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On the properties of a set of lines drawn through all points in space according to a continuous law

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I.

When lines are distributed throughout space, and the direction of each of them is determined by the coordinates of its starting point, one can study them from two different viewpoints: First of all, if one considers their totality then one can try to divide them into groups that are characterized by some geometric situation, such as being normal to a family of surfaces. In the second place, if one considers the ones that are infinitely close, or more precisely ones whose starting points are infinitely close, then one might wish to know their relative relationships.

In one case and the other, one will be led to some laws that belong to general geometry, because they are, as one will see in this article, independent of the particular form of the functions that determine the direction of the corresponding line at each point of space.

That double viewpoint is also encountered in a theory that has a close connection with the one that I would like to address, moreover. I would like to speak of the theory of normals to the same surface. On the one hand, the set of those normals is divided into groups that are characterized by the fact that all of the ones in the same group are generators of the same developable surface, and on the other hand, in the neighborhood of the same point of the surface, they will present a mutual relationship that c an be expressed in various ways. Before commencing with the principal topic of this article, I would like to present one of those ways that will be useful in what follows.

II.

As usual, I shall take the point *A* where one studies the curvature of a surface to be the origin of the coordinates, the normal at that point to be the *Z*-axis, and in the tangent plane, I shall take the *X* and *Y* axes to be the generally-unique rectangular system for which the coefficient $\frac{d^2z}{dx\,dy}$ is zero, or in other words, the system that relates to *two principal sections.*

Under those conditions, the normal that relates to a point that is infinitely close to the origin, i.e., a point whose coordinates *dx*, *dy*, and 0 are infinitely small of second order, that normal will have the equations:

$$\xi - dx + (r\,dx)\,\zeta = 0\,, \qquad \eta - dy + (t\,dy)\,\zeta = 0\,, \tag{1}$$

in which ξ, η, ζ are the running coordinates, while r and t are the values of the two coefficients $\frac{d^2 z}{dx^2}$, $\frac{d^2 z}{dy^2}$ at the origin.

I would like to express the idea that this normal meets a line that is drawn through a point on the *Z*-axis parallel to the tangent plane, which consequently has the equations:

$$\frac{\xi}{\cos\omega} = \frac{\eta}{\sin\omega}\,, \qquad \zeta - R = 0\,, \tag{2}$$

and I intend that the word "meet" should mean that the shortest distance between the two lines is a higher-order infinitesimal, as one always does with this type of question. Now, one can prove with the desired condition that:

$$R = \frac{\cos\omega\cdot dy - \sin\omega\cdot dx}{t\cos\omega\cdot dy - r\sin\omega\cdot dx}\,. \tag{3}$$

Hence, the value of R, i.e., the distance from the line (2) to the tangent plane, depends upon both the angle ω and the ratio dy / dx, at least whenever the angle ω does not have one of the two special values $\omega = \pi / 2$ or $\omega = 0$, because in each of those cases, the value of R is independent of dy / dx. It is equal to $R_1 = 1 / r$ when $\omega = \pi / 2$ and to $R_2 = 1 / t$ when $\omega = 0$.

It follows from this that if the first of the two lines has the equation:

$$\xi = 0\,, \qquad \zeta = R_1\,,$$

and the second one has the equation:

$$\eta = 0\,, \qquad \zeta = R_2$$

then they will be met by all of the neighboring normals to *AZ*. That is the geometric situation that characterizes the relationship between neighboring normals most concisely. One can express it as follows:

Let AN be the normal to a surface at the point A. All of the normals that relate to the infinitely-close points to A will meet the two lines that are drawn perpendicular to AN in the planes of the two principal sections and through the centers of curvature of those two sections, respectively (*).

(*) Kummer proved this property of infinitely-close normals in a recently-published paper. (see "Allgemeine Theorie der gradlinigen Strahlensystem," Crelle's Journal **57** (1860). See also "Mémoire sur la théorie de la vision" by Sturm in Comptes rendus de l'Académie des Sciences.)

I will call those two lines the *directors* of the infinitely-close normals.

That will immediately imply Monge's theorem that relates to the two directions of the lines of curvature, because if the distances R_1 and R_2 are unequal, which will be true except at singular points, then it will be easy to see that a neighboring normal to *AN*, due to the fact that it must be supported by the two directors, can meet *AN* only if it relates to a point that is situated in one of the two normal planes that contain the first or second director.

Furthermore, equation (3) will suffice to prove Euler's theorem on normal sections. Indeed, let α be the angle whose tangent is equal to dy / dx, and consider the normal section that contains the point (dx, dy, 0), i.e., the normal section whose plane makes an angle of α with the *ZX*-plane. Its center of curvature is at the intersection of *AZ* and the projection of the normal (1) into the plane of that section itself. It will therefore also be at the intersection of *AZ* with the line (2), provided that one supposes that $\omega = \pi / 2 + \alpha$ in its equations, which implies that $\sin \omega = \cos \alpha$ and $\cos \omega = -\sin \alpha$. One will have $\dfrac{dx}{\cos\alpha} = \dfrac{dy}{\sin\alpha}$, moreover. Under those conditions, one will find that:

$$R_\alpha = \frac{1}{r\cos^2\alpha + t\sin^2\alpha},$$

i.e.:

$$\frac{1}{R_\alpha} = \frac{1}{R_1}\cos^2\alpha + \frac{1}{R_2}\sin^2\alpha,$$

which is precisely Euler's formula.

III.

Let X, Y, Z be the functions that determine the direction of a line that is drawn through the point x, y, z in such a way that if one calls the angles between that line and the three rectangular axes λ, μ, ν then one will have:

(1) $$\cos\lambda = X, \qquad \cos\mu = Y, \qquad \cos\nu = Z.$$

Furthermore, let Θ represent the following function:

$$X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right).$$

One knows that if that function is zero then the total differential equation:

(2) $$X\,dx + Y\,dy + Z\,dz = 0$$

will be integrable, and it will correspond to a finite equation that contains an arbitrary constant. At the same time, each of the surfaces that result from attributing a particular value to that constant will enjoy the property that the line that is constructed at each of its points under the condition (1) will be normal to it, and since due to the arbitrariness in the constant, there is generally no point in space at which one can draw one of the surfaces that one deals with, it will follow that the integrability of equation (2) will imply the existence of a way of dividing all of the given lines into groups that are perpendicular to a family of surfaces.

However, it would be wrong for one to believe that conversely, the possibility such a distribution will be subordinate to the integrability of equation (2), and as a result, to the geometric fact that characterizing the set of given lines will imply that integrability. Quite the contrary, one sees that no matter what the functions X, Y, Z are, there are always an infinitude of ways to distribute the corresponding lines into groups that are normal to a family of surfaces in such a way that in the case where equation (2) is integrable, the surfaces that are included in its general integral can be procured for such a distribution only in a very special manner from among an infinitude of other ones that are equally possible.

Indeed, let O be the point whose coordinates are x, y, z. Imagine the line that pertains to that point, i.e., the one whose angles with the axes have the given functions X, Y, Z for their cosines. Along that line, I start from O and measure out a variable length $OO' = \rho$, and I let ξ, η, ζ denote the coordinates of the point O' in such a way that one has:

(3) $$\xi = x + \rho X , \quad \eta = y + \rho Y , \quad \zeta = z + \rho Z .$$

Indeed, the latter situation supposes the relation:

$$X\,d\xi + Y\,d\eta + Z\,d\zeta = 0 ,$$

in which one must replace $d\xi$, $d\eta$, $d\zeta$ with their values that are deduced from equations (3). When one considers the condition $X^2 + Y^2 + Z^2 = 1$, such a substitution will lead to the equation:

$$X\,dx + Y\,dy + Z\,dz + d\rho = 0 ,$$

in which one can, moreover, recognize an expression for the desired fact by writing it in the equivalent form:

$$ds\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) + d\rho = 0 .$$

One can write it in another form:

(4) $$\left(X + \frac{d\rho}{dx}\right)dx + \left(Y + \frac{d\rho}{dy}\right)dy + \left(Z + \frac{d\rho}{dz}\right)dz = 0 .$$

Now, that is a total differential equation whose integrability condition is the following:

(5) $$\left(\frac{dY}{dz} - \frac{dZ}{dy}\right)\frac{d\rho}{dx} + \left(\frac{dZ}{dx} - \frac{dX}{dz}\right)\frac{d\rho}{dy} + \left(\frac{dX}{dy} - \frac{dY}{dx}\right)\frac{d\rho}{dz} = 0 .$$

Finally, the latter equation is a partial differential equation for the unknown function ρ. It will therefore include (and with no further restricting condition) a general integral that is represented by a finite equation between *x*, *y*, *z*, and ρ with an arbitrary function. When one attributes a well-defined form to that function, equation (4) will include a general integral with an arbitrary constant, i.e., an integral that includes a family of surfaces that are loci of the point *O*, and each of those surfaces will correspond to another one that is the locus of points O'.

I shall call the surfaces that are loci of points *O resolvents*, because they are given by means of *resolving* the given set of lines into groups that are normal to a family of surfaces. I shall call the other surfaces that are loci of the points O' *directors*, because each of them is characterized or *directed* by one of those groups, and I shall summarize that by saying that each form that is attributed to the arbitrary function that is included in the integral of equation (5) will give rise to a family of resolvent surfaces, and consequently, a family of director surfaces, and ultimately to a particular way of dividing them that will exhaust all of the given lines.

IV.

Several remarks also need to be made in regard to equation (5) of the preceding paragraph.

First of all, since it includes only the partial derivatives of the function ρ, it is easy to see that one can add an arbitrary constant (either positive or negative) to any convenient value of that function in such a way that a resolvent surface will not correspond to a unique director surface, but to a *set of parallel surfaces* that are all directors of the same group, as it must.

In the second place, if we suppose that $\Theta = 0$, which is the case of integrability for the equation $X\,dx + Y\,dy + Z\,dz = 0$, then we can take ρ to have the value zero (or a constant), and that will give the only way of distributing lines has been accepted up to now (or at least pointed out) by the geometers who have addressed that question. However, since ρ can also take on an infinitude of values in that same case where Θ is zero, one will even see that this way is not unique, and all that one can say in particular is that the resolvent surfaces are, at the same time, director surfaces.

Finally, if one considers exclusively these two of the three simultaneous equations that are collectively equivalent to that same equation:

$$\frac{dx}{\frac{dY}{dz} - \frac{dZ}{dy}} = \frac{dy}{\frac{dZ}{dx} - \frac{dX}{dz}} = \frac{dz}{\frac{dX}{dy} - \frac{dY}{dx}}$$

then they will characterize the resolvent surface completely and exclusively since the function ρ is absent from them. That has some further consequences:

1. The resolvent surfaces are represented by the partial differential equation:

$$\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)p+\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)q = \frac{dX}{dy}-\frac{dY}{dx},$$

in which *p* and *q* are the partial derivatives of *z* with respect to *x* and *y*, resp., as usual.

2. The planes that touch the various resolvent surfaces that pass through the point *x*, *y*, *z* at that point will all contain the line:

$$(\alpha) \qquad \frac{\xi - x}{\dfrac{dY}{dz}-\dfrac{dZ}{dy}} = \frac{\eta - y}{\dfrac{dZ}{dx}-\dfrac{dX}{dz}} = \frac{\zeta - z}{\dfrac{dX}{dy}-\dfrac{dY}{dx}} .$$

Now, the equation $\Theta = 0$ expresses precisely the idea that this line is perpendicular to the line of the system that refers to the same point, i.e., the line:

$$\frac{\xi - x}{X} = \frac{\eta - y}{Y} = \frac{\zeta - z}{Z} .$$

That situation gives the geometric interpretation of the integrability condition for total differential equations for the first time. However, in order to appreciate that interpretation better, one must see how the line (α) will become a means for coordinating the lines whose starting points are infinitely close. We shall now study that means of coordination.

V.

Let us first recall what was done before in regard to that question.

Monge proved (*Mémoires de l'Académie*, 1781) that "if one imagines drawing some lines in space through all points of a plane according to an arbitrary law and one considers one of those lines, in particular, then of all of the ones that surround it and are infinitely close to it, there will generally be only two of them that are in the same plane as it."

Later on, Malus, in his "Mémoire sur l'optique," studied the properties of a system of lines that are drawn through all points in space, and he established this theorem: "In order for the line OO' that relates to an initial point *O* to be met by the lines that relate to the contiguous points, it is necessary that those infinitely-close points to *O* belong to a second-degree cone that has its summit at *O* and has OO' for one edge." (Journal de l'École Polytechnique, Cahier XIV).

That beautiful theorem, which includes that of Monge as a corollary (*), is not enough for our purposes.

(*) Or that one can, conversely, deduce from Monge's theorem. (See Comptes rendus de l'Académie des Sciences, 20 May 1861, report by Chasles.)

As above, let OO' denote the line of the system that relates to the point *O*. Imagine the line *OR* along which the tangent planes to the resolvent surface that pass through that point intersect [viz., the point (α) in the preceding paragraph].

Any plane that is drawn through that line *OR* contains a group of neighboring points to *O*, and the lines of the system that pertain to those points, along with OO', will collectively form a group (or pencil) that is normal to the same surface. Another plane that is also drawn through *OR* will correspond to a similar pencil, and since a moving plane that turns around *OR* will successively meet all of the infinitely-close points to *O*, none of the neighboring lines to OO' will escape that association.

Therefore, here is how one, already with the aid of planes that are drawn through *OR* and which I will call *resolvent planes*, resolves the set of lines that are infinitely close to OO' into distinct groups or pencils that are characterized by the fact that all of the ones in the same pencil are normal to the same surface, and it will only remain for us to define the situation that relates to the line of such a pencil. We will easily succeed in doing that with the aid of the two directors (*see* above, § **II**).

I shall place the coordinate origin at the point *O* and take OO' to be the *Z*-axis. For that origin, the three functions *X*, *Y*, *Z* will take the values $X_0 = 0$, $Y_0 = 0$, $Z_0 = 1$. The three functions $\frac{dZ}{dx}$, $\frac{dZ}{dy}$, $\frac{dZ}{dz}$ are zero there, since they represent the partial derivatives of the constant function $X^2 + Y^2 + Z^2$ at that same point ($x = y = z = 0$).

At a neighboring point to the origin that has the coordinates *dx*, *dy*, *dz*, the functions *X*, *Y*, *Z* will take the following values, in which one does not write the infinitesimals of order higher than one:

$$(1)\qquad \begin{cases} X_1 = \left(\dfrac{dX}{dx}\right)_0 dx + \left(\dfrac{dX}{dy}\right)_0 dy + \left(\dfrac{dX}{dz}\right)_0 dz, \\[2ex] Y_1 = \left(\dfrac{dY}{dx}\right)_0 dx + \left(\dfrac{dY}{dy}\right)_0 dy + \left(\dfrac{dY}{dz}\right)_0 dz, \\[2ex] Z_1 = 1, \end{cases}$$

which are expressions in which the index 0 expresses the idea that one must replace the symbols *x*, *y*, *z* with the values $x = 0$, $y = 0$, $z = 0$ in the affected functions.

The equations of the line *OR* [line (α) in § **IV**] will then be:

$$\frac{\xi}{\left(\dfrac{dY}{dz}\right)_0} = \frac{\eta}{-\left(\dfrac{dZ}{dz}\right)_0} = \frac{\zeta}{\left(\dfrac{dX}{dy}\right)_0 - \left(\dfrac{dY}{dx}\right)_0}.$$

However, one can take the *ZX*-plane to be a plane that contains that same line, and one must then set $\left(\frac{dX}{dz}\right)_0$ = 0 in those equations, as one also did with the value of X_1 above, in such a way that the equations of *OR* will be, by definition:

$$\left(\frac{dY}{dz}\right)_0 \zeta - \left[\left(\frac{dX}{dy}\right)_0 - \left(\frac{dY}{dx}\right)_0\right] \xi = 0\,, \quad \eta = 0\,.$$

Now, any plane that is drawn through that line will have the equation:

$$(2) \qquad \left(\frac{dY}{dz}\right)_0 \zeta - \left[\left(\frac{dX}{dy}\right)_0 - \left(\frac{dY}{dx}\right)_0\right] \xi - \lambda \cdot \eta = 0\,,$$

in which ξ, η, ζ are the running coordinates, and λ is undetermined. That is then the general equation of the resolvent plane that relates to the present origin, and we must study the nature of the two directors of the group that corresponds to a particular value of λ.

Since any one of those directors will meet the *OZ*-axis at a right angle, it will have equations of the form:

$$(3) \qquad \frac{\xi}{\cos\omega} = \frac{\eta}{\sin\omega}\,, \qquad \zeta - R = 0\,,$$

and it must be met by any line such that:

$$(4) \qquad \frac{\xi - dx}{X_1} = \frac{\eta - dy}{Y_1} = \frac{\zeta - dz}{Z_1}\,,$$

provided that the starting point *dx*, *dy*, *dz* is in the plane (2), which supposes the relation:

$$(5) \qquad \left(\frac{dY}{dz}\right)_0 \cdot dz = \left[\left(\frac{dX}{dy}\right)_0 - \left(\frac{dY}{dx}\right)_0\right] dx + \lambda \cdot dy\,.$$

Now, (3) will intersect (4) if *R* has the value:

$$R = -\frac{\cos\omega\, dy - \sin\omega\, dx}{\cos\omega \cdot Y_1 - \sin\omega \cdot X_1}\,,$$

and upon considering equations (1) and (5), that will be true for any starting point of the line (4) on the plane (2), provided that the angle ω is determined by the condition:

(6) $$\left(\frac{dX}{dy}\right)_0 \tan^2\omega + \left[\left(\frac{dX}{dy}\right)_0 - \left(\frac{dY}{dx}\right)_0 - \lambda\right]\tan\omega - \left(\frac{dX}{dy}\right)_0 = 0 .$$

The two directions thus-determined are rectangular, and we understand that to mean (if we do not know this in advance) that the lines that have the starting points on the plane (2), along with *OZ*, have the same nature that characterizes the infinitely-close normals on the same surface.

Meanwhile, the value of *R* will, at the same time, take one or the other of the equivalent forms:

$$R = -\frac{\cos\omega}{\cos\omega\left[\left(\frac{dY}{dy}\right)_0 + \lambda\right] - \sin\omega\left(\frac{dX}{dy}\right)_0} = -\frac{\sin\omega}{\sin\omega\left(\frac{dX}{dx}\right)_0 - \cos\omega\left(\frac{dX}{dy}\right)_0},$$

of which I will keep only the second one since it is independent of λ. It can be written as follows:

(7) $$R = -\frac{\tan\omega}{\left(\frac{dX}{dx}\right)_0 \tan\omega - \left(\frac{dX}{dy}\right)_0} .$$

The combination of that value with (6) will give the relationship between the two directors of the group that correspond to the particular value of λ.

If one replaces *R* with *z* and tan ω with *y* / *x* in (7) then the result:

(8) $$z\left[\left(\frac{dX}{dy}\right)_0 y - \left(\frac{dY}{dx}\right)_0 x\right] + y = 0$$

will be the equation of the locus of the directors. One sees that it is a hyperbolic paraboloid that has the *XY*-plane for one of its director planes and the *Z*-axis for a generator.

If one replaces tan ω with *y* / *x* in (6) then the resulting equation:

(9) $$\left(\frac{dX}{dy}\right)_0 y^2 + \left[\left(\frac{dX}{dy}\right)_0 - \left(\frac{dY}{dx}\right)_0 - \lambda\right] x\,y - \left(\frac{dX}{dy}\right)_0 x^2 = 0$$

will represent the system of two rectangular planes that contain the two directors of the group that are characterized by the present value of λ, which are planes that, by that fact itself, also contain the only two lines of the group that meet the *OZ*-axis. Hence, if one eliminates λ from equations (2) and (9) then one will find the conic surface whose edges indicate the directions along which one must pass from the point *O* to the neighboring points in order for the lines that relate to those points to meet *OZ*.

The result of that elimination is the following:

(10)
$$\left(\frac{dY}{dx}\right)_0 x^2 - \left(\frac{dX}{dy}\right)_0 y^2 + \left(\frac{dY}{dz}\right)_0 z\,x + \left[\left(\frac{dY}{dx}\right)_0 - \left(\frac{dX}{dy}\right)_0\right] x\,y = 0 .$$

One recognizes a second-order cone whose *Z*-axis is one edge… That result was given on another occasion by Malus, but the circumstances are different here, because the combination of the cone (10) and the paraboloid (8) will allow us to procure a complete picture of the relationships between all of the lines close to OO'.

Indeed, suppose that one has constructed the cone (10) and the paraboloid (8), and imagine that a right dihedral angle has the line OO' for its edge, which is, as we have pointed out, a generator of the cone, as well as the paraboloid. If that dihedral angle pivots around its edge then its two faces will meet the cone along two generators whose plane will be a resolvent plane, and they will also meet the paraboloid along two generators that will be the directors of the corresponding group.

I shall conclude by remarking to the reader that from the construction of the resolvent plane with the aid of the Malus cone and the right dihedral angle that pivots around OO', the fact that all of those planes intersect along the same line is found to result from a well-known property of second-order cones.

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