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On an application of the theory of linear differential equations to the calculus of variations.

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The process for examining the second variation of a simple integral with one function to be determined that **Jacobi** gave the foundations of in Bd. **17** of this journal is known to have been developed thoroughly by **Hesse** in Bd. **54** of this journal. The present treatise is connected with those investigations of **Jacobi** and **Hesse** and couples them with the theory of linear differential equations with analytic functions as coefficients.

Let the expression under the integral sign be $f(x, y, y^{(1)}, \ldots, y^{(n)})$, where x is the independent real variable that lies between the limits a and b, y is the unknown real function of x, and $y^{(r)} = d^r y / dx^r$. The desired function y will be obtained integrating the differential equation that emerges by setting the first variation of the integral equal to zero, while the constants that enter into it are determined by the endpoint conditions. Let the function y thus-found be a single-valued and continuous analytic function of x in a strip in the construction plane of the complex variable x that includes the segment along the real axis from a to b in its interior. Let that also be the case in regard to the functions $\frac{\partial f}{\partial y^{(p)}}$, $\frac{\partial^2 f}{\partial y^{(p)} \partial y^{(q)}}$. Let the **Jacobi** condition be fulfilled, such that $\frac{\partial^2 f}{\partial y^{(n} \partial y^{(n)}}$ does not vanish along the segment from a to b.

Under that assumption, one will see the following from the outset on the basis of some simple considerations from the theory of linear differential equations with analytic functions as coefficients:

There always exist families of curves that are infinitely-close to the curve y that was found such that the present integral will be a maximum (minimum, respectively, according to the sign of $\frac{\partial^2 f}{\partial y^{(n} \partial y^{(n)}}$*) for the curve y, and indeed the families of neighboring curves to y will have that property in general.*

That will be shown in the first section. In the second section, it will be proved that the corresponding statement is also true for the isoperimetric problems. The theorem above will be applied to some examples in the third section.

Section One

1. – Statement of the theorems of Jacobi and Hesse that will be used here.

Let the integral:

$$\int_a^b f(x, y, y^{(1)}, \ldots, y^{(n)})\, dx \tag{1}$$

be given, in which f is a real function of x, y, $y^{(1)}$ to $y^{(n)}$, in which $y^{(r)} = d^r y / dx^r$, and a and b are real. Set y equal to $y + \varepsilon z$. ε is a real quantity that varies in the neighborhood of zero, while z is an arbitrary real function of x that remains finite and continuous between a and b. Let the derivatives of z with respect to x behave similarly for every order up to $2n$. z shall vanish, along with its first n- 1 derivatives, for $x = a$ and b.

If a maximum (minimum, respectively) of the integral (1) is to occur for $\varepsilon = 0$ then the first differential quotients of the integral with respect to ε, viz., the first variation, must vanish for $\varepsilon = 0$. When one sets:

$$\frac{\partial f}{\partial y^{(p)}} = f'(y^{(p)}), \tag{2}$$

that will lead to the following differential equation in the known way (cf., **Hesse**, *loc. cit.*, pp. 231):

$$f'(y) - \frac{d}{dx} f'(y^{(1)}) + \frac{d^2}{dx^2} f'(y^{(2)}) - \cdots (-1)^n \frac{d^n}{dx^n} f'(y^{(n)}) = 0 . \tag{3}$$

That differential equation has order $2n$ when $\dfrac{\partial^2 f}{\partial y^{(n)}\, \partial y^{(n)}}$ does not vanish, which will be required as a condition from now on. y shall emerge from that differential equation as a result function of x with $2n$ constants. Let the constants be determined such that the given real limiting values of y and its first $n - 1$ derivatives at a and b will be obtained.

The second differential quotient of the integral (1) with respect to ε, viz., the second variation, will be given by the following expression. Let:

$$\frac{\partial^2 f}{\partial y^{(p)}\, \partial y^{(q)}} = a_{pq} , \tag{4}$$

(5) $$\frac{d^r z}{dx^r} = z^{(r)},$$

and

(6) $$2\psi = a_{00}\, z\, z + 2a_{01}\, z\, z^{(1)} + a_{11}\, z^{(1)}\, z^{(1)} + \cdots + 2a_{n-1,n}\, z^{(n-1)}\, z^{(n)} + a_{nn}\, z^{(n)}\, z^{(n)},$$

so the second differential quotient will become:

(7) $$2\int_a^b \psi\, dx.$$

That differential quotient shall have one and the same sign for $\varepsilon = 0$ for the various functions z. In the **Taylor** development of the integral (1) in powers of ε, with the remainder term in ε^2, that sign will decide whether a maximum or minimum occurs.

The integral $\int_a^b \psi\, dx$ will be equal to the following integral for $\varepsilon = 0$ since z vanishes for $x = a$ and b, along with its first $n - 1$ derivatives. Set:

(8) $$\frac{\partial \psi}{\partial z^{(r)}} = \psi'(z^{(r)})$$

and

(9) $$\psi'(z) - \frac{d}{dx}\psi'(z^{(1)}) + \frac{d^2}{dx^2}\psi'(z^{(2)}) - \cdots (-1)^n \frac{d^n}{dx^n}\psi'(z^{(n)}) = \Psi(z),$$

so:

(10) $$2\int_a^b \psi\, dx = \int_a^b z\,\Psi(z)\, dx$$

(**Hesse**, *loc. cit.*, pps. 231, 247), and one will further come down to the treatment of:

(11) $$\int_a^b z\,\Psi(z)\, dx.$$

The differential expression of order $2n$ in (9), in which z is left arbitrary, can always be put into the form (**Hesse**, *loc. cit.*, pp. 236):

(12) $$\mathfrak{A}_0\, z - \frac{d}{dx}\mathfrak{A}_1\, z^{(1)} + \frac{d^2}{dx^2}\mathfrak{A}_2\, z^{(2)} - \cdots (-1)^n \frac{d^n}{dx^n}\mathfrak{A}_n\, z^{(n)}.$$

By setting the coefficients equal to higher derivatives of z in (9) and (12), that will determine the quantities $\mathfrak{A}_n$, $\mathfrak{A}_{n-1}$, down to $\mathfrak{A}_0$, in succession, uniquely as entire rational expressions in the quantities a_{pq} (4) and their derivatives with respect to x. When one sets $\frac{d^r a_{pq}}{dx^r} = a_{pq}^{(r)}$, that will make:

(13) $$\begin{cases} \mathfrak{A}_n = a_{nn}, \\ \mathfrak{A}_0 = a_{00} - a_{01}^{(1)} + a_{02}^{(2)} - \cdots - (-1)^n a_{0n}^{(n)}. \end{cases}$$

One substitutes:

(14) $$z = u\, z_1$$

for z. Let $\frac{d^r u}{dx^r}$ be denoted by $u^{(r)}$, and $\frac{d^r z_1}{dx^r}$ by $z_1^{(r)}$, so:

$$z^{(1)} = u^{(1)} z_1 + u\, z_1^{(1)}, \qquad \text{etc.}$$

In that way, the function (6) 2ψ, which is a function in the quantities z, $z^{(1)}$, up to $z^{(n)}$, will go to a function $2\ \psi_1$ in the quantities z_1, $z_1^{(1)}$, up to $z_1^{(n)}$. The coefficient of $z_1^{(n)} z_1^{(n)}$ in $2\ \psi_1$ is:

(15) $$a_{nn} u^2 .$$

The differential expression that corresponds to (8) and (9):

(16) $$\psi_1'(z_1) - \frac{d}{dx}\psi_1'(z_1^{(1)}) + \frac{d^2}{dx^2}\psi_1'(z_1^{(2)}) - \cdots (-1)^n \frac{d^n \psi_1'(z_1^{(n)})}{dx^{(n)}} = \Phi\,(z_1)\,,$$

can be put into the following form that corresponds to (12):

(17) $$\mathfrak{B}_0\, z - \frac{d}{dx}\mathfrak{B}_1\, z^{(1)} + \frac{d^2}{dx^2}\mathfrak{B}_2\, z^{(2)} - \cdots (-1)^n \frac{d^n}{dx^n}\mathfrak{B}_n\, z^{(n)} .$$

One now starts from the differential expression (12) [let it now be denoted by $\Psi\,(z)$], and one must further address the integral:

(18) $$\int_a^b z\,\Psi(z)\,dx .$$

$\Psi\,(z)$ will coincide with the form (9) when one sets $2\,\psi\,(z)$ equal to:

(19) $$2\,\psi = \mathfrak{A}_0\, z^2 + \mathfrak{A}_1 (z^{(1)})^2 + \mathfrak{A}_2 (z^{(2)})^2 + \cdots + \mathfrak{A}_n (z^{(n)})^2 .$$

Now, under the substitution (14), $z = u\,z_1$, 2ψ will be defined by the expression $2\psi_1$, which will then define the expressions (16), (17), which determine the quantities $\mathfrak{B}_0$ to $\mathfrak{B}_n$, and according to (13) and (15), one will have:

(20) $$\mathfrak{B}_n = \mathfrak{A}_n\,u^2 = a_{nn}\,u^2 .$$

Set:

(21) $$\mathfrak{B}_1\,z_1^{(1)} - \frac{d}{dx}\mathfrak{B}_2\,z_1^{(2)} + \frac{d^2}{dx^2}\mathfrak{B}_3\,z_1^{(3)} - \cdots (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\mathfrak{B}_n\,z_1^{(n)} = \Psi_1(z_1^{(1)}),$$

which will then make $\Psi_1(z_1^{(1)})$ a differential expression of order 2 (n – 1) order relative to $z_1^{(1)}$. The relation:

(22) $$u\,\Psi\,(z) - z\,\Psi\,(u) = -\,\frac{d}{dx}\Psi_1(z_1^{(1)})$$

will then exist for arbitrary u and z_1 (**Hesse**, *loc. cit.*, pps. 241, 242).

Now set $z = u\,z_1$ in the integral (18), in which $\Psi\,(z)$ is the linear differential expression (12) of order $2n$ and take u to be an integral of the differential equation $\Psi\,(u) = 0$. In so doing, assume that this integral u is real, finite, continuous, and nowhere-vanishing along the interval of z from a to b. Now, z is the function of x that is given by (1). z_1 will then vanish for $x = a$ and b, along with its first n – 1 derivatives. By means of the relation (22), the integral (18) will become:

(23) $$\int_a^b z\,\Psi(z)\,dx = \int_a^b z_1\,u\,\Psi(z)\,dx = \int_a^b z_1^{(1)}\,\Psi_1(z_1^{(1)})\,dx .$$

The integral:

(24) $$\int_a^b z_1^{(1)}\,\Psi_1(z_1^{(1)})\,dx$$

has the same type as the integral (18) and will then be treated in the same way. $\Psi_1(z_1^{(1)})$ is the linear differential expression (21), which has order 2 (n – 1) relative to $z_1^{(1)}$. Set $z_1^{(1)} = v_1^{(1)}\,z_2^{(1)}$. Let $\Psi_2(z_2^{(2)})$ be the linear differential expression of order 2 (n – 2) relative to the $z_2^{(2)}$ that arises from by the substitution $z_1^{(1)} = v_1^{(1)}\,z_2^{(1)}$ in precisely the same way that $\Psi_1(z_1^{(1)})$ arises from $\Psi\,(z)$ by the substitution $z = u\,z_1$. Let that differential expression $\Psi_2(z_2^{(2)})$ be:

(25) $$\mathfrak{C}_2\,z_2^{(2)} - \frac{d}{dx}\mathfrak{C}_3\,z_2^{(3)} + \frac{d^2}{dx^2}\mathfrak{C}_4\,z_2^{(4)} - \cdots (-1)^{n-2}\frac{d^{n-2}\,\mathfrak{C}_n\,z_2^{(n)}}{dx^{n-2}} = \Psi_2(z_2^{(2)}),$$

in which:

(26) $$\mathfrak{C}_n = \mathfrak{B}_n (v_1^{(1)})^2 .$$

The relation:

(27) $$v_1^{(1)} \Psi_1(z_1^{(1)}) - z_1^{(1)} \Psi_1(v_1^{(1)}) = - \frac{d}{dx} \Psi_2(z_2^{(2)})$$

will then exist for arbitrary $v_1^{(1)}$ and $z_2^{(1)}$.

Now let $v_1^{(1)}$ be an integral of the differential equation $\Psi_1(z_1^{(1)}) = 0$. In that way, the assumption that this integral $v_1^{(1)}$ is real, finite, continuous, and nowhere-vanishing along the interval of x from a to b will again be valid. The $z_1^{(1)}$ in the integral (24) is the first differential quotient of z_1, and z_1 arises from z by way of $z = u\, z_1$, where z is the function that was named in (1). $z_2^{(1)}$ will then vanish for $x = a$ and b, along with its first $n - 2$ differential quotients. Now, the integral (24) will imply:

(28) $$\int_a^b z_1^{(1)} \Psi_1(z_1^{(1)})\, dx = \int_a^b z_2^{(1)} v_1^{(1)} \Psi_1(z_1^{(1)})\, dx = \int_a^b z_2^{(2)} \Psi_2(z_2^{(2)})\, dx$$

by means of the relation (27).

According to (13), (20), (26), one has:

(29) $$\mathfrak{A}_n = a_{nn} = \frac{\partial^2 f}{\partial y^{(n)}\, \partial y^{(n)}}, \qquad \mathfrak{B}_n = a_{nn}\, u^2, \qquad \mathfrak{C}_n = a_{nn} (u\, v_1^{(1)})^2,$$

One always proceeds in the same way, under the same assumptions as for u, $v_1^{(1)}$ in regard to the new integrals that enter into successively-appearing linear differential equations. After the nth transformation, the integral (18) will be given by the expression (**Hesse**, *loc. cit.*, pp. 248):

(30) $$\int_a^b a_{nn} (u\, v_1^{(1)}\, w_2^{(2)} \cdots z_n^{(n)})^2\, dx .$$

The substitutions will then exist:

(31) $$z = u\, z_1 , \qquad z_1^{(1)} = v_1^{(1)} z_2^{(1)} , \qquad z_2^{(2)} = w_2^{(2)} z_3^{(2)} , \qquad \text{etc.},$$

in which u, $v_1^{(1)}$, $w_2^{(2)}$, etc., are integrals of the n linear differential equations:

(32) $$\Psi(z) = 0 , \qquad \Psi_1(z_1^{(1)}) = 0 , \qquad \Psi_2(z_2^{(2)}) = 0 ,$$

respectively, $\Psi(z)$ is the differential expression (12), $\Psi_1(z_1^{(1)})$ is the differential expression (21):

(33) $$u\,\Psi\,(z) = -\,\frac{d}{dx}\,\Psi_1(z_1^{(1)})\,,$$

$\Psi_2(z_2^{(2)})$ is the differential equation (25):

(34) $$v_1^{(1)}\,\Psi_1(z_1^{(1)}) = -\,\frac{d}{dx}\,\Psi_2(z_2^{(2)})\,,$$

and so on. n functions are constructed from the functions (31):

(35) $$\begin{cases} u = u\,, \\ v = u\int v_1^{(1)}\,dx\,, \\ w = u\int dx\,v_1^{(1)}\int w_2^{(2)}\,dx\,, \\ \quad\cdots \end{cases}$$

The determinant of those n functions and their first $n - 1$ derivatives:

(36) $$\begin{vmatrix} u & u^{(1)} & \cdots & u^{(n-1)} \\ v & v^{(1)} & \cdots & v^{(n-1)} \\ w & w^{(1)} & \cdots & w^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix} = \Delta_n$$

is

(37) $$\Delta_n = u^n\,(v_1^{(1)})^{n-1}(w_2^{(2)})^{n-2}\cdots$$

It will then emerge from (31) that:

(38) $$z = u\int dx\,v_1^{(1)}\int dx\,w_2^{(2)}\int\cdots\int z_n^{(n)}\,dx\,.$$

The determinant of the $n + 1$ functions (35) and (38) and their first n derivatives:

(39) $$\begin{vmatrix} u & u^{(1)} & \cdots & u^{(n)} \\ v & v^{(1)} & \cdots & v^{(n)} \\ w & w^{(1)} & \cdots & w^{(n)} \\ \vdots & \vdots & \cdots & \vdots \\ z & z^{(1)} & \cdots & z^{(n)} \end{vmatrix} = \Delta$$

is

(40) $$\Delta = u^{n+1}(v_1^{(1)})^n(w_2^{(2)})^{n-1}\cdots z_n^{(n)}.$$

That will give the following expression that occurs in the integral (30):

(41) $$u\,v_1^{(1)}\,w_2^{(2)}\cdots z_n^{(n)} = \frac{\Delta}{\Delta_n}.$$

That follows from the differential equations (32), with the relations (31), (33), (34), etc., in relation to the functions (35). Let u be any integral of the differential equation $\Psi(z) = 0$ and define the differential expression $\Psi_1(z_1^{(1)})$ with u in a region where u does not vanish. Now let $v_1^{(1)}$ be any integral of $\Psi_1(z_1^{(1)}) = 0$ and define the differential expression $\Psi_2(z_2^{(2)})$ with $v_1^{(1)}$ in a region where $v_1^{(1)}$ does not vanish. Then let $w_2^{(2)}$ be any integral of $\Psi_2(z_2^{(2)}) = 0$, etc. According to (31), (33), (34), etc., the functions (35) u, v, w, etc., are n integrals of the $2n^{\text{th}}$-order differential equation $\Psi(z) = 0$ then (**Hesse**, *loc. cit.*, pps. 249, 250).

2. – Application of the theory of linear differential equations.

A basic theorem from the theory of linear differential equations with analytic functions of the independent variable x as coefficients and one dependent variable finds an application here:

In a simply-connected region of the construction plane of the complex variable x, let the coefficients of the differential quotients be single-valued and continuous analytic functions of x, while the coefficient of the highest derivative is equal to 1. The homogeneous linear differential equation of order m will then have an integral that is a single-valued and continuous analytic function of x and possesses prescribed values at a point inside of that region along with its first $m-1$ derivatives and will be determined uniquely in that way.

Let n linearly-independent functions y_1, y_2, up to y_n be single-valued and continuous analytic in a given region of x. They are brought into the form:

(1) $$\begin{cases} y_1 = v_1, \\ y_2 = v_1\int v_2\,dx, \\ w = v_1\int dx\,v_2\int v_3\,dx, \\ \quad\vdots \\ y_n = v_1\int dx\,v_2\int dx\,v_3\int\cdots\int v_n\,dx, \end{cases}$$

in which there is also a domain in each part of the region of x considered such that the v are nowhere-vanishing in it since the y are linearly independent. It will then emerge from the expressions for the v:

(2) $$v_2 = \frac{d}{dx}\frac{y_2}{v_1}, \quad v_3 = \frac{d}{dx}\frac{1}{v_2}\frac{d}{dx}\frac{y_3}{v_1}, \quad v_3 = \frac{d}{dx}\frac{1}{v_3}\frac{d}{dx}\frac{1}{v_2}\frac{d}{dx}\frac{y_4}{v_1}, \quad \text{etc.}$$

that they have the form $\frac{\chi(x)}{\psi(x)}$, where $\chi(x)$ and $\psi(x)$ are single-valued and continuous analytic functions in the original region of x that do not vanish identically. In a simply-connected region S that includes the boundary curve and lies inside of the previously-considered simply-connected region of x, the functions $\chi(x)$ and $\psi(x)$ will be zero at a finite number of points. The quantities v will also become zero or infinite at only a *finite* number of points in S.

The application that we are treating here is based upon those theorems.

Take the integral [no. **1**, (1)]:

(3) $$\int_a^b f(x, y, y^{(1)}, \ldots, y^{(n)})\, dx,$$

in which $y^{(r)} = \frac{d^r y}{dx^r}$. Upon integrating the differential equation [no. **1**, (3)], by means of which the first variation of the integral (3) vanishes, y will emerge as a function of x with $2n$ constants. Let those constants be determined in such a way that y takes given real values for the real values $x = a$ and b, along with its derivatives up to order $n - 1$, so y will then be a real function along the segment of x from a to b.

The following assumption will now be made: In a strip T in the construction plane of the complex variables x inside of which the segment along the real axis from a to b lies, let the function y that is found be a *single-valued and continuous analytic* function of x. y is real for $x = a$ to b, so that will be true for every real x in T at all.

The values that $y^{(0)} = y$, $y^{(r)} = \frac{d^r y}{dx^r}$ assume for x from a to b might lie in the interval from $\alpha^{(r)}$ to $\beta^{(r)}$, $\alpha^{(r)} < \beta^{(r)}$. Let each of the functions:

(4) $$f(x, y, y^{(1)}, \ldots, y^{(n)}), \quad \frac{\partial f}{\partial y^{(p)}}, \quad \frac{\partial^2 f}{\partial y^{(p)}\, \partial y^{(q)}}$$

be denoted by:

(5) $$\varphi(x, y, y^{(1)}, \ldots, y^{(n)}) \ .$$

κ shall be positive (> 0) in such a way that $\varphi(x, y, y^{(1)}, \ldots, y^{(n)})$ is a finite and continuous real function of the variables x, y, $y^{(1)}$, up to $y^{(n)}$, which are taken to be independent, when x remains inside the interval from a to b, y remains in the interval $\alpha - \kappa$ to $\beta + \kappa$, and remains in the interval from $\alpha^{(r)} - \kappa$ to $\beta^{(r)} + k$.

The expressions:

(6) $$\frac{\partial f}{\partial y^{(p)}}, \frac{\partial^2 f}{\partial y^{(p)}\,\partial y^{(q)}},$$

in which the function y that was found is substituted, shall be *single-valued and continuous analytic functions of* x in the strip T.

Therefore, in order to apply the **Jacobi-Hesse** theory that was presented in no. **1**, one assumes that the function:

(7) $$\frac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$$

does not vanish anywhere *along the interval of x from a to b*. That function will then remain non-zero inside of a strip like T due to the assumption in (6).

From the cited assumption that preceded (7), one is given that the differentiation of the integral [no. **1**, (1)] with respect to ε allows differentiation under the integral sign, that the first variation vanishes for $\varepsilon = 0$, and that the second variation is expressed by the integral [no. **1**, (7)], and then by the integral [no. **1**, (18)]. The latter integral must now be treated further.

In the homogeneous linear differential expression $\Psi(z)$ of order $2n$ [no. **1**, (12)], the coefficients $\mathfrak{A}$ are single-valued and continuous analytic functions in the strip T, and the coefficient of the highest derivative $\mathfrak{A}_n = \dfrac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$ is non-zero inside of that strip. The homogeneous linear differential equation $\dfrac{1}{\mathfrak{A}_n}\Psi(z) = 0$, with the coefficient of the highest derivative equal to 1, will then have coefficients of its differential quotients that are single-valued and continuous analytic functions inside of the strip T that are real functions of x from a to b, and therefore for all real x in T.

Now take an integral u of that differential equation that has real values for a certain real value x inside of T, along with its first $2n - 1$ derivatives, and therefore it is a real, single-valued, and continuous analytic function in T for real x. The homogeneous linear differential expression of order $2(n-1)$ [no. **1**, (21)] $\Psi_1(z_1^{(1)})$ is defined by means of u according to what was given in [no. **1**, (14) – (21)]. The coefficients $\mathfrak{B}$ that appear in it are single-valued and continuous analytic functions of x that are real when x is real. The coefficient of the highest derivative is $\mathfrak{B}_n = \mathfrak{A}_n u^2$. and it will have a finite number of zero-points in a region inside of T. Take a region in T in which $\mathfrak{B}_n$ does not vanish and which includes a segment along the real axis. In that region of x, the homogeneous linear differential equation $\dfrac{1}{\mathfrak{B}_n}\Psi_1(z_1^{(1)}) = 0$ whose highest derivative has a coefficient equal to 1 will have coefficients that are single-valued and continuous analytic functions there and are real when x is real. In that same region of x, take an integral $v_1^{(1)}$ of the latter differential equation then that is real for real x and is a single-valued, continuous analytic

function. The homogeneous linear differential expression of order 2 (n – 2) [no.**1**, (25)] $\Psi_2(z_2^{(2)})$ is defined by means of $v_1^{(1)}$ in the last region of x that was considered. Its coefficients $\mathfrak{C}$ are single-valued and continuous analytic functions there that are real for real x. The coefficient of the highest derivative is $\mathfrak{C}_n = \mathfrak{A}_n(u v_1^{(1)})^2$. Once again, take a region within that region such that $\mathfrak{C}_n$ does not vanish in it, a segment of the real axis is included in it, and then take an integral $w_2^{(2)}$ of the differential equation $\frac{1}{\mathfrak{C}_n}\Psi_2(z_2^{(2)}) = 0$ that is real for real x, etc.

The expressions [no. **1**, (35)] u, v, w, etc., will be represented as n linearly-independent integrals of $\Psi(z) = 0$ that are real when x is real by means of the functions u, $v_1^{(1)}$, $w_2^{(2)}$, etc., that are obtained in that way. Each of those integrals of the differential equation $\frac{1}{\mathfrak{A}_n}\Psi(z) = 0$ is a single-valued and continuous analytic function inside of the strip T. Those n functions are brought into the form (1) by way of [no. **1**, (35)]. From what was said in (2), one can infer that the functions u, $v_1^{(1)}$, $w_2^{(2)}$, etc., that were just cited are the same single-valued, analytic functions in T that will become zero or infinite at a *finite* number of points in any region inside of T that include its boundary and is real when x is real.

The functions u, $v_1^{(1)}$, $w_2^{(2)}$, etc., will become zero or infinite at a finite number of points along the segment of x from a to b. Let those points be ξ_1, ξ_2, up to ξ_λ; they can also include a or b. Now select consecutive pieces of the segment a to b that each include a point ξ. They can be taken to be arbitrarily small, but they shall be fixed. Let the piece that includes ξ_r be η_r. Then take the subsegments:

$$\eta_1, \eta_2, \ldots, \eta_\lambda. \tag{8}$$

The function z in the integral [no. **1**, (1)] shall be an arbitrary real function of x that is finite and continuous from a to b, along with its derivatives up to order $2n$, and which vanishes at $x = a$ and b, along with its derivatives up to order $n - 1$.

Now let that function z be set equal to zero in the subinterval from η_1 *to* η_λ (8). *In a piece that lies between two consecutive intervals* η_{r-1} *to* η_r *(in the piece between a or b and the next* η*, respectively), which has the endpoints* a' *and* b', *let z be an arbitrary real function of x that is finite and continuous and vanishes at x =* a' *and* b', *along with its derivatives up to order* $2n$. If the point a or b is one of the points a' or b' then the derivatives of z up to order $n - 1$ shall vanish there.

One such function is:

$$(x-a')^{2n+1}(x-b')^{2n+1}w, \tag{9}$$

in which w is an arbitrary real function that remains finite and continuous from a' to b', along with its derivatives up to order $2n$.

The function z then fulfills the previously-cited conditions along the interval of x from a to b.

The integral [no. **1**, (18)]:

$$\int_a^b z\,\Psi(z)\,dx \tag{10}$$

needs to be dealt with further. That integral decomposes into a sum of a finite number of integrals due to the fact that integration interval from a to b is decomposed into the pieces (8) η_1 to η_λ and the pieces that lie between them. The integral over a segment η_r is zero since z was taken to be zero in it. What will remain is the integral over each segment that lies between two segments η_{r-1} and η_r (between a or b and the next η, respectively). One must now address:

$$\int_{a'}^{b'} z\,\Psi(z)\,dx\,, \tag{11}$$

in which z is an arbitrary real function that remains finite and continuous from a' to b' and vanishes at a' and b', along with its derivatives up to order $2n$. (If $a' = a$ or $b' = b$ then the derivatives vanish up to order $n - 1$ there.)

The conditions are fulfilled by the integral (11) that led to its expression in [no. **1**, (30)]:

$$\int_{a'}^{b'} a_{nn}\,(u\,v_1^{(1)}\,w_2^{(2)}\cdots z_1^{(1)})^2\,dx\,. \tag{12}$$

The n functions u, $v_1^{(1)}$, $w_2^{(2)}$, etc., [the integrals of the differential equations no. **1**, (32), respectively] are real, finite, continuous, and nowhere-vanishing from a' to b'. From [no. **1**, (41)], one has the expression:

$$u\,v_1^{(1)}\,w_2^{(2)}\cdots z_1^{(1)} = \frac{\Delta}{\Delta_n}\,. \tag{13}$$

From [no. **1**, (37)]:

$$\Delta_n = u^n\,(v_1^{(1)})^{n-1}\,(w_2^{(2)})^{n-2}\cdots, \tag{14}$$

so it will be real, finite, continuous, and nowhere equal to zero along the interval from a' to b'. Δ is the determinant [no. **1**, (39)].

The determinant Δ is a homogeneous linear differential expression of order n relative to z. The coefficient of $\frac{d^n z}{dx^n}$ is the quantity Δ_n in (14). The homogeneous linear differential equation of order n, $\Delta\,/\,\Delta_n = 0$, with the coefficient of the highest derivative equal to 1, has single-valued and continuous analytic functions as coefficients inside of a strip in the construction plane of x, inside of which lies the segment along the real axis from a' to b'. It would then emerge from the expression for Δ that its integrals are the n functions u, v, w, etc., that are constructed from the

functions u, $v_1^{(1)}$, $w_2^{(2)}$, etc., using the expressions in [no. **1**, (35)]. From what was said above, the latter expressions are single-valued analytic functions in the strip T. They will be finite and non-zero along the segment of the real axis from a' to b', and therefore along a strip that includes that segment. It emerges from the expressions of the form [no. **1**, (35)] that every integral of the differential equation $\Delta = 0$ will be a homogeneous linear combination with constant coefficients of the integral u, v, w, etc., in that region.

Now when the quantity z fulfills the equation $\Delta = 0$ along the entire interval of x from a' to b', it will be a homogeneous linear combination of u, v, w, etc. with constant coefficients. If those constants are determined at one of the points a' or b' by differentiating that linear combination $n-1$ times and solving the system of n linear equations that thus arises, in which the determinant of the system Δ_n is non-zero, then that will imply that the constants must be zero since z vanishes there along with its first $n - 1$ derivatives.

If z is not equal to zero along the entire interval from a' to b' then Δ will not be zero either, so the integral (12) will be non-zero and have the same sign as a_{nn}.

When the assumption that was made in (3) is fulfilled, that will then imply the following:

If z is an arbitrary finite and continuous real function along the interval of x from a to b whose derivatives up to order $2n$ remain finite and continuous there, and z vanishes along with its first $n-1$ derivatives at the points $x = a$ and b, and furthermore z is equal to zero along the subinterval from η_1 to η_λ that was cited in (8), *but does not vanish along the entire interval from a to b, then the integral* [no. **1**, (18)] *will always keep the same sign as* $\dfrac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$.

For all families of neighboring curves $y + \varepsilon z$ to y, where ε is a real quantity that varies close to zero, the integral [no. 1, (1)] *will be a maximum (minimum, respectively) according to the sign of* $\dfrac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$.

3. – Generalizing the results of no. 2.

The function z of x was set equal to zero along the subinterval from η_1 to η_λ in [no. **2**, (8)] is set equal to zero. The family of neighboring curves to the curve y that is determined by setting the first variation of the integral [no. **1**, (1)] equal to zero, which is given by $y + \varepsilon x$, therefore coincides with y along the subinterval from η_1 to η_λ. From the result that was obtained, one can then go to a more general family of neighboring curves to y that satisfies the requirements that were posed.

In place of z, one now sets:

$$\left\{\begin{array}{l} z+\varepsilon Z, \\ Z=(x-a)^n(x-b)^n\,\omega(x,\varepsilon), \end{array}\right. \tag{1}$$

in which:

$$\omega(x,\varepsilon), \qquad \frac{d^r\omega(x,\varepsilon)}{dx^r} \qquad (r = 1, \ldots, n) \tag{2}$$

vary close to zero and are real, finite, and continuous functions of x and ε for x from a to b and when ε varies in the neighborhood of zero. If either of those functions is denoted by $\varphi(x, \varepsilon)$ then they shall have the property that:

$$\frac{d\varphi}{d\varepsilon}, \quad \frac{d^2\varphi}{d\varepsilon^2}. \tag{3}$$

That will imply that the first variation of the integral [no. **1**, (1)] will have the same expression as before when $\varepsilon = 0$, so it will vanish, and that the second variation will again have the previous expression [no. **1**, (7)] by means of the differential equation [no. **1**, (3)] and due to the fact that Z vanishes for $x = a$ and b, along with its derivatives with respect to x up to order $n - 1$.

Thus, when the assumption that was in given in [no. **2**, after (3)] is fulfilled, the integral [no. **1**, (1)] will be a maximum (minimum, respectively, according to the sign of $\frac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$) under the transition from the curve y that is determined by setting the first variation of the integral equal to zero and leaves y unchanged at the endpoints, along with its first $n - 1$ derivatives, *so those curves can generally be taken from the neighboring curves of the type that is ordinarily considered.*

Section Two

4. – Isoperimetric problems.

In these problems, the integral [no. **1**, (1)]:

$$\int_a^b f(x, y, y^{(1)}, \ldots, y^{(n)})\,dx \tag{1}$$

will be a maximum (minimum, respectively) under the transition from a curve y to the infinitely-close curves, while the value of another integral:

$$\int_a^b F(x, y, y^{(1)}, \ldots, y^{(m)})\,dx \tag{2}$$

is given in which F is a real function of $x, y, \ldots, y^{(m)}$. It shall not be required of this integral (2), in which $m \leq n$, that it must remain constant under the transition from the curve y to the infinitely-close curves. Rather, the less-demanding condition shall be made here that when ε is a real quantity in the neighborhood of zero that brings about that transition (nos. **1**, **3**), the change in the integral (2) in comparison to ε will be infinitely small when ε is infinitely small.

y is set equal to $y + \varepsilon z$ [no. **1**, near (1)]. Let the differential expression [no.**1**, (3)]:

(3) $$f'(y)-\frac{d}{dx}f'(y^{(1)})+\frac{d^2}{dx^2}f'(y^{(2)})-\cdots(-1)^n\frac{d^n}{dx^n}f'(y^{(n)})$$

be denoted by Q, and let the differential expression:

(4) $$F'(y)-\frac{d}{dx}F'(y^{(1)})+\frac{d^2}{dx^2}F'(y^{(2)})-\cdots(-1)^m\frac{d^m}{dx^m}F'(y^{(m)}),$$

in which one sets $\frac{\partial F}{\partial y^{(r)}} = F'(y^{(r)})$, be denoted by S. Since the values of z and its first $n-1$ derivatives vanish at $x =$ and b, the first variation of (1) will be:

(5) $$\int_a^b z\,Q\,dx$$

for $\varepsilon = 0$, and the first variation of (2) will be:

(6) $$\int_a^b z\,S\,dx$$

for $\varepsilon = 0$. In order for the latter integral to vanish, according to **Cauchy** (see **Duhamel**'s *Cours d'Analyse* or **Serret**'s *Cours de Calcul diff. et int.*), one must set:

(7) $$z\,S = \varphi'(x) = \frac{d\varphi(x)}{dx},$$

and take:

(8) $$\varphi(x) = (x-a)^{n+1}(x-b)^{n+1}w.$$

It will emerge from this that:

(9) $$z = \frac{\varphi'(x)}{S}.$$

When that is substituted in (5), that will give:

(10) $$\int_a^b \varphi'(x)\frac{Q}{S}dx = -\int_a^b \varphi(x)\frac{d}{dx}\frac{Q}{S}dx.$$

Since w is arbitrary, it will follow that:

(11) $$\frac{d}{dx}\frac{Q}{S} = 0 ,$$

(12) $$\frac{Q}{S} = c ,$$

in which c is a constant. y will be obtained as a function of x, c, and $2n$ other constants that are real for $x = a$ to b by integrating the $2n^{\text{th}}$-order differential equation:

(13) $$Q - c\,S = 0 .$$

The constants are determined from the given values of y and its first $n - 1$ derivatives at $x = a$ and b and from the given value of the integral (2).

The same assumptions will then be made in regard to the function y and the functions:

$$f\,(x, y, y^{(1)}, \ldots, y^{(n)}) , \quad \frac{\partial f}{\partial y^{(p)}} , \quad \frac{\partial^2 f}{\partial y^{(p)}\,\partial y^{(q)}} , \qquad \frac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$$

that were made in [no.**2**, after (3)]. *The same assumptions that were made in* [no. **2**, near (4), (5), (6)] *will be made in regard to the functions* $F\,(x, y, y^{(1)}, \ldots, y^{(m)})$, $\dfrac{\partial F}{\partial y^{(p)}}$.

Moreover, the differential expression S (4) *shall be a single-valued and continuous analytic function of x that does not vanish identically in the strip that was denoted by T in* [no. **2**, after (3)].

S will then be equal to zero at a finite number of points along the interval of x from a to b. Let those points be ζ_1 to ζ_μ. Select consecutive subintervals θ along the interval from a to b, each of which includes only one point ζ; let them be:

(14) $$\theta_1, \theta_2, \ldots, \theta_\mu .$$

Now let w in (8) be an arbitrary finite and continuous real function along the interval of x from a to b whose derivatives up to order $2n + 1$ are likewise finite and continuous there and which is equal to zero along the aforementioned subintervals from θ_1 to θ_μ, but does not vanish everywhere along the interval from a to b. (Such a function can be constructed from the functions [no. **2**, (9), in which one sets $2n + 2$, instead of $2n + 1$, and $2n + 1$, instead of $2n$.)

Since $\varphi\,(x)$ in (8) is not constantly equal to zero everywhere from $x = a$ to b, $\varphi'\,(x)$ will not vanish everywhere either.

The expression (9) *for z is therefore a function that is finite and continuous along the interval of x from a to b, along with its derivatives up to order* 2*n*, *and vanishes for x = a and b, along with*

its derivatives up to order $n - 1$, *and it is equal to zero along the subintervals* η_1 *to* η_λ [no. **2**, (8)] *and* θ_1 *to* θ_μ (14), *but it does not vanish everywhere from a to b*.

The first variation (6) of the integral (1) is equal to zero for $\varepsilon = 0$ since it follows from (13) that:

$$\int_a^b z\,Q\,dx - c\int_a^b z\,S\,dx = 0\,. \tag{15}$$

Since the first variation of the integral (2) vanishes for $\varepsilon = 0$, the requirement that was imposed upon that integral is fulfilled. Since the first variation of the integral (1) vanishes for $\varepsilon = 0$, one must now deal with the second variation of that integral from now on. That variation is treated as in no. **2** by means of the assumption that was made and properties of z that were given there, and the result will be the same as in no. **2**.

When the assumption that was given above is fulfilled, the final result will therefore be this:

The curve that enters into the integrals (1) *and* (2) *is not the curve y that is determined by integrating the differential equation* (13), *but a curve* $y + \varepsilon z$, *where z is the function with the properties that were indicated before. For those families of neighboring curves to y that come about as* ε *becomes infinitely small, the integral* (1) *will be a maximum (minimum, respectively) according to the sign of* $\dfrac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}$, *which will make the change in the integral* (2), *divided by* ε, *become infinitely small*.

5. – Generalizing the results of no. 4.

The generalization takes the same form as in no. **3**. One replaces the z in $y + \varepsilon z$ with:

$$\begin{cases} z + \varepsilon Z\,, \\ Z = (x-a)^n\,(x-b)^n\,\omega(x,\varepsilon)\,, \end{cases} \tag{1}$$

in which $\omega\,(x, \varepsilon)$ is a function with the properties [no. **3**, (2) and (3)]. That implies that the first variation of the integral [no. **4**, (1) and (2)] will again have the expression [no. **4**, (5) or (6), resp.], so it will vanish. The second variation of the integral [no. **4**, (1)] has the previous expression for $\varepsilon = 0$, to which one adds the integral [no. **4**, (5)], multiplied by 2, and in which $Z\,(\varepsilon = 0)$ replaces z. In order for the latter integral to vanish, one sets:

$$\omega\,(x, \varepsilon) = h\,(x) + \varepsilon\,\Omega\,(x, \varepsilon)\,, \tag{2}$$

in which $\Omega(x, \varepsilon)$ is a function of the same type as $\omega(x, \varepsilon)$ [no. **3**, (2), (3)]. (An example of such a thing is $\sum_{\lambda=0}^{\infty} \varepsilon^{\lambda} \psi_{\lambda}(x)$, where $\psi_{\lambda}(x)$ and $\frac{d^r \psi_{\lambda}(x)}{dx^r}$ $(r = 1, \ldots, n)$ are real, finite, and continuous for x from a to b, and their absolute values remain below one and the same positive constant G):

$$(x-a)^n (x-b)^n h(x) \tag{3}$$

is an expression of the form [no. **4**, (9)], in which $w = w_0(x)$, and $w_0(x)$ and $\frac{d^r w_0(x)}{dx^r}$ $(r = 1, \ldots, n + 1)$ are real, finite, and continuous for $x = a$ to b, *and are equal to zero along the subinterval* [no. **4**, (14)].

Hence, when the assumption in no. **4** has been made, *the general families* of curves $y + \varepsilon(z + \varepsilon Z)$ that are neighboring to y will fulfill the requirement that for an ε that becomes infinitely small, the integral [no. **4**, (1)] will be a maximum (minimum, respectively), and the change in the integral [no. **4**, (2)], when divided by ε, will become infinitely small.

Section Three

6. – Examples from Section One.

I. – *Shortest line between two points*.

The integral [no. **1**, (1)] is:

$$\int_a^b \sqrt{1+(y^{(1)})^2}\, dx\,, \tag{1}$$

so

$$f = \sqrt{1+(y^{(1)})^2}\,, \qquad f'(y) = 0\,, \qquad f'(y^{(1)}) = \frac{y^{(1)}}{\sqrt{1+(y^{(1)})^2}}\,. \tag{2}$$

The differential equation [no. **1**, (3)]:

$$f'(y) - \frac{d}{dx} f'(y^{(1)}) = 0 \tag{3}$$

gives $y^{(1)} = \text{const.}$, $y = c_1 x + x_2$. For $x = a$, $y = \alpha$, and $x = b$, $y = \beta$:

$$y - \alpha = \frac{\beta - \alpha}{b - a}(x - a)\,, \tag{4}$$

(5) $$\frac{\partial^2 f}{\partial y\,\partial y} = 0\,, \qquad \frac{\partial^2 f}{\partial y\,\partial y^{(1)}} = 0\,, \qquad \frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{1}{(1+(y^{(1)})^2)^{1/2}}\,.$$

The assumption in [no. **2**, after (3)] in regard to y and the expressions (2), (5) is fulfilled. The sign of $\frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}}$ is positive, so one is dealing with a minimum.

II. – *Meridian curve of a surface of revolution of smallest area.*

Except for the factor $2n$, and when the x-axis is taken to be the axis of rotation, while y is positive, the integral [no. **1**, (1)] will be:

(6) $$\int_a^b y\sqrt{1+(y^{(1)})^2}\,dx\,,$$

(7) $$f = y\sqrt{1+(y^{(1)})^2}\,, \qquad f'(y) = y\sqrt{1+(y^{(1)})^2}\,, \qquad f'(y^{(1)}) = \frac{y\,y^{(1)}}{\sqrt{1+(y^{(1)})^2}}\,.$$

The differential equation [no. **1**, (3)]:

(8) $$f'(y) - \frac{d}{dx} f'(y^{(1)}) = 0$$

has the first integral:

(9) $$f - f'(y^{(1)})\,y^{(1)} = c_1\,,$$

so

(10) $$\frac{y}{\sqrt{1+(y^{(1)})^2}} = c_1\,.$$

That implies:

(11) $$y = \tfrac{1}{2}\,c_1[e^{(x-c_2)/c_1} + e^{-(x-c_2)/c_1}]\,.$$

By displacing the origin of the coordinate system along the x-axis, (11) will become:

(12) $$y = \tfrac{1}{2}\,c_1[e^{x/c_1} + e^{-x/c_1}]\,.$$

The curve (12) is based upon the integral (6), so c_1 is positive since y is supposed to be positive, while the endpoints of y shall lie along the catenary (12) at $x = a$ and b.

(13) $$\frac{\partial^2 f}{\partial y\,\partial y} = 0\,, \qquad \frac{\partial^2 f}{\partial y\,\partial y^{(1)}} = \frac{y^{(1)}}{(1+(y^{(1)})^2)^{1/2}}\,, \qquad \frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{y}{(1+(y^{(1)})^2)^{1/2}}\,.$$

The assumption in [no. **2**, after (3)] in regard to y and the expressions (7) and (13) is fulfilled. The sign of $\dfrac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}}$ is positive, so one is dealing with a minimum.

III. – *Brachistochrone*.

The positive y-axis points in the direction of gravity, g denotes the weight per unit mass, and v is the velocity of the mass-point that moves along the curve. Let $y = 0$ when $x = a = 0$, and let $y = \beta$ for $x = b$. One has:

(14) $$\tfrac{1}{2}v^2 = g\,y + \text{const.},$$

or

(15) $$\tfrac{1}{2}v^2 = g\,(y + k)\,.$$

Let the constant $k > 0$ here. The integral [no. **1**, (1)], which will express the time elapsed during the motion when it is multiplied by $1/\sqrt{2g}$, is:

(16) $$\int_a^b \sqrt{\frac{1+(y^{(1)})^2}{y+k}}\,dx\,,$$

(17) $$f = \sqrt{\frac{1+(y^{(1)})^2}{y+k}}\,, \qquad f'(y) = -\frac{\sqrt{1+(y^{(1)})^2}}{2\,(y+k)^{3/2}}\,, \qquad f'(y^{(1)}) = \frac{y^{(1)}}{\sqrt{(y+k)(1+(y^{(1)})^2)}}\,.$$

The differential equation [no. **1**, (3)]:

(18) $$f'(y) - \frac{d}{dx}\,f'(y^{(1)}) = 0$$

has the first integral:

(19) $$f - f'(y^{(1)})\,y^{(1)} = c_1\,,$$

so

(20) $$(y+k)(1+(y^{(1)})^2) = \frac{1}{c_1^2} = 2\,R\,.$$

Thus:

(21) $$(y^{(1)})^2 = \frac{2R-(y+k)}{y+k}\,,$$

(22) $$\frac{dy}{dx} = \sqrt{\frac{2R-(y+k)}{y+k}}\,.$$

The differential equations (22), and therefore (20), will be satisfied by the coordinates of the cycloid:

(23) $$x + h = R\,(\theta - \sin\,\theta)\,,$$

(24) $$y + h = R\,(1 - \cos\,\theta)\,.$$

Take the interval for the angle θ to go from θ_0 to θ_1, where:

(25) $$0 < \theta_0 < \theta_1 < 2\pi\,.$$

For a given positive value R and arbitrarily-taken values of θ_0, k can be determined from the equation:

(26) $$k = R\,(1 - \cos\,\theta_0)\,,$$

and h can be determined from the equation:

(27) $$h = R\,(\theta - \sin\,\theta_0)\,.$$

One then takes:

(28) $$b + h = R\,(\theta_1 - \sin\,\theta_1)\,,$$

(29) $$\beta + h = R\,(1 - \cos\,\theta_1)$$

for θ_1 . x is a single-valued and continuous analytic function of θ, and $dx\,/\,d\theta = R\,(1 - \cos\,\theta)$ does not vanish in the interval from θ_0 to θ_1. It will then follow that as a function of x, θ gives a strip T in the construction plane of the complex variable x inside of which the segment $x = 0$ to b lies, and in which θ is a single-valued and continuous analytic function of x. y is also a single-valued and continuous analytic function of x within that strip then. $y + k$ is non-zero from θ_0 to θ_1, so from $x = 0$ to $x = b$, and it will therefore remain non-zero inside of such a strip T. One has:

(30) $$\frac{\partial^2 f}{\partial y\,\partial y} = \frac{3}{4}\frac{\sqrt{1+(y^{(1)})^2}}{(y+k)^{1/2}}, \quad \frac{\partial^2 f}{\partial y\,\partial y^{(1)}} = -\frac{y_1}{2(y+k)^{1/2}\sqrt{1+(y^{(1)})^2}}, \quad \frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{1}{\sqrt{y+k}\,(1+(y^{(1)})^2)^{3/2}}.$$

The assumption in [no. **2**, after (3)] in regard to y and the expressions (17) and (30) is fulfilled. The sign of $\dfrac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}}$ is positive, so one is dealing with a minimum.

The point $\theta = 0$ on the cycloid (23), (24) has the coordinates $x = -h$, $y = -k$, with the relations (26) to (29), When $v = 0$ at this point, equation (15) will follow from equation (14), so the mass-point will achieve the velocity at $x = 0$, $y = 0$ that emerges from equation (15), which is used as a basis here. The neighboring curves have the points θ_0 or $x = 0$, $y = 0$ and θ_1 or $x = b$, $y = \beta$ in common with the cycloid. In order for the tangents to the neighboring curves to coincide with that of the cycloid at those two points, since n is equal to 1 here, from [no. **1**, (1)], z [no. **1**] and Z [no. **3**], which vanish at $x = a$ and b along with their first $n - 1$ derivatives, must be chosen such that the nth derivative also vanishes at $x = a$ ($x = 0$, here) and $x = b$. Now, when the mass-point moves along the cycloid from the point $\theta = 0$ to $\theta = \theta_0$ with an initial velocity of zero, then goes over to a neighboring curve, and again comes back to the cycloid at the point $\theta = \theta_1$, that will give the case that is treated here.

7. – Examples from Section Two.

I – *Meridian curves of given length on a surface of revolution of least area.*

Except for the factor 2π, when the x-axis is taken to be the axis of rotation and y is taken to be positive, the integral [no. **4**, (1)] will be:

(1) $$\int_a^b y\sqrt{1+(y^{(1)})^2}\,dx,$$

while the integral [no. **4**, (2)]:

(2) $$\int_a^b \sqrt{1+(y^{(1)})^2}\,dx,$$

One has:

(3) $$f = y\sqrt{1+(y^{(1)})^2}, \qquad f'(y) = \sqrt{1+(y^{(1)})^2}, \qquad f'(y^{(1)}) = \frac{y\,y^{(1)}}{\sqrt{1+(y^{(1)})^2}},$$

(4) $$F = \sqrt{1+(y^{(1)})^2}, \qquad F'(y) = 0, \qquad F'(y^{(1)}) = \frac{y^{(1)}}{\sqrt{1+(y^{(1)})^2}}.$$

The differential equation [no. **4**, (13)]

(5) $$Q - c\,S = 0$$

has the first integral:

(6) $$f - f'(y^{(1)})\,y^{(1)} - c\,(F - F'(y^{(1)})\,y^{(1)}) = k_1 ,$$

so

(7) $$\frac{y-c}{\sqrt{1+(y^{(1)})^2}} = k_1 .$$

Hence:

(8) $$y - c = \tfrac{1}{2}\,k_1[e^{(x-k_2)/k_1} + e^{-(x-k_2)/k_1}] .$$

Upon shifting the origin of the coordinate system along the x-axis, one will get:

(9) $$y - c = \tfrac{1}{2}\,k_1[e^{x/k_1} + e^{-x/k_1}] .$$

The curve (9) is based upon the integrals (1) and (2), while y is positive. Let the points $x = a$ and b be such that the given curve length applies to the catenary $k_1 > 0$ or $k_1 < 0$:

(10) $$Y = \tfrac{1}{2}\,k_1[e^{x/k_1} + e^{-x/k_1}]$$

between $x = a$ and b. The endpoints shall then lie on the curve (9).

(11) $$\frac{\partial^2 f}{\partial y\,\partial y} = 0 , \qquad \frac{\partial^2 f}{\partial y\,\partial y^{(1)}} = \frac{y^{(1)}}{\sqrt{1+(y^{(1)})^2}} , \qquad \frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{y}{(1+(y^{(1)})^2)^{3/2}} .$$

(12) $$S = -\frac{d}{dx}\frac{y^{(1)}}{\sqrt{1+(y^{(1)})^2}} .$$

The assumption in no. **4** in regard to y and the expressions (3), (4), (11), (12) is fulfilled. S is not identically zero since otherwise $y^{(1)}$ would be constant. The sign of $\frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}}$ is positive, so (cf., no. **5**) one is dealing with a minimum.

(13) $$\frac{\partial^2 F}{\partial y\,\partial y} = 0 , \qquad \frac{\partial^2 F}{\partial y\,\partial y^{(1)}} = 0, \qquad \frac{\partial^2 F}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{1}{(1+(y^{(1)})^2)^{3/2}} .$$

At the same time, the integral (2) will be a minimum when z is zero along not only the subintervals [no. **2**, (8)], but also along the analogous subinterval according to the treatment of the integral (2) that was described in no. **2**.

II. – *Meridian curves of shortest length on a surface of revolution of given volume.*

The integral [no. **4**, (1)] is:

(14) $$\int_a^b \sqrt{1+(y^{(1)})^2}\, dx\,.$$

Except for the factor of π, when the x-axis is taken to be the rotational axis, the integral [no. **4**, (2)] will be:

(15) $$\int_a^b y^2\, dx\,.$$

One has:

(16) $$f = \sqrt{1+(y^{(1)})^2}\,, \qquad f'(y) = 0\,, \qquad f'(y^{(1)}) = \frac{y^{(1)}}{\sqrt{1+(y^{(1)})^2}}\,,$$

(17) $$F = y^2\,, \qquad F'(y) = 2\,y, \qquad F'(y^{(1)}) = 0\,.$$

The differential equation [no. **4**, (13)]:

(18) $$Q - c\,S = 0$$

has the first integral:

(19) $$f - f'(y^{(1)})\,y^{(1)} - c\,(F - F'(y^{(1)})\,y^{(1)}) = k_1\,,$$

so:

(20) $$\frac{1}{\sqrt{1+(y^{(1)})^2}} - c\,y^2 = k\,.$$

Thus:

(21) $$(y^{(1)})^2 = \frac{1-(cy^2+k)^2}{(cy^2+k)^2}\,,$$

(22) $$\frac{dy}{dx} = \pm\sqrt{\frac{1-(cy^2+k)^2}{(cy^2+k)^2}}\,,$$

(23) $$x = \int_\alpha^y \pm\sqrt{\frac{(cy^2+k)^2}{1-(cy^2+k)^2}}\, dy + a\,.$$

c and k are real constants. y shall move along the interval from α to β, which does not include zero, such that $(cy^2+k)^2$ will not vanish and will be less than 1. The sign of the square root is chosen such that the quantity under the integral sign will prove to be positive. x is a single-valued and continuous analytic function of y along the interval of y from α to β and in its neighborhood that will be equal to a for $y = \alpha$ and equal to b for $y = \beta$. Since dx / dy does not vanish there, the inverse function y will be a single-valued and continuous analytic function of x that is real for real x in a strip of the complex variable x that includes the interval of x from a to b in its interior. Let the volume of the body of revolution between the planes $x = a$ and $x = b$ be given.

(24) $$\frac{\partial^2 f}{\partial y\,\partial y} = 0\,, \qquad \frac{\partial^2 f}{\partial y\,\partial y^{(1)}} = 0\,, \qquad \frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{1}{(1+(y^{(1)})^2)^{3/2}}\,,$$

(25) $$S = 2\,y\,.$$

The assumption in no. **4** in regard to y and the expressions (16), (17), (24), (25) is fulfilled. S is nowhere-zero, so the exceptional locations for θ [no. **4**, (14)] do not exist here. The sign of $\dfrac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}}$ is positive. When the family of neighboring curves to y in no. **5** do not come under consideration, one will be dealing with a minimum.

III. – *Meridian curves on a surface of revolution of least area over a given volume.*

When the x-axis is the rotational axis and y is taken to be positive, except for the factor of 2π, the integral [no. **4**, (1)] will be:

(26) $$\int_a^b y\sqrt{1+(y^{(1)})^2}\,dx\,.$$

Except for the factor of π, the integral [no. **4**, (2)] will be:

(27) $$\int_a^b y^2\,dx\,.$$

One has:

(28) $$f = y\sqrt{1+(y^{(1)})^2}\,, \qquad f'(y) = \sqrt{1+(y^{(1)})^2}\,, \qquad f'(y^{(1)}) = \frac{y\,y^{(1)}}{(1+(y^{(1)})^2)^{3/2}}\,,$$

(29) $$F = y^2\,, \qquad F'(y) = 2\,y, \qquad F'(y^{(1)}) = 0\,.$$

The differential equation [no. **4**, (13)]:

(30) $$Q - c\,S = 0$$

has the first integral:

(31) $$f - f'(y^{(1)})\,y^{(1)} - c\,(F - F'(y^{(1)})\,y^{(1)}) = k\,,$$

so:

(32) $$\frac{y}{\sqrt{1+(y^{(1)})^2}} - c\,y^2 = k\,.$$

Thus:

(33) $$(y^{(1)})^2 = \frac{y^2 - (cy^2+k)^2}{(cy^2+k)^2}\,,$$

(34) $$\frac{dy}{dx} = \pm\sqrt{\frac{y^2 - (cy^2+k)^2}{(cy^2+k)^2}}\,,$$

(35) $$x = \int_\alpha^y \pm\sqrt{\frac{(cy^2+k)^2}{1-(cy^2+k)^2}}\,dy + a\,.$$

c and k are real constants. y, which is positive, shall move along the segment from α to β such that $(cy^2+k)^2$ will not vanish, and $y^2 > (cy^2+k)^2$. The sign of the square root is such that the quantity under the integral sign will prove to be positive. x is a single-valued and continuous analytic function of y along the interval of y from α to β and in the neighborhood of it that is equal to a for $y = \alpha$ and equal to b for $y = \beta$. dx / dy does not vanish there so the inverse function y will be a single-valued and continuous analytic function of x that is real when x is real inside of a strip that includes the segment of x from a to b in its interior. Let the volume of the body of rotation between the planes $x = a$ and $x = b$ be given.

(36) $$\frac{\partial^2 f}{\partial y\,\partial y} = 0\,, \qquad \frac{\partial^2 f}{\partial y\,\partial y^{(1)}} = \frac{y^{(1)}}{\sqrt{1+(y^{(1)})^2}}\,, \qquad \frac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}} = \frac{y}{(1+(y^{(1)})^2)^{3/2}}\,,$$

(37) $$S = 2\,y\,.$$

The assumption in no. **4** in regard to y and the expressions (28), (29), (36), (37) is fulfilled. S is nowhere-zero, so the exceptional locations for θ [no. **4**, (14)] do not exist here. The sign of $\dfrac{\partial^2 f}{\partial y^{(1)}\,\partial y^{(1)}}$ is positive. One will be dealing with a minimum (cf., no. **5**).