observes in ordinary optical systems (e.g., mirrors, prisms, telescopes, microscopes), that will now pose the following problem:

Determine all oriented contact transformations that take parallel unions to other such things (and also wave surfaces to other wave surfaces ***in general****), or (which obviously amounts to the same thing), find all oriented contact transformations that the group of dilatations leaves fixed.*

When monochromatic light goes through, e.g., a system of lenses, every wave element E within a certain region will, in fact, be assigned to another one E' by a well-defined time evolution. If one displaces the first one along its normal then the second one will displace in the same way, and indeed when C and C' mean the reciprocal values of the speed of light in object space and image space, the following equation will exist between the corresponding coordinates and the time Δt that the light needs in order to go from E to E_1 or from E' to E'_1 ([2]):

$$C\,(\Omega_1 - \Omega) = C'(\Omega'_1 - \Omega) = \Delta t\,.$$

Now, the solution of the cited problem cannot indeed be achieved explicitly, in general, but probably by implicit formulas, i.e., the question that was posed can be reduced to an elimination problem (which insoluble, as a rule). Thus, it would be appropriate to distinguish three cases I, II, III, according to whether the ∞^2 elements of a planar union in general position are associated with a union of the above type I, II, or III, resp., because that structure is invariant under dilatations ([3]).

([1]) That concept differ from **Lie**'s concept of a "contact transformation" by the choice of spatial elements, or from a different viewpoint, by the geometric interpretation of the formula. Every **Lie** contact transformation will be oriented by orienting the **Lie** (i.e., unoriented) elements that it is applied to. Two different (i.e., analytically distinct) oriented contact transformations can then arise, as in the example of dilatations, but also just a single one. On the other hand, any **Lie** contact transformation (with a different interpretation of the formulas) will imply an oriented contact transformation.

([2]) Here, we assume that the light propagates in the positive direction of the wave normals.

([3]) That distinction was already suggested in **Hamilton**'s third supplement. By contrast, to our knowledge, it is missing from all later authors.

However, in all three cases, the solution is very easy to effect with the help of the methods that **Lie** gave (to the extent that it is even possible, from what was said). For that reason, it might suffice here to treat only *Case I*, which is the so-called general case. In order to do that, we appeal to the superfluous coordinates that were introduced, but point out that the form of the solution might be changed in many ways by that (and also simplified analytically), such that one will satisfy equations (1) and (6) with suitable so-called parameters. We can also sketch out the argument in order to show that the derivation is exhaustive and rigorous, so it would require a somewhat more substantial presentation.

It then follows from the foregoing that an equation of the following special form must exist between the coordinates of associated elements E, E':

$$C'\Omega' - C\,\Omega = W_0\,(X, Y, Z\,;\, X', Y', Z')\,. \tag{10}$$

C and C' are *constants* here, neither of which can have the value zero. However (although we cannot go into more detail about this), W is an arbitrary analytic function of the direction cosines X, Y, Z and X', Y', Z', moreover, that is constrained by a single restriction, namely, that the matrix:

$$\left\| \frac{\partial^2 W_0}{\partial X\,\partial X'} \quad \frac{\partial^2 W_0}{\partial Y\,\partial Y'} \quad \frac{\partial^2 W_0}{\partial Z\,\partial Z'} \right\|$$

must have a rank of at least two ([1]).

Now, equation (10) already determines the entire contact transformation. Namely, if one fixes the oriented plane (X, Y, Z ; Ω) then that will yield the point of an oriented element E' when one intersects one of the associated planes (X', Y', Z' ; Ω) with the neighboring likewise-associated planes. If one proceeds in the opposite sense then one will obtain the point of the element E that corresponds to the element E' (or rather, several elements E, since W does not need to be a single-valued function). The process requires only quite elementary arguments. One comes to the equations:

$$\begin{aligned}
C\cdot\Xi &= \Phi\cdot X - \frac{\partial W_0}{\partial X}, & C'\cdot\Xi' &= -\Phi'\cdot X' - \frac{\partial W_0}{\partial X'},\\
C\cdot\mathrm{H} &= \Phi\cdot Y - \frac{\partial W_0}{\partial Y}, & C'\cdot\mathrm{H}' &= -\Phi'\cdot Y' - \frac{\partial W_0}{\partial Y'},\\
C\cdot\mathrm{Z} &= \Phi\cdot Z - \frac{\partial W_0}{\partial Z}, & C'\cdot\mathrm{Z}' &= -\Phi'\cdot Z' - \frac{\partial W_0}{\partial Z'}\,;
\end{aligned} \tag{11}$$

$$\begin{aligned}
x &= \Xi + \Omega\cdot X\,, & x' &= \Xi' + \Omega'\cdot X',\\
y &= \mathrm{H} + \Omega\cdot Y\,, & y' &= \mathrm{H}' + \Omega'\cdot Y',\\
z &= \mathrm{Z} + \Omega\cdot Z\,, & z' &= \mathrm{Z}' + \Omega'\cdot Z',
\end{aligned} \tag{12}$$

([1]) All differentiations are performed as if the quantities X, Y, Z and X', Y', Z' were independent of each other.

in which Φ and Φ' mean:

$$\Phi = X\frac{\partial W_0}{\partial X} + Y\frac{\partial W_0}{\partial Y} + Z\frac{\partial W_0}{\partial Z}, \tag{13}$$
$$\Phi' = X'\frac{\partial W_0}{\partial X'} + Y'\frac{\partial W_0}{\partial Y'} + Z'\frac{\partial W_0}{\partial Z'},$$

so when W_0 is doubly-homogeneous to any degrees ρ, ρ' (which can always be arranged), they will have the values $\rho\, W_0$, $\rho' W_0$.

Equations (10)-(13) together contain the complete solution of our problem, whenever case I is in question. Equations (11) and (13), by themselves, imply the association of wave normals ($S \to S'$). However, all solutions of our problem that belong to the same association of rays ($S \to S'$) will be found when we replace equation (10) with:

$$C'\Omega' - C\,\Omega = W = W_0 + t \tag{10.b}$$

(as we would like to do, moreover), in which t means an arbitrary constant.

In the applications to optics, the parameter t can be identified immediately with the optical distance between the elements E, E', that is, with the *time* that the emitted light needs in order to go from E to E'. If one assumes that the light in object space and image space propagates in the positive direction of the associated rays S, S' then the constants C, C' will be set equal to positive values, namely, the reciprocal values of the two speeds of light. Naturally, it should be observed that the optical distance t will include an undetermined additive constant as long as one is not dealing with a well-defined dioptric apparatus.

If one assumes that t is variable in the replacement of W_0 with $W = W_0 + t$ that was carried out then one can consider the coordinates x, y, z and x', y', z' to be independently-varying quantities (within a certain region). t will then be a function of those quantities:

$$t = V(x, y, z\,;\, x', y', z') = \int \{C'(X'dx' + Y'dy' + Z'dz') - C\,(X\,dx + Y\,dy + Z\,dz)\}\ . \tag{14}$$

That is **Hamilton**'s *characteristic function*. It satisfies the partial differential equations:

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = C^2, \tag{15}$$
$$\left(\frac{\partial V}{\partial x'}\right)^2 + \left(\frac{\partial V}{\partial y'}\right)^2 + \left(\frac{\partial V}{\partial z'}\right)^2 = C'^2.$$

They likewise determine the association of the rays ($S \to S'$) by means of the equations:

$$C \cdot X = -\frac{\partial V}{\partial x}, \qquad C' \cdot X' = -\frac{\partial V}{\partial x'}, \tag{16}$$

etc. Furthermore, they determine the associated ∞^1 contact transformations by means of the further equations:

$$
(17) \qquad \begin{aligned} &X\,x + Y\,y + Z\,z = \Omega\,, \qquad X'x' + Y'y' + Z'z' = \Omega'\,, \\ &V\,(x, y, z\,;\, x', y', z') = t = \text{const.} \end{aligned}
$$

Finally, they once more determine *the values* of the functions W, W_0 :

$$
(18) \qquad W = \frac{\partial V}{\partial x}x + \frac{\partial V}{\partial y}y + \frac{\partial V}{\partial z}z + \frac{\partial V}{\partial x'}x' + \frac{\partial V}{\partial y'}y' + \frac{\partial V}{\partial z'}z'\,,
$$

$$
(19) \qquad W_0 = W - V\,.
$$

If one of the functions:

$$
W_0\,(X, Y, Z\,;\, X', Y', Z')\,, \qquad V\,(x, y, z\,;\, x', y', z')
$$

is known then the determination of the other one will require an elimination that cannot be performed, in general.

Moreover, if one would like to consider the function V to be given then that would require some other restrictions that must be fulfilled by the way that one chooses to create it from a function W_0 or W. Namely, first of all, the determinant:

$$
\begin{vmatrix} 0 & V_{x'} & V_{y'} & V_{z'} \\ V_x & V_{xx'} & V_{xy'} & V_{xz'} \\ V_y & V_{yx'} & V_{yy'} & V_{yz'} \\ V_z & V_{zx'} & V_{zy'} & V_{zz'} \end{vmatrix}
$$

cannot vanish, and secondly, the six-rowed matrix of second differential quotients of V must have rank four in case I (which is the only one that is treated here).

In regard to the description of the effect of an optical instrument, both functions V, W prove to be same in a theoretical context, and they also both prove to be especially simple in any case, but in a *natural* way, since points and planes are the simplest unions of elements. However, one probably cannot rate their practical value very highly, since one must be satisfied with approximations, which is, in fact, appropriate. Even when it is easy to trace a given ray through a dioptric system, exhibiting the easily-accessible function W will already encounter great difficulties, as a rule.

In order to explain at least one part of the argument that was just demonstrated, we shall appeal to the example of *refraction at a spherical surface:*

$$
\xi^2 + \eta^2 + \zeta^2 = r^2 \qquad (r > 0).
$$

Determining the function V in this case already requires solving an equation of degree six, but the function V still has an entirely simple expression here.

Let C and C' be the reciprocal values of the speeds of light outside and inside the sphere. Should a light ray that enters from the outside in the direction X, Y, Z be able to continue its path into the interior of the sphere in the direction X', Y', Z' then if ϑ means the angle between the two directions, one must have:

$$C' \cos \vartheta > C, \qquad \text{when} \qquad C' > C,$$

$$C' \cos \vartheta > C', \qquad \text{when} \qquad C > C'.$$

One now finds by an easy calculation that:

$$C'\Omega' - C\,\Omega = W = W_0 + t = r\sqrt{(C'X' - C\,X)^2 + (C'Y' - C\,Y)^2 + (C'Z' - C\,Z)^2} + t,$$

in which t means the (arbitrarily-prescribed) optical distance between the associated wave elements E, E', and in which the square root is assigned the negative value in the case of $C' > C$ and the positive value in the case of $C > C'$.

$$W_0 = r\sqrt{C^2 - 2C\,C' \cos \vartheta + C'^{\,2}}$$

is the value of $C'\Omega' - C\,\Omega$ for pairs of wave elements whose points (x, y, z) and (x', y', z') are united at the same point (ξ, η, ζ) of the sphere. One then has:

$$W_0 \cdot \xi = r^2\,(C'X' - C\,X), \qquad \text{etc.}$$

The formula above for W also serves to represent *reflection at a spherical surface*. The quantities X, Y, Z, X', Y', Z' are then coupled by only the inequality $\cos \vartheta < 1$. If the mirror is convex then one replaces C' with C and chooses the positive value for the square root. In the case of a concave mirror, one replaces C with C' and assigns the negative value to the root.

As we said, we would not like to go into the cases II, III here, the last of which corresponds to the so-called telescopic map and has especially simple properties. We would also like to discuss the remarkable structure of the infinite group of all dilatations of mutually-commuting contact transformations (namely, in the real domain), and first in a thorough representation in which we shall come to speak of further applications and new problems, several of which deserve attention. However, we still cannot discuss a closely-related question of fundamental significance here.

The contact transformations that were discussed will imply associations ($S \to S'$) of the oriented rays (i.e., light paths) that can make normal congruences emerge from normal congruences in a region of regular behavior *by shortening* them, one might say. Will all transformations of ray space that take normal congruences to other such things be found in that way?

The question that was posed can be resolved in the affirmative. From the criterion that was known already to **Hamilton**, a congruence of oriented rays is a normal congruence when the **Pfaff** expression:

$$\Xi\, dX + \mathrm{H}\, dY + \mathrm{Z}\, dZ$$

is a complete differential (= $d\Omega$) (cf., no. **9**). Now, if $(S \to S')$ is a transformation that does not disturb that property then it will follow that it will imply an equation of the special form:

$$C'\{\Xi'\, dX' + \mathrm{H}'\, dY' + \mathrm{Z}'\, dZ'\} = C\,\{\Xi\, dX + \mathrm{H}\, dY + \mathrm{Z}\, dZ\} + dF\,(X, Y, Z, \Xi, \mathrm{H}, \mathrm{Z})\,,$$

where C and C' are different *constants*. If one then sets:

$$F + t = C'\Omega' - C\,\Omega\,,$$

in which Ω is understood to mean a new independent variable, then that will yield:

$$C'\{d\Omega' - \Xi'\, dX' + \mathrm{H}'\, dY' + \mathrm{Z}'\, dZ'\} = C\,\{d\Omega - \Xi\, dX + \mathrm{H}\, dY + \mathrm{Z}\, dZ\}\,.$$

That is, the transformation $(S \to S')$ can be extended to an oriented contact transformation $(E \to E')$, and indeed in ∞^1 ways.

It deserves to be pointed out that an analogous theorem still does not exist in plane geometry.

Bonn, in July 1905.