"La classification des immersions," Séminaire Bourbaki (1956-8), exp. 157, pp. 279-289.

The classification of immersions

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(following SMALE [**2**])

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1. Review of definitions. – Let V^n, M^p be two differentiable manifolds of class C^r ($r \geq 2$) and dimension n (p, resp.). Suppose that $n < p$. In general, "differentiable" here will mean "r-times differentiable," where r is a fixed integer that is ≥ 2, and possibly infinite.

A differentiable map $f : V^n \to M^p$ is called an *immersion* if the rank of the map f is equal to n (viz., the dimension of V) at every point of V^n.

A map f of V^n into M^p is called an *embedding* if f is an immersion and f is *bijective*, as well.

Two immersions (embeddings, resp.) f, g of V into M will be called *equivalent* (or also *regularly homotopic*) if there exists a differentiable map F of $V \times I$ into M such that:

a. $F \mid V \times 0 = f$, $F \mid V \times 1 = g$.

b. The map $f_t(V)$ that is defined by $F(V \times t) \to M$ is an immersion (embedding, resp.) for any $t \in I$.

In the case where V^n is *compact*, the classes of the embeddings of V into M are identified with *isotopy* classes of V into M. Later on, one will see why it is that when two embeddings f, g of V into M are regularly homotopic, there exists a global diffeomorphism H of M such that $g = H \circ f$. One of the most classical problems in topology (but still one of the less well-known ones) is to classify the embeddings of V into M. By way of example, we cite the Schönflies problem: Determine whether two embeddings of the n-sphere S^n in $\mathbb{R}^{n+1}$ are isotopic or not.

It is clear that before classifying the embeddings, it would be important to classify the immersions as a preliminary step. In that regard, one can formulate the following conjecture [**1**], which was proposed by EHRESMANN: Let P_V be the bundle of n-frames on V_n , and let P_M be the analogous bundle on M. Any immersion $f : V \to M$ prolongs to a map $F : P_V \to P_M$. In order for two immersions f, $g : V^p \to M^m$ to be regularly homotopic, it is necessary and sufficient that the corresponding maps F, $G : P_V \to P_M$ should be homotopic (in the usual sense).

The work of SMALE represented important progress along that path. He gave a complete solution in the particular where $V = S^n$, $M = \mathbb{R}^p$, which has fundamental importance.

SMALE's results appeared in his thesis and in an abstract in the Bulletin of the American Mathematical Society [**2**]. The most complete and most recent result that he has communicated to me (viz., Theorem 4) has not been the subject of any publication up to now. Here, I propose to give a proof that was derived from some conversations that I had with SMALE this Spring in Chicago. It gives a result that is perhaps a little more general than his theorem, but it undoubtedly does not have the explicit character that it would probably need to subsume SMALE's final proof.

2. Generalities on function spaces. – Given two manifolds V^n, X^m, one endows the set $L\ (V, X)$ of maps from V into X with the C^r topology, which is defined by the differences between the partial derivatives up to order r over any compact subset of V^n. Such a space is locally contractible. More precisely, any point $f \in L\ (V, X)$ admits a neighborhood U_f that one can "locally" endow with the structure of a Frechet space: Let $T\ (X)$ be the bundle of tangent vectors to X, and let $\Theta\ (f)$ be the induced bundle of $T\ (X)$ on V by f. Any map that is sufficiently close to f can be put into the form $f + \delta f$, where δf is a function of $x \in V$ that takes its values in the vectors that are tangent to X at the point $y = f\ (x)$. As a result, a neighborhood U_f of f in $L\ (V, X)$ is homeomorphic to the vector space of sections of $\Theta\ (f)$ over V, when endowed with the C^r topology.

Let a map $j : V \to M$ be given. If one is given a third manifold X then the map j will induce a canonical map $j^* : L\ (M, X) \to L\ (V, X)$. A systematic study of that map does not seem to have been attempted up to now. Such a map is "locally linear" in the following sense: If $x \in L\ (M, X)$ and $y \in j^*\ (x) \in L\ (V, X)$ then there will exist neighborhoods U_x, V_y of x, y such that the restriction of j^* to U_x in V_y will be a linear map when they are provided with their Frechet space structures. It seems (and I would be grateful to the specialists if they could establish this assertion rigorously) that the image of $L\ (M, X)$ in $L(M, V)$ under j^* is always a closed subspace W (viz., a submanifold) that generally has infinite codimension. It likewise seems that the map $j^* : L\ (M, X) \to W$ can be considered to be a fibration (one does have local triviality). One proposes to show that under certain conditions the map j^* will satisfy the homotopy lifting property for cubes (or more generally, for compacta).

3. The lifting of homotopies for spaces of embeddings. – Suppose that one is given an embedding $i : V^n \to M^p$, where V and M are compact manifolds. Suppose that X is a third manifold whose dimension m is strictly greater than the dimension p of M. Let $Pl\ (M, X)$ denote the space of all embeddings of M in X. It is an open subset of $L\ (M, X)$ that one can endow with the induced topology (viz., the C^r topology). It is clear that if $f : M \to X$ is an embedding then the induced map $g = f \circ i$ will likewise be an embedding. There is then an induced map $i^* : Pl\ (M, X) \to Pl\ (V, X)$.

Theorem 1: *The map $i^* : Pl\ (M, X) \to Pl\ (V, X)$ satisfies the homotopy lifting property for cubes.*

Let T be a fixed tubular neighborhood of the submanifold $g(V)$ in X. Consider a neighborhood W_g of g in $Pl(V, X)$ such that for any $h \in W_g$, the image $h(V)$ is in T. Let a deformation g_t of g in W_g be given. It will permit one to a vector field $J_t = dg_t(x) / dt$ that is defined on $g_t(V)$ at any point x of V and for any t. In the product $T \times I$, due to the Whitney extension theorem, the vector field J_t can be extended to a field K_t that is defined in $T \times I$ and reduces to the unit vector that is carried by the t-axis on the boundary $\partial T \times I$. The integration of the differential system that is defined by the vector field K_t defines a group of homeomorphisms H_t with one parameter t in $T \times I$ that preserves the points of the boundary point-wise. H_t is a diffeomorphism of T that reduces to the identity on the boundary ∂T of T, and takes $g(V)$ to $g_t(V)$. As a result, H_t can be extended to a global diffeomorphism (which is likewise denoted by H_t) of X onto itself, and one will have:

$$g_t = H_t \circ g \,. \tag{1}$$

The same construction can be performed if the deformation g_t depends differentiably upon one parameter u, for example, in a cube I^k. One obtains a family of diffeomorphisms $H_{t,u}$ of X such that:

$$g_{t,u} = H_{t,u} \circ g \,. \tag{1'}$$

The lifting of homotopies for cubes in the neighborhood W_g then shows the following: Let $f_{0,u}$ be a cube of maps of M into X, and let $g_{0,u}$ be its projection onto $Pl(V, X)$. Suppose that a deformation $g_{t,u}$ of $g_{0,u}$ in W_g is given. One lifts that deformation by setting:

$$f_{t,u} = H_{t,u} \circ H_{0,u}^{-1} \circ g_{0,u} \,. \tag{2}$$

That succeeds in proving Theorem 1.

4. Factorization by jets of order one. – Two maps f, f' of M into X define the same jet of order one along V if the restrictions of f, f' to V, along with all of the partial derivatives of order one, are equal. The set of equivalence classes form a space $J_V^1(M, X)$ that is fibered over $L(V, X)$. Indeed, the fiber is the vector space of sections of the bundle whose base is V and whose fiber is the jets of the vector space that is normal to V in M with target X. The map $i^* : Pl(M, X) \to Pl(M, X)$ factorizes into $Pl(M, X) \xrightarrow{h} J_V^1(M, X) \to Pl(M, X)$. In order for the map h to be an epimorphism, one replaces $J_V^1(M, X)$ with the image H of h. At any point x of $g(V)$, any embedding $f : M \to X$ will induce a map $x \mapsto z(f)(x)$, in which $z(f)(x)$ denotes the jet of order one and rank p that is defined by the embedding f. Let H be that image. It is easy to see that the map $h : Pl(M, X) \to H$ also satisfies the homotopy lifting property for cubes. It suffices to repeat the proof of the theorem while observing the following fact: One demands that the homeomorphism $H_{t,u}$ of T must transform a given p-jet z (of rank p) at the point $x \in g_0(V)$ into a given p-jet z (likewise of rank p)

at the point $x' = g_{t,u}(x)$. That amounts to imposing certain linear conditions on the extension K_t of the vector field J_t by way of its jet of order one on the submanifold $g_t(V)$ of T . Now such an extension is always possible, due to the Whitney extension theorem. We thus obtain:

Theorem 2: - *The map $h : Pl(M, X) \to H$ that associates any extension of M into X with its jet of order one, restricted to the submanifold V, satisfies the homotopy lifting property for cubes.*

Remark. – One can just as well consider the jets of order higher than one. However, that will not lead to any result that is interesting from a topological standpoint, due to the well-known fact that, from the topological standpoint, the jets of order r and rank p of $\mathbb{R}^p$ into $\mathbb{R}^m$ ($p < m$) are the products of jets of order one with elements of a vector space.

5. Lifting of homotopies for spaces of immersions. – Now let $g : V \to X$ be an immersion of V into X. Since the map g has rank n everywhere, at every point of the image $g(V)$, one can define the fiber bundle of normal vectors (when X is endowed with a Riemannian metric), namely, N. One forms the bundle Q over V that is induced by N along g . It is clear that if one gives a very small value to the "radius" of the bundle N – namely, $a > 0$ – then the immersion $V \to X$ can be extended to an immersion $\gamma_1 : Q \to X$. The image $\gamma_1(Q)$ is called a *tubular neighborhood* of $g(V)$ in X of radius a. The manifold V is embedded in the bundle Q as the zero section $s_0(V) \subset Q$. One obtains an open neighborhood W_g of g in $L(V, X)$ by considering the effect on V in Q of the translation that is defined by a vector field J_t that is defined on V in Q . As in the case of embeddings, it then results that any deformation g_t of g that belongs to W_g can be written:

$$g_t = \gamma_1 \circ H_t \circ s_0, \tag{3}$$

in which H_t denotes a diffeomorphism of the fiber Q that depends continuously (for the C^r topology) on the deformation g_t and reduces to the identity on the boundary ∂Q of Q .

6. "Good position" of an immersion $f : M \to X$ with respect to a tubular neighborhood. – Suppose that one has an immersion $f : M \to X$ such that $g = f \circ i$. One can always find a tubular neighborhood C of $i(V)$ in M of very small radius r such that C can be embedded bijectively in the bundle Q with maximum rank in such a way that one will have the diagram:

$$V \to C \to \underbrace{Q \to T \to X}_{\gamma_I}$$

One says that the immersion f is *in good position* with respect to the tubular neighborhood T if the radius r of the neighborhood C can be taken to be large enough that the boundary of C can be lifted

into the boundary bundle Q (or a sufficiently small neighborhood of it). The reason that this definition is interesting is based upon the following lemma:

Let $Im\ (V, X)$ denote the space of immersions of V into X. Let $g : V \to X$ be an immersion, let T be a tubular neighborhood of $g\ (V)$, and let W_g be a neighborhood of g in $Im\ (V, X)$ such that $h(V)$ is contained in T for any $h \in W_g$. One proposes to show that the induced map $i^* : Im\ (M, X) \to Im(V, X)$ satisfies the homotopy lifting property for cubes. One first shows that:

Lemma 1: *If B denotes the subset of $i^{-1}\ (W_g)$ that consists of immersions of M into X that are in good position relative to T then $i^* : B \to W_g$ satisfies the homotopy lifting property for cubes.*

Let $f_{0,k}$ be a cube of immersions of B, and let $g_{0,k} = f_{0,k} \circ i$ be its projection onto $Im\ (V, X)$. By hypothesis, there exists a tubular neighborhood C_k of V in M such that the boundary of C lifts to the boundary of Q for any $f_{0,k}$. Let Q_1 be a bundle that is concentric to Q with a radius of $2a / 3$, for example. One can suppose that the cube k of immersions was taken to be small enough to satisfy the following condition: There exists a tubular neighborhood C_0 of V in M such that for any $f_{0,k}$, C_0 lifts to Q, and the boundary of C_0 lifts to Q between Q and Q_1 (in the "annular collar" that is defined by $2a / 3 < \Gamma < a$). Furthermore, one can suppose that the homeomorphisms $H_{t,k}$ that appear in formula (3) reduce to the identity outside of Q_1 in Q.

Let a deformation $g_{t,k}$ of $g_{0,k}$ be given then that is small enough for it to be in W_g . One lifts $g_{t,k}$ to $f_{t,k}$ as follows: Outside of C_0, one sets $f_{t,k} = f_{0,k}$; inside of C_0 one sets:

$$f_{t,k} = \gamma_1 \circ H_{t,k} \circ H_{0.t}^{-1} \circ f_{0.k} \ . \tag{4}$$

That succeeds in proving Lemma 1.

If one now factorizes the map $i^* : B \to W_g$ by means of jets of order one along V, namely, $B \xrightarrow{h} J_V^1(M, X) \to W_g$, then one can prove, as before, that the map h satisfies the homotopy lifting property for cubes (on the image of h).

Recall that one proposes to show that the map:

$$Im(M, X) \xrightarrow{h} J_V^1(M, X) \to W_g$$

satisfies the homotopy lifting property for cubes. One can simplify the proof greatly by using the lemma:

Lemma 2 (localization lemma): *Any sufficiently-small deformation of a map $g : V \to X$ is a sum (which is finite if V is compact) of local deformations.*

One intends that the term "local deformation" should mean a deformation that reduces to the identity outside of a (small) compact subset K of V. The lemma results from the following fact: As one has seen, the sufficiently-small deformations of a map g of V into X are in bijective and

bicontinuous correspondence with the sections of a bundle Θ. If (K_i) denotes a covering of V then a partition of unity that is subordinate to that covering will permit one to consider any section of Θ to be the sum of sections (s_i) that are zero outside of K_i . Such a section s_i obviously defines a local deformation whose "support" is K_i .

As a result, it will suffice to consider the case in which the cube of immersions k considered has a fixed compact subset K of V for its support. One can suppose that K is small enough that $g(K)$ is contained in a given chart of X for any $g \in k$.

7. Normalization of an immersion (in a given chart). – An immersion $f : M \to X$ will be called *normalized* [in a neighborhood of a compact subset K of V and for a chart on X that contains $g\ (K)$] if there exists a tubular neighborhood C of V in M such that the geodesic rays of C are mapped to *lines* (in the chart on X) by f . The normalization of an immersion consists of replacing a geodesic ray in a tubular neighborhood C of $i\ (V)$ in M, when it is embedded in X, with its tangent at the origin. Such a deformation is obviously possible for a tube of sufficiently-small radius, and that will be uniformly true for a compactum k of immersions.

We now enter into the finest part of Smale's theory, which consists of deforming a cube of immersions into immersions that are "in good position" with respect to a given tubular neighborhood T of $g\ (V)$ in X. In the chart on X that contains the compact image $g(K)$ that is of interest to us, we suppose that the tube T has a Euclidian radius of $2a$, that for any $g \in W_g$ one has $| g - g'\ (x) | < a / 10$, and that the homeomorphisms $H_{t,k}$ are associated, as in (3), with deformations g of g that reduce to the identity outside of the tube T_1, which is concentric to T and has a radius a.

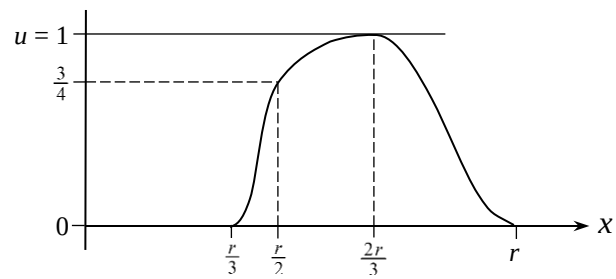
First, observe that if one has a *normalized* curve $P\ (s)$ that is traced in V, or rather on M, and if ν denotes a unit vector that has its origin at P and is normal to M (and thus, to V) and is a function of P, and if s denotes the curvilinear abscissa to the curve $P\ (s)$ then the derived vector $d\nu / ds$ will have a tangential component (in the tangent plane to M) whose modulus is less than a fixed limit A, which is a limit that is greater than the principal curvatures of M at P. Since those principal curvatures are continuous functions of the second derivatives of the map g' considered, they will also differ as little as one desires from the corresponding curvatures for the immersion g (one uses the C^2 topology here), and the radius $2a$ of the tube T must be itself such that $1 / 2a > A$. If that is true then on all of the image $f'(M)$, one can choice a vector field Y that is normal to $f'(M)$ and has length $2a$. Then associate f' with any map of the form $f'' = f' + u\ (P) \cdot Y$, where $u\ (P)$ is a positive scalar function that is much smaller than unity. I say that all the maps f'' are again immersions. Indeed, for any curve $P\ (s)$ that is traced on $f'(M)$ (with curvilinear abscissa s), one will have:

$$\frac{df''}{ds} = \frac{df}{ds} + u'_s Y + u \frac{dY}{ds} .$$

The tangential component of that vector cannot be zero, because it is $\geq 1 + u$ | tangential component of dY / ds |, and that component will have the form:

$$|Y| \cdot \left| \frac{d\nu}{ds} \right| = 2a \left| \frac{d\nu}{ds} \right| < 2a \cdot A < 1 .$$

The field Y permits one to define a deformation DV that transforms any (normalized) cube k of immersions in $i^{*-1}(W_g)$ into immersions that are *in good position* with respect to T . Indeed, let r be the Euclidian radius of the neighborhood C of V in M that has been normalized (for all $f' \in k$); if one desires, one can take $r < a / 10$. One then lets $u(r)$ denote a function C of the real variable x that is zero for $x < r / 3$, equal to 3/4 for $x = r / 2$, attains its maximum for $x = 2r / 3$, and is zero for $x \geq r$. (Cf. the Figure)



The deformation D_V is the defined by the formula:

$$D_V f = f + v \cdot u(r(P)) \cdot Y .$$

For $v = 0$, $D_0 f = f$. For $v = 1$, the map $D_1 f$ is an immersion, as well as all the $D_V f$, from the calculation that was made above. Moreover, all of the $D_1 f$ for $f \in k$ are in good position with respect to T. Indeed, if one makes $x = r / 2$ then all points at a distance of $r / 2$ from C are mapped by $D_1 f$ to points that are located at a normal distance from $g'(V)$ that is found between $\frac{3}{4} \times 2a \pm \frac{r}{2}$ $= \frac{3}{2} a \pm \frac{r}{2}$, so at a normal distance from $g(V)$ that is found between a and $2a$, since $r < a / 10$, and $| g'(p) - g(p) | < a / 10$ for any $g' \in W_g$.

It would seem to result from the preceding constructions that the map i^* does not permit one to immediately lift homotopies of cubes of immersions. Indeed, even if one deals with normalized maps from the outset, before one can apply formulas (1) for good position, it will be necessary to perform the deformation D_V in the bundle in order to put the cube k into "good position" (which is possible without changing its projection onto the base space W_g). That will demand a change of parameterization for the homotopy, and one will not be dealing with a fibration, but a quasi-fibration. Indeed, that is not true, and a very simple artificial technique (which is too lengthy to describe here) will permit one to perform the two deformations in some way simultaneously, in such a way that i^* will indeed be a fibration (in the SERRE sense). Finally, one has:

Theorem 3: *The map* $i^* : Im(M, X) \to Im(V, X)$ *that is induced by an embedding* $i : V \to M$ *satisfies the homotopy lifting property for cubes. The same thing is true for the map* $h : Im(M, X) \to J^1_V(M, X)$ *that is defined by taking the jet of order one of the restriction to V.*

Applications.

Definition. – An immersion $g : V \to X$ is said to "have a base point" if it takes a base point $p \in V$ to a base point $q \in X$, and if the first-order jet $J^1(g)$ of g is given at the point p .

Let B^k be the unit ball of dimension k . Let $Im\,(B^k, X\,;\,p)$ be the space of immersions of the k-ball B^k with a base point p. The base point p can be a point on the boundary S^{k-1} of B^k, moreover. One has:

Lemma: *The space* $Im\,(B^k, X\,;\,p)$ *is contractible.*

Proof: Normalize the immersion in a neighborhood of p and then retract B^k differentiably onto a sufficiently-small neighborhood of p .

Thanks to that lemma, SMALE determined the classes of immersions with a base point from the sphere S^k into $X = \mathbb{R}^m$, $k < m$. If one is given two immersions f, f_1 of S^k into $\mathbb{R}^m$ then after "normalizing" f and f_1 , one can suppose that f and f_1 coincide on a small ball whose center is p and a very small radius r. The restrictions of f, f_1 to the complementary ball D define (upon passing to the jets of order one) a map from the sphere S^k into the Stiefel manifold $\mathfrak{s}_k^{m-k}$ of k-frames in $\mathbb{R}^m$. Hence, one can define a difference element:

$$c\,(f, f_1) \in \pi_k(\mathfrak{s}_k^{m-k})\,,$$

which obviously depends upon only the class of the immersions f, f_1 . Smale's final result is stated:

Theorem 4:

1. *In order for two immersions with base point* f, f_1 *of* S^k *into* $\mathbb{R}^m$ *to be regularly homotopic (with base point), it is necessary and sufficient that the difference element* $c\,(f, f_1) \in \pi_k(\mathfrak{s}_k^{m-k})$ *should be zero.*

2. *If one is given an element in* $\pi_k(\mathfrak{s}_k^{m-k})$ *and an immersion f (with base point then there will exist an immersion g (with base point) such that:*

$$c\,(f, g) = a.$$

One considers the space F of immersions of the k-ball B^k into $\mathbb{R}^m$ whose jets of order one are given on the boundary sphere S^{k-1}. Such a space is a fiber bundle with a fibration (in the SERRE sense):

$$Im\,(B^k, \mathbb{R}^m\,; p) \to J^1_{S^k}(Im(B^k, X))\,.$$

One essentially wants to know $\pi_0\,(F)$. That "group" is isomorphic to $\pi_1(J^1_{S^k}(Im(B^k, \mathbb{R}^m); p)$, since $Im\,(B^k, \mathbb{R}^m)$ is contractible. One then applies the same process to the boundary sphere S^{k-1}. If one makes a choice of "equator" S^{k-2} in S^{k-1} then one can exhibit the isomorphism:

$$\pi_1(J^1_{S^{k-1}}(Im(B^k, \mathbb{R}^m); p) \cong \pi_2(J^1_{S^{k-2}}(Im(B^k, \mathbb{R}^m); p)\,,$$

as before, so after k isomorphisms of that type, one will have:

$$\pi_0\,(F) \cong \pi_1(J^1_{S^{k-1}}(Im(B^k, \mathbb{R}^m); p) \cong \pi_k(J^1_{S^0}(Im(B^k, \mathbb{R}^m); p)\,.$$

(In all of those formulas, J^1 denotes the jets of maximum rank k.)

Of the two points that constitute the sphere S^0, one of them is the base point p where the jet is given. The jet is arbitrary at the other point, provided that it has rank k. It then results that the latter space is identified with the Stiefel manifold $\mathfrak{s}_k^{m-k}$. That proves theorem 4.

Corollaries. – *Classification of immersions of S^k into $\mathbb{R}^{k+1}$:*

S^k can be immersed in $\mathbb{R}^{k+1}$ with an arbitrary normal degree if and only if S^k is parallelizable.

Two immersions of S^k into $\mathbb{R}^{2k}$ are regularly homotopic if and only if they have the same number of self-intersections.

Remark. – In the case of immersions of S^2 into $\mathbb{R}^3$, $\pi_2(\mathfrak{s}_2^1) = 0$. Two arbitrary immersions are regularly homotopic. For example, the usual embedding of S^2 in $\mathbb{R}^3$ and the antipodal embedding are regularly homotopic then. Meanwhile, the effective description of that regular homotopy gives the impression of being a daunting problem.

Another Chicago mathematician R. HIRSCH has extended Smale's theory of the sphere to manifolds. His procedure consists of realizing an immersion of V in X by proceeding with successive (differentiable) skeletons. He then met up with some obstructions with values in the homotopy groups of the Stiefel manifolds, from Theorem 4. For example, in order to immerse a V^k into $\mathbb{R}^{2k-2}$, one meets up with an obstruction that is defined on the $(k-1)$-skeleton with values in $\pi_k(\mathfrak{s}_k^{k-2})$. One can show the identity of that obstruction with the normal Stiefel-Whitney $\bar{w}_{k-1}$.

For example, for a three-dimensional manifold, that obstruction, which is equal to $\bar{w}_2$, is identically zero. *Hence, any three-dimensional (compact) manifold can be immersed in* $\mathbb{R}^4$.

In conclusion, Smale's result seems to give an essential tool for the study of immersions. Passing from immersions to embeddings probably necessitates some other methods. Meanwhile, it is not out of the question that the partial results, notably in regard to the homotopy of certain space of embeddings, can already be obtained with only that theory.

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