"Über eine Anwendung der Theorie der linearen Differentialgleichungen in der Variationsrechnung," J. reine angew. Math. 128 (1905), 33-44.

# On an application of the theory of linear differential equations to the calculus of variations. 

Supplement to the author's article in Bd. 125 of this journal.

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Translated by D. H. Delphenich

The theory of the second variation of a simple integral with one unknown function that Jacobi founded in volume 17 of this journal and Hesse developed further in volume 54 was connected with the theory of linear differential equations with analytic functions as coefficients in the treatise by the author in Bd. 125, and indeed the foundations of the latter theory come under consideration in that way.

The assumption was made there that in a strip in the construction plane of the complex independent variables that includes the segment on the real axis between the limits of integration in its interior, the real function that is determined by setting the first variation of the integral along that segment equal to zero is a single-valued and continuous analytic function. The first and second-order partial derivatives of the expression under the integral sign that one takes with respect to the function and its differential quotients shall be single-valued and continuous analytic functions of the independent variables. Let the Jacobi condition be fulfilled that the second partial derivatives of that expression with respect to the highest differential quotients does not vanish between the limits of integration.

If those conditions are fulfilled then, as was proved in the aforementioned treatise, the curve that is found will have the property that the integral will be a maximum or minimum for it (according to the sign of the last second partial derivative) when one appeals to the family of neighboring curves for the sake of comparison, in general. In the present article, it will be shown that on the basis of Jacobi's theory and the previously-applied theorems from the theory of linear differential equations, the restrictions that were made before can be essentially omitted.

Thus, under the given assumption, the maximum (minimum, respectively) of the integral will be assured from the outset with the originally-given integration limits when the curve that was found belongs to the family of neighboring curves and that family of neighboring curves is taken to be fully general.

These investigations into the calculus of variations might be dedicated to the memory of Jacobi as a contribution to the centennial of his birthday that falls in the year 1904.

## 1. - Review of the contents of Section One in the author's treatise in Bd. 125.

There (no. 1), the integral:

$$
\begin{equation*}
\int_{a}^{b} f\left(x, y, y^{(1)}, \ldots, y^{(n)}\right) d x \tag{1}
\end{equation*}
$$

is given, in which $f$ is a real function of $x, y, y^{(1)}$, to $y^{(n)} . y$ is the unknown real function of $x, y^{(r)}$ $=d^{r} y / d x^{r}$, and $a$ and $b$ are real. $y$ is set equal to $y+\varepsilon z$, in which $\varepsilon$ is a quantity that varies in the neighborhood of zero, $z$ is an arbitrary real function of $x$ that remains finite and continuous from $a$ to $b$, along with its derivatives up to order $2 n$, and vanishes at $x=a$ and $b$, along with its first $n-1$ derivatives. Initially, $z$ might be zero in arbitrary neighborhoods of a finite number of points between $a$ and $b$.

The first differential quotient of the integral (1) with respect to $\varepsilon$ - viz., the first variation must vanish for $\varepsilon=0$. That leads to the differential equation, in which one sets:

$$
\begin{equation*}
\frac{\partial f}{\partial y^{(p)}}=f^{\prime}\left(y^{(p)}\right) \tag{2}
\end{equation*}
$$

namely:

$$
\begin{equation*}
f^{\prime}(y)-\frac{d}{d x} f^{\prime}\left(y^{(1)}\right)+\frac{d^{2}}{d x^{2}} f^{\prime}\left(y^{(2)}\right)-\cdots(-1)^{n} \frac{d^{n}}{d x^{n}} f^{\prime}\left(y^{(n)}\right)=0 . \tag{3}
\end{equation*}
$$

$y$ emerges from that differential equation as a real function between $a$ and $b$ that has given values at the endpoints $x=a$ and $b$, along with its first $n-1$ derivatives.

The second differential quotient of the integral (1) with respect to $\varepsilon-$ viz., the second variation - will be given by the following expression for $\varepsilon=0$. Let the notations:

$$
\begin{gather*}
\frac{\partial^{2} f}{\partial y^{(p)} \partial y^{(q)}}=a_{p q},  \tag{4}\\
\frac{d^{r} z}{d x^{r}}=z^{(r)}, \tag{5}
\end{gather*}
$$

and

$$
\begin{gather*}
2 \psi=a_{00} z z+2 a_{01} z z^{(1)}+2 a_{11} z^{(1)} z^{(1)}+\cdots+2 a_{n-1, n} z^{(n-1)} z^{(n)}+a_{n n} z^{(n)} z^{(n)},  \tag{6}\\
\frac{\partial \psi}{\partial z^{(r)}}=\psi^{\prime}\left(z^{(r)}\right)
\end{gather*}
$$

be given. In order to convert the expression for the second variation of (1), which comes about by differentiation under the integral sign at $\varepsilon=0$, while taking into account the fact that $z$ vanishes at $x=a$ and $b$, along with its first $n-1$ derivatives, one considers:

$$
\begin{equation*}
\psi^{\prime}(y)-\frac{d}{d x}\left(z^{(1)}\right)+\frac{d^{2}}{d x^{2}} \psi^{\prime}\left(z^{(2)}\right)-\cdots(-1)^{n} \frac{d^{n}}{d x^{n}} \psi^{\prime}\left(z^{(n)}\right) . \tag{8}
\end{equation*}
$$

The latter expression can always be put into the form:

$$
\begin{equation*}
\mathfrak{A}_{0} z-\frac{d}{d x} \mathfrak{A}_{1} z^{(1)}+\frac{d^{2}}{d x^{2}} \mathfrak{A}_{2} z^{(2)}-\cdots(-1)^{n} \frac{d^{n}}{d x^{n}} \mathfrak{A}_{n} z^{(n)}=\Psi(z), \tag{9}
\end{equation*}
$$

in which the $\mathfrak{A}_{n}, \mathfrak{A}_{n-1}$, down to $\mathfrak{A}_{1}$, are entire rational functions of the quantities $a_{p q}$ in (6), and whose derivatives with respect to $x$ are:

$$
\begin{equation*}
\mathfrak{A}_{n}=a_{n n}=\frac{\partial^{2} f}{\partial y^{(n)} \partial y^{(n)}} . \tag{10}
\end{equation*}
$$

The second variation of the integral (1) for $\varepsilon=0$ will now have the expression:

$$
\begin{equation*}
\int_{a}^{b} z \Psi(z) d x \tag{11}
\end{equation*}
$$

One must now investigate whether the expression (11) will keep one and the same sign for the various functions $z$. Since $z$ vanishes at $x=a$ and $b$, along with its first $n-1$ derivatives, according to Jacobi's theory, that will give the following representation for the integral (11):

$$
\begin{equation*}
\int_{a}^{b} a_{n n}\left(u v_{1}^{(1)} w_{2}^{(2)} \ldots z_{n}^{(n)}\right)^{2} d x \tag{12}
\end{equation*}
$$

[Treatise in Bd. 125 of this Journal, no. 1, (30)] Here, $u$, $v, w$, etc., are $n$ integrals of the homogeneous linear differential equation of order $2 n$ :

$$
\begin{equation*}
\frac{1}{\mathfrak{A}_{n}} \Psi(z)=0 \tag{13}
\end{equation*}
$$

by means of which the $n$ functions $u, v_{1}^{(1)}, w_{2}^{(2)}, \ldots$ that were given in loc. cit. are constructed. The latter functions shall be real, finite, continuous, and nowhere-vanishing along the interval from $a$ to $b . z_{n}^{(n)}$ emerges from $z$ by the substitutions:

$$
\begin{equation*}
z=u z_{1}, \quad \frac{d z_{1}}{d x}=z_{1}^{(1)}=v_{1}^{(1)} z_{1}^{(1)}, \quad \frac{d z_{2}^{(1)}}{d x}=z_{2}^{(2)}=w_{2}^{(2)} z_{3}^{(2)}, \tag{14}
\end{equation*}
$$

The expression that enters into (12):

$$
\begin{equation*}
u v_{1}^{(1)} w_{2}^{(2)} \ldots z_{n}^{(n)} \tag{15}
\end{equation*}
$$

was represented by Hesse (cf., the author's articles in Bd. 125 of this journal, pps. 7, 8) as the quotient:

$$
\begin{equation*}
\frac{\Delta}{\Delta_{n}} \tag{16}
\end{equation*}
$$

in which:

$$
\begin{equation*}
\Delta_{n}=u^{n}\left(v_{1}^{(1)}\right)^{n-1}\left(w_{2}^{(2)}\right)^{n-2} \ldots \tag{17}
\end{equation*}
$$

is the determinant of the $n$ functions $u, v, w, \ldots$, and its first $n-1$ derivatives, and:

$$
\begin{equation*}
\Delta=u^{n+1}\left(v_{1}^{(1)}\right)^{n}\left(w_{2}^{(2)}\right)^{n-1} \ldots z_{n}^{(n)} \tag{18}
\end{equation*}
$$

is the determinant of the $n+1$ functions $u, v, w$, up to $z$, and their first $n$ derivatives.
$y$ will then emerge as a function of $x$ with $2 n$ constants by integrating the differential equation (3), and the latter shall be determined such that $y$, along with its first $n-1$ derivatives, assumes prescribed values at $x=a$ and $b$, where $y$ is now real between $a$ and $b$.

As was said in the introduction (for the details, see the article in Bd. 125 of this journal, pp. 10), one then makes the following assumption: Let the function $y$ to be determined be a singlevalued and continuous analytic function of $x$ in a strip $T$ in the construction plane of the complex variables $x$ that includes the segment along the real axis between $a$ and $b$. When $y, y^{(1)}$, up to $y^{(n)}$ are taken to be independent, the function $f\left(x, y, y^{(1)}, \ldots, y^{(n)}\right)$ in (1) and its first and second-order partial derivatives $y, y^{(1)}$, up to $y^{(n)}$ shall be finite and continuous real functions of $x, y, y^{(1)}$, up to $y^{(n)}$ along real intervals that include the values of those variables in their interior when $x$ varies from $a$ to $b$. Moreover, let the functions $\frac{\partial f}{\partial y^{(p)}}, \frac{\partial^{2} f}{\partial y^{(p)} \partial y^{(q)}}(p, q=0, \ldots, n)$ be single-valued and continuous analytic functions of $x$ in the strip $T$, and let $\frac{\partial^{2} f}{\partial y^{(n)} \partial y^{(n)}}=a_{n n}$ be nowhere-vanishing along the interval of $x$ from $a$ to $b$.

The quantity $\mathfrak{A}_{n}=a_{n n}$ in (13) will not vanish in a strip like $T$ then, and the theory of linear differential equations with analytic functions as coefficients could then be used as the basis for an application to the differential equation (13) in the treatise in Bd. 125 of this journal. That led to the following result:

Select a finite number of points $\xi_{1}, \xi_{2}$, up to $\xi_{\lambda}$ along the interval of $x$ from $a$ to $b$ and then take consecutive subintervals:

$$
\begin{equation*}
\eta_{1}, \eta_{2}, \ldots, \eta_{\lambda} \tag{19}
\end{equation*}
$$

each of which includes a point $\xi$ in its interior. Those subintervals can be chosen to be arbitrarily small, but they shall be fixed. There are $n$ integrals $u, v, w, \ldots$ of the differential equation (13) in a strip like $T$ that are single-valued and continuous analytic functions and are real when $x$ is real, and $n$ functions $u, v_{1}^{(1)}, w_{2}^{(2)}, \ldots$ as in (12) will emerge from them that will be single-valued and continuous non-zero analytic functions that are real when $x$ is real in a region inside of $T$ that includes the segment of the real line from $a$ to $b$ in its interior, except for the points $\xi(a, b$ can also belong to them).

The integration path from $a$ to $b$ in (11) is divided into pieces, namely, the segments $\eta$ in (19) and the segments between any two $\eta$ (between $a$ or $b$ and the neighboring $\eta$, respectively). The function $z$, which is real, finite, and continuous from $a$ to $b$, along with its first $2 n$ derivatives, and vanishes at $a$ and $b$, along with its first $n-1$ derivatives, is set equal to zero on the segments $\eta$, but shall not vanish everywhere. The conditions on the functions $u, v_{1}^{(1)}, w_{2}^{(2)}, \ldots$ that allow one to express the integral in the form (12) are fulfilled on each of the remaining segments. That integral will have the sign of $a_{n n}$ as long as $z$ does not vanish everywhere there. The integral (11) will then have the sign of $a_{n n}$ for every such function $z$.

In the present article, we shall also deal with functions $z$ in the integral (11) that are finite and continuous from $a$ to $b$, along with their first $2 n$ derivatives, and vanish at $a$ and $b$, along with their first $n-1$ derivatives and do not vanish on the subintervals $\eta$ in (19).

The generalization that was given in no. 3 of the treatise in Bd. 125 of this journal, which replaced the $y+\varepsilon z$ that originally replaced $z$ in the integral (1) with $z+\varepsilon Z$, where $z$ is the previous function, and $Z$ is a function of $x$ and $\varepsilon$ with the behavior that was cited in loc. cit., will also remain valid for the $z$ that is to be constructed now.

## 2. - Addressing the problem that was posed in the conclusion of no. 1 .

I. - At each of the points $\xi$ on the subinterval $\eta\left[\right.$ no. 1, (19)], take $n$ functions $u, v_{1}^{(1)}, w_{2}^{(2)}, \ldots$ as in [no. 1, (12)] that are single-valued, continuous, and non-zero analytic functions of $\xi$ and in the neighborhood of $\xi$ and are real when $x$ is real by means of the homogeneous linear differential equation $\frac{1}{\mathfrak{A}_{n}} \Psi(z)=0$ in [no. 1, (13)]. That will happen as a result of what was given on pp. 11 in the treatise in Bd. 125 of this journal. Let the segment $\eta$ in [no. 1, (19)] be a segment in that region along the real axis between $a$ and $b$ that includes only one of the points $\xi$ in its interior. Let the endpoints of $\eta$ be $a^{\prime}$ and $b^{\prime}$. Now, the conditions on the integral:

$$
\begin{equation*}
\int_{a^{\prime}}^{b^{\prime}} z \Psi(z) d x \tag{1}
\end{equation*}
$$

that lead to its representation as in [no. 1, (12)] are fulfilled by means of those functions $u, v_{1}^{(1)}$, $w_{2}^{(2)}, \ldots$ Thus, if the function $z$ is an arbitrary real function along the interval $\eta$ that remains finite and continuous, along with its first $2 n$ derivatives, and vanishes at $a^{\prime}$ and $b^{\prime}$, along with its first $n$ -1 derivatives, but not everywhere, then the integral (1) will have the same sign as $a_{n n}$.

Let the endpoints of the interval $\eta$ in [no. 1, (19)] be denotes by $\xi^{\prime}$, in general. (The points $a$ or $b$ can also belong to the points $\xi^{\prime}$.) The integral [no. 1, (11)] is equal to the sum of the integrals between each two successive points $\xi^{\prime}$ and the integral between $a$ ( $b$, respectively) and the next point $\xi^{\prime}$. Let the function $z$ in [no. 1, (11)], which is real, finite, and continuous from $a$ to $b$, but not equal to zero everywhere, along with its first $2 n$ derivatives, be equal to zero at the points $\xi^{\prime}$, along with its first $\mathrm{n}-1$ derivatives, in addition to the points $a$ and $b$, but otherwise arbitrary. From what was said in no. $\mathbf{1}$ and in the foregoing, the integral [no. 1, (11)] will then have the same sign as $a_{n n}$.
II. - Once more, exhibit $n$ functions $u, v_{1}^{(1)}, w_{2}^{(2)}, \ldots$ by means of the differential equation $\frac{1}{\mathfrak{A}_{n}} \Psi(z)=0$ in [no. 1, (12)] that are single-valued, continuous, and non-zero analytic functions that are real when $x$ is real at any point $\xi^{\prime}$ in $\mathbf{I}$ and in a neighborhood of that point. Let $\eta^{\prime}$ be a segment in that region along the real axis between $a$ and $b$ that includes only one of the points $\xi^{\prime}$ in its interior. The subintervals $\eta^{\prime}$ around the various points $\xi^{\prime}$ shall be taken to lie consecutively. Let the endpoint of one interval $\eta^{\prime}$ be $a^{\prime \prime}$ and $b^{\prime \prime}$.

Let the $z$ in [no. 1, (11)] be replaced with $s+(1+\rho) t$ on the interval $\eta^{\prime}$, where $\rho$ is a real constant. The real functions $s$ and $t$ shall be finite and continuous from $a^{\prime \prime}$ to $b^{\prime \prime}$, along with their first $2 n$ derivatives, $t$ shall vanish at $a^{\prime \prime}$ and $b^{\prime \prime}$, along with the same derivatives, but it shall not be zero everywhere. That will then give:

$$
\left\{\begin{array}{l}
\int_{a^{\prime \prime}}^{b^{\prime \prime}}(s+(1+\rho) t) \Psi(s+(1+\rho) t) d x  \tag{2}\\
=\int_{a^{\prime \prime}}^{b^{\prime \prime}} s \Psi(s)+(1+\rho)^{2} \int_{a^{\prime \prime}}^{b^{\prime \prime}} t \Psi(t) d x+(1+\rho) \int_{a^{\prime \prime}}^{b^{\prime \prime}}(s \Psi(t)+t \Psi(s)) d x .
\end{array}\right.
$$

The integral:

$$
\begin{equation*}
\int_{a^{\prime \prime}}^{b^{\prime \prime}} t \Psi(t) d x \tag{3}
\end{equation*}
$$

will again have the same sign as $a_{n n}$ then. The expression:

$$
\begin{equation*}
(1+\rho)^{2} \int_{a^{\prime \prime}}^{b^{\prime \prime}} t \Psi(t) d x+(1+\rho) \int_{a^{\prime \prime}}^{b^{\prime \prime}}(s \Psi(t)+t \Psi(s)) d x \tag{4}
\end{equation*}
$$

will have:

$$
\begin{equation*}
2(1+\rho) \int_{a^{\prime \prime}}^{b^{\prime \prime}} t \Psi(t) d x+\int_{a^{\prime \prime}}^{b^{\prime \prime}}(s \Psi(t)+t \Psi(s)) d x \tag{5}
\end{equation*}
$$

for its differential quotient with respect to $\rho$. That quantity might vanish for $\rho=\rho_{1}$. For $\rho>\rho_{1}$, the change in the expression (4) as $\rho$ increases will take place in the direction that is determined by the sign of $a_{n n}$. For positive values of $1+\rho$ for which $1+\rho>2\left(1+\rho_{1}\right)$, the expression (4) will itself have the sign of $a_{n n}$.
III. - Now let the function $z$ in the integral [no. 1, (11)] be an arbitrary real function that remains finite and continuous from $x=a$ to $b$, along with its first $2 n$ derivatives and is equal to zero at $a$ and $b$, along with its first $n-1$ derivatives, but does not vanish everywhere The $(2 n+1)^{\text {th }}$ derivative of $z$ shall be finite and continuous along the interval $x=a$ to $b$, except for isolated points that appear in a finite or infinite number.

The points $\xi^{\prime}$ are the points that were defined in $\mathbf{I}$. When $z$ does not vanish at a point $\xi^{\prime}$, along with its first $n-1$ derivatives, that point will be relocated as follows: By reducing the adjacent interval $\eta$ in $\mathbf{I}$, one can succeed in making that point $\xi^{\prime}$ lie inside of an interval where $z$ is nowherevanishing and where, at the same time, the $(2 n+1)^{\text {th }}$ derivative of $z$ is finite and continuous. Now, the interval denoted by $\eta^{\prime}$ in II is now taken at one such point $\xi^{\prime}$ such that it also lies inside of the aforementioned interval. The endpoints of $\eta^{\prime}$ were denoted by $a^{\prime \prime}$ and $b^{\prime \prime}$ in II.

The given function $z$ is set equal to $s+t$ on this interval $\eta^{\prime}$. The function $s$ is determined such that it is real, finite, and continuous from $a^{\prime \prime}$ to $b^{\prime \prime}$, along with its first $2 n$ derivatives (coincides with $z$ and its first $2 n$ derivatives at $a^{\prime \prime}$ and $b^{\prime \prime}$, respectively), and that $s$ vanishes at the point $\xi^{\prime}$, along with its first $n-1$ derivatives. One such function $s$ for $x=a^{\prime \prime}$ to $b^{\prime \prime}$ (the corresponding situation is true for $x=\xi^{\prime}$ to $b^{\prime \prime}$ ) is the following one:

$$
\left\{\begin{align*}
s & =(x-\xi)^{2 n+1}\left\{\varphi(x)+(x-a)^{2 n+1} \chi(x)\right\},  \tag{6}\\
\varphi(x) & =c_{0}+c_{1}\left(x-a^{\prime \prime}\right)+c_{2}\left(x-a^{\prime \prime}\right)^{2}+\cdots+c_{2 n}\left(x-a^{\prime \prime}\right)^{2 n},
\end{align*}\right.
$$

in which $\chi(x)$ is a real function that is finite and continuous from to $a^{\prime \prime}$ to $b^{\prime \prime}$, along with its first $2 n$ derivatives. The constants $c_{0}, c_{1}$, up to $c_{2 n}$ are determined from the following equations in succession:

$$
\left\{\begin{array}{l}
(z)_{x=a^{\prime \prime}}=\left\{\left(x-\xi^{\prime}\right)^{2 n+1} \varphi(x)\right\}_{x=a^{\prime \prime}},  \tag{7}\\
\left(\frac{d z}{d x}\right)_{x=a^{\prime \prime}}=\left\{\left(x-\xi^{\prime}\right)^{2 n+1} \frac{d \varphi(x)}{d x}+\varphi(x) \frac{d}{d x}\left(x-\xi^{\prime}\right)^{2 n+1}\right\}_{x=a^{\prime \prime}},
\end{array}\right.
$$

by means of the values of $z$ and its first $2 n$ derivatives at $x=a^{\prime \prime}$.

The function $z=f(x)$ has the following Taylor series development along the interval from $x=$ $a^{\prime \prime}$ to $\xi^{\prime}$ with a remainder term in $\left(x-a^{\prime \prime}\right)^{2 n+1}$, and in which one sets $\frac{d^{r} f(x)}{d x^{r}}=f^{(r)}(x)$ :

$$
\begin{equation*}
z=f\left(a^{\prime \prime}\right)+\left(x-a^{\prime \prime}\right) f^{\prime}\left(a^{\prime \prime}\right)+\frac{\left(x-a^{\prime \prime}\right)}{1 \cdot 2} f^{\prime \prime}\left(a^{\prime \prime}\right)+\cdots+\frac{\left(x-a^{\prime \prime}\right)^{2 n+1}}{(2 n+1)!} f^{(2 n+1)}\left(a^{\prime \prime}+\theta\left(x-a^{\prime \prime}\right)\right) \tag{8}
\end{equation*}
$$

and $0 \leq \theta \leq 1$. When the function $\left(x-\xi^{\prime}\right)^{2 n+1} \varphi(x)$ is developed in powers of $x-a^{\prime \prime}$, that will give a polynomial in $x-a^{\prime \prime}$ whose terms up to $\left(x-a^{\prime \prime}\right)^{2 n}$ coincide with the ones in the development of $z$ in (8), which is followed by an expression $\left(x-a^{\prime \prime}\right)^{2 n+1} \psi(x)$.

One has:

$$
\begin{equation*}
z-s=\left(x-a^{\prime \prime}\right)^{2 n+1}\left\{\frac{f^{(2 n+1)}\left(a^{\prime \prime}-\theta\left(x-a^{\prime \prime}\right)\right)}{(2 n+1)!}-\psi(x)-\left(x-\xi^{\prime}\right)^{2 n+1} \chi(x)\right\} . \tag{9}
\end{equation*}
$$

The expression:

$$
\begin{equation*}
\left(x-a^{\prime \prime}\right)^{2 n+1}\left\{\frac{f^{(2 n+1)}\left(a^{\prime \prime}-\theta\left(x-a^{\prime \prime}\right)\right)}{(2 n+1)!}-\psi(x)\right\}=z-\left(x-\xi^{\prime}\right)^{2 n+1} \varphi(x) \tag{10}
\end{equation*}
$$

has the same sign as $z$ for $x=\xi^{\prime}$. If it is zero along the interval from $\xi^{\prime}$ to $a^{\prime \prime}$ then let that first be true at $x_{1}$. Along the interval from $x_{1}$ to $\xi^{\prime}$, the absolute value of:

$$
\begin{equation*}
\frac{f^{(2 n+1)}\left(a^{\prime \prime}-\theta\left(x-a^{\prime \prime}\right)\right)}{(2 n+1)!}-\psi(x) \tag{11}
\end{equation*}
$$

might lie below $M . \chi(x)$ will then be set equal to a constant $C$ such that for $x_{1}$ up to $a^{\prime \prime}$, the absolute value of:

$$
\begin{equation*}
\left(x-\xi^{\prime}\right)^{2 n+1} C \tag{12}
\end{equation*}
$$

will lie above a value larger than $M$, and the sign of:

$$
\begin{equation*}
-\left(x-a^{\prime \prime}\right)^{2 n+1}\left(x-\xi^{\prime}\right)^{2 n+1} C \tag{13}
\end{equation*}
$$

inside of the interval $x=a^{\prime \prime}$ to $\xi^{\prime}$ will the same as that of $z$. Now, the difference $z-s$, which vanishes for $a^{\prime \prime}$, will have the same sign as $z$ along the interval from $a^{\prime \prime}$ to $\xi^{\prime}$, moreover.

Now, the function $t$ in $z=s+t$ is real, finite, and continuous along the interval from $a^{\prime \prime}$ to $b^{\prime \prime}$, along with its first $2 n$ derivatives, and equal to zero at $a^{\prime \prime}$ and $b^{\prime \prime}$. $t$ has the same sign as $z$ everywhere inside the interval from $a^{\prime \prime}$ to $b^{\prime \prime} . s$ is non-zero at $a^{\prime \prime}$ and $b^{\prime \prime}$ since that is true for $z$.
IV. - In III, the arbitrary function $z$ was set equal to $z=s+t$ along every interval that was called $\eta^{\prime}$ there. $z$ will now be taken to be the function that is equal $s$ along each of those intervals $\eta^{\prime}$ and equal to the original function $z$ on the remaining intervals between $a$ and $b$. Let the function that is determined in that way be denoted by $(z)$. It is real, finite, and continuous from $x=a$ to $b$, along with the first $2 n$ derivatives, and vanishes at $a, b$, and the point $\xi^{\prime}$ (I, III), but is not equal to zero everywhere, along with its first $n-1$ derivatives. $(z)$ then fulfills the conditions in $\mathbf{I}$. When $z$ is replaced with $(z)$, the integral [no. 1, (11)] will then have the same sign as $a_{n n}$. When the original $z$ from III occurs in the integral [no. 1, (11)], according to II:

$$
\begin{equation*}
\int_{a^{\prime}}^{b^{\prime \prime}} t \Psi(t) d x+\int_{a^{\prime}}^{b^{\prime \prime}}(s \Psi(t)+t \Psi(s)) d x \tag{14}
\end{equation*}
$$

will be added to it along each of the intervals $\eta^{\prime}$. Now, $z=s+t$ can be replaced with $s+(1+\rho) t$ along one such interval $\eta^{\prime}$, where $\rho$ is a real constant that is greater than or equal to zero and can vary from one interval $\eta^{\prime}$ to another. The original $z$ applies to the remaining intervals between $a$ and $b$. Let the function thus-obtained be denoted by $((z))$. It is real, finite, and continuous from $a$ to $b$, along with its first $2 n$ derivatives, and it vanishes at $a$ and $b$, along with its first $n-1$ derivatives, but not everywhere.

In the integral [no. 1, (11)], in which $((z))$ now stands in for $z$, one takes the positive quantity $\rho$ to be above the value that was denoted by $\rho_{1}$ in II on each interval $\eta^{\prime}$, such that the change in the total integral [no. 1, (11)], in which $z=((z))$, with increasing $\rho$ will proceed in the direction that corresponds to the sign of $a_{n n}$. According to II, there will then be further values such that the integral [no. 1, (11)] will continually have the same sign as $a_{n n}$ for $z=((z))$ as soon as the positive quantity $\rho$ grows beyond that value along the interval $\eta^{\prime}$.

The original function $z$ will then take on an increase $\rho t$ along an interval $\eta^{\prime}$ that has the same sign inside of the interval $\eta^{\prime}$ that $z$ possesses along that interval.

The information about the function $z$ in the integral [no. 1, (11)] that was obtained in I and IV produces the generalization of the case that was considered in no. $\mathbf{1}$, in which $z$ was set equal to zero along the individual subintervals between $a$ and $b$.

## 3. - The isoperimetric problems. (See Section Two of the treatise in Bd. 125).

The first variation of the integral [no. 1, (1)], in which $y$ is set equal to $y+\varepsilon z$, shall vanish for $\varepsilon=0$, while at the same time, the first variation of a different integral [treatise in Bd. 125 , no. 4 , (2)] shall vanish for $\varepsilon=0$. From [treatise in Bd. 125, no. 4], the function $z$ can be expressed as:

$$
\begin{equation*}
z=\frac{\varphi^{\prime}(x)}{S} \tag{1}
\end{equation*}
$$

in which $\varphi^{\prime}(x)=d \varphi(x) / d x$, and:

$$
\begin{equation*}
\varphi(x)=(x-a)^{n+1}(x-b)^{n+1} w, \tag{2}
\end{equation*}
$$

while $S$ is the differential expression that was defined in loc. cit, (4). $w$ is a real function from $x=$ $a$ to $b$ with derivatives up to order $2 n+1$ that are finite and continuous and equal to zero in an arbitrary neighborhood of a finite number of points along the interval between $a$ and $b$. Among those points are found the ones for which $S=0$ since $S$ is supposed to vanish at only a finite number of points along that interval; $w$ is otherwise arbitrary. The vanishing of the first variation of the integral [no. 1, (1)] will then lead to the differential equation that was given in loc. cit. (13):

$$
\begin{equation*}
Q-c S=0 . \tag{3}
\end{equation*}
$$

Some assumptions are made about the function $y$ that emerges from the differential equations that correspond to the ones that were given in the introduction and were exhibited in [treatise in Bd . 125, no. 4]. Let the finite number of points between $a$ and $b$ at which $S$ vanishes be $\zeta_{1}$ to $\zeta_{\mu}$. Let the consecutive subintervals between $a$ and $b$, each of which includes only one of the points $\zeta$ be:

$$
\begin{equation*}
\Theta_{1}, \Theta_{2}, \ldots, \Theta_{\mu} \tag{4}
\end{equation*}
$$

They can be chosen to be arbitrarily small, but fixed. $w$ will now be zero equal to zero on those subintervals and on the subintervals $\eta$ in [no. 1, (19)], but it shall not vanish everywhere between $a$ and $b$. The first variation of the two integrals will then vanish for $e=0$, and the second variation of the integral [no. 1, (1)] will have the same sign as $a_{n n}$.

Since $\varphi(x)$ vanishes for $x=a$ and $b, \varphi^{\prime}(x)$ must go through zero between $a$ and $b$. The generalization of what was said in [treatise in Bd. 125, no. 4] consists of applying the information in no. 2, I here. The function $w$ in (2) shall now be real and continuous from $x=a$ to $b$, and let the same thing be true of its derivatives up to order $2 n+1 . w$ shall vanish on the subintervals $\Theta$, and it shall be equal to zero at the point $\xi^{\prime}$ that was defined in no. 2, I, along with its first $n$ derivatives. $w$ is otherwise arbitrary, but not equal to zero everywhere between $a$ and $b$. The integral [no. 1, (11)] will then have the same sign as $a_{n n}$.

The generalization that was given in [treatise in Bd. 125, no. 5], which replaced the $z$ in $z+\varepsilon$ $z$ with $z+\varepsilon Z$, where $z$ is the previous function (1), and $Z$ has the behavior that was given in loc. cit., will also remain valid.

