"Sur les invariants intégraux de la propagation par ondes," Bull. Soc. math. France **42** (1914), 142-167.

On the integral invariants of wave propagation

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In a recent paper that appeared in this Bulletin ([1]), Dontot directed his attention to an integral invariant of geometric optics that was pointed out by Hadamard ([2]) some years ago, and he then studied the relationships between that invariant and the transformations of lines to lines that are called *Malus transformations*; i.e., the ones that satisfy the condition that they change normal congruences into normal congruences.

In the following pages, I shall point out how these questions can be treated quite easily when one imagines, in a completely general manner, the propagation of waves in an arbitrary medium, instead of the rectilinear propagation of classical optics. One even obtains the most immediate proofs of the existence of the invariants for the case of rectilinear rays. At the same time, one accounts for the true reason for the success of the preceding methods that were employed for the study of Malus transformations.

The mathematical image of the propagation of waves in a given medium of a fixed nature is, as I have shown ([3]), a one-parameter group of contact transformations. The characteristic function $H(x_1, x_2, x_3; p_1, p_2, p_3)$ of the infinitesimal homogeneous transformation of that group characterizes the medium. If one equates it to unity then one will get the condition for the plane $\sum_{i=1}^{3} p_i (X_i - x_i) = 1$ to contact the wave surface that has the point (x_1, x_2, x_3) for its origin.

The contact elements that the group in question (G) operates on are defined by the rectangular coordinates x_1, x_2, x_3 of a point A and the direction coefficients p_1, p_2, p_3 of a plane that passes through that point. Upon subjecting the latter to the condition $H = 1$, one defines a normal vector to the contact element (E) considered, and one calls it the *index vector,* because its length N is the inverse $1 / V$ of the velocity V of the normal displacement at A of the front of any wave that has (E) as a contact element.

By definition, any homogeneous contact transformation leaves $\omega = \sum_{i=1}^{3} p_i \, dx_i$ invariant. One immediately deduces the series of symbolic differential expressions that Cartan introduced in his theory of Pfaff expressions ([4]) by the process of calculation and

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([1]) Tome XLII, 1914, fasc. 1, pp. 53.
([2]) C. R. Acad. Sc., 14 March 1898.
([3]) Bull. Soc. math. France, t. XXXIV, 1906; Ann. Éc. Norm. sup. (3), t. XXVI, 1909.
([4]) Ann. de l'Éc. Norm. sup. (3), t. XVI, 1899, pp. 239. – This mode of deduction was employed by De Donder in an article in this volume of the *Bulletin de la Société mathématique,* but in a form that is much more complicated. *See* pp. 91.

symbolic differentiation that he used in order to deduce some beautiful results. They are, in turn, along with the expression ω itself, the differential elements of a sequence of integral invariants whose form is common to all contact transformations. All that is left is to interpret them in order to obtain, in particular, integral invariants for any mode of wave propagation.

That interpretation is done effortlessly by means of the index vector: The first invariants J_1 and J_2 , which are of order 1 and 2, resp., are simply the work and the flux of that index vector.

Each contact element (E) defines a ray (R), which is curvilinear, in general, along which the point of that element displaces during the propagation of any wave that has (E) for one its elements, and in such a way that, conversely, any point A of a ray (R) is associated with the index vector (N) of the element (E), which then displaces along the ray (R). We say that (E) and (R) are *pseudo-orthogonal*. The rays (R) that are pseudo-orthogonal to the wave elements are called the *pseudo-normals* to that wave.

The invariants J_1 and J_2 are thus related to the curves that are situated in the surfaces that are coupled to the rays and to the surfaces that are transverse to the ray congruences. However, one confirms that J_1 is a relative invariant for the infinitesimal transformations $\varphi(H\ f)$, where the factor φ is arbitrary. These transformations slide the elements (E) along the rays (R) according to an arbitrary law, so it results that J_1 is an invariant for a tube of rays and J_2 is an invariant for a congruence of rays. The condition for a congruence of rays to be a congruence of pseudo-normals is expressed by the vanishing of these invariants ([1]).

The invariant J_3 (as a relative invariant) and the invariant J_4 possess the same invariance property for the transformations $\varphi(H\ f)$, and are thus attached to closed families of ∞^3 rays and systems of ∞^4 rays. One deduces an interpretation for J_4 from this that generalizes the one that Hadamard gave for the rectilinear rays of the optics of homogeneous and isotropic media.

The study of ray transformations results very simply from the reduction of any group (G) to the canonical form of a group of translations by a contact transformation. The Malus transformations are divided into two classes according to whether they are or are not compatible with the conservation of the elastic properties of the medium considered.

The one and the other can be realized by subjecting the family of waves to contact transformations. One then has the ones that leave the characteristic function H invariant and the ones that leave invariant the differential system that defines the trajectories of propagation.

Since any contact transformation can be obtained by means of an infinitesimal contact transformation, and it corresponds to the propagation of waves in a conveniently-chosen medium, it seems legitimate to conclude that all of the Malus transformations can be obtained by making a layer of a convenient auxiliary medium pass through the rays that are produced in the given medium, such that when the layer leaves that medium, it re-enters the given medium or a second medium with wave surfaces that are homothetic to the first ones (according to whether one is dealing with transformations of the first or second class). However, the rigorous proof of that result will necessitate a new investigation.

([1]) This result was pointed out by Cartan for rectilinear rays and normal congruences [Bull. Soc. math. France **25** (1898), pp. 140].

I. – INTEGRAL INVARIANTS.

1. First, recall some principles of the propagation of waves in a given plane ([1]).

At each instant t_0, any point A of the medium is capable of emitting a wave that is a certain surface S at any later instant $t_0 + t$, from the geometric viewpoint. The homothetic image of that surface S, when taken with respect to A, and with a homothety ratio $1 / t$, tends to a limiting surface Σ when t tends to zero. If the medium is stationary then that limiting surface Σ will depend upon only the point A; i.e., independently of time, which is what we assumed. One calls it the *wave surface* that has A for its origin.

The mode of propagation is characterized entirely by the family of ∞^3 wave surfaces that have various points of space for their origins. One defines it analytically in the following manner:

Let (x_1, x_2, x_3) be the rectangular coordinates of a point A; write down the general equation of a plane in the form:

$$p_1 (X_1 - x_1) + p_2 (X_2 - x_2) + p_3 (X_3 - x_3) = 1. \tag{1}$$

The wave surface whose origin is A will be represented by the equation that expresses the idea that the plane (1) is tangent to it, and one suppose that this tangential equation is presented in the form:

$$H(x_1, x_2, x_3 ; p_1, p_2, p_3) = 1, \tag{2}$$

in which H is a homogeneous function of first degree in p_1, p_2, p_3 .

The corresponding mode of propagation then has its geometric expression in the one-parameter group (G) of homogeneous contact transformations that is generated by the infinitesimal homogeneous contact transformation ([2]) that has H for its characteristic function; i.e., its symbol is the Poisson-Jacobi bracket:

$$(H\,f) = \sum_{i=1}^{3}\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial f}{\partial p_i}\right). \tag{3}$$

Here is what we mean by that: The general transformation of the group $(H\,f)$ is given by the formulas:

$$x_i = f_i(x_1^0, x_2^0, x_3^0; p_1^0, p_2^0, p_3^0 \mid t - t_0) \qquad (i = 1, 2, 3), \tag{4}$$

$$p_i = g_i(x_1^0, x_2^0, x_3^0; p_1^0, p_2^0, p_3^0 \mid t - t_0) \qquad (i = 1, 2, 3) \tag{5}$$

that one gets by integrating the canonical system:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, 3), \tag{6}$$

([1]) *Das Eikonal*, Abh. der. k. sächsischen Ges. der Wiss., 1895.
([2]) S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, t. I, pp. 262-264.

(7) $$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2, 3),$$

with the initial conditions:

(8) $$x_i = x_i^0, \qquad p_i = p_i^0, \qquad (i = 1, 2, 3), \qquad \text{for } t = t_0 .$$

The parameter of the group is:

(9) $$\theta = t - t_0 .$$

The functions f_i and the quotients of the function g_i are homogeneous of degree zero in p_1, p_2, p_3. One can consider the group as operating upon the variables x_1, x_2, x_3, and the ratios of the variables p_1, p_2, p_3, which define a contact element that is composed of the point whose coordinates are x_1, x_2, x_3 and the plane that passes through that point that is normal to the direction whose direction coefficients are p_1, p_2, p_3.

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Having said that, the wave that one gets after a time θ from an arbitrary original wave that is given at time t_0 is deduced precisely from that original wave by applying the transformation (4), (5) to its contact elements. The principle of enveloping waves – or Huygens's principle – is equivalent to the fact that this transformation is a contact transformation; i.e., the fact that the propagation of wave involves contact elements.

Equations (4) define ∞^4 curves that are the *rays* of propagation. Under the displacement of a contact element that is defined by the equations of propagation (4), (5), the point of that element will describe a ray, and the plane of that element will take on an orientation at each point of the ray that is determined by formulas (6).

These formulas, which can be written:

(10) $$\frac{dx_1}{\frac{\partial H}{\partial p_1}} = \frac{dx_2}{\frac{\partial H}{\partial p_2}} = \frac{dx_3}{\frac{\partial H}{\partial p_3}},$$

are solved for p_1, p_2, p_3, and give:

(11) $$p_i = \frac{\partial \Omega}{\partial\, dx_i} \qquad (i = 1, 2, 3),$$

upon introducing the point-like equation of the wave surface with origin (x_1, x_2, x_3) in the form:

(12) $$\Omega(x_1, x_2, x_3 ; X_1 - x_1, X_2 - x_2, X_3 - x_3) = 1,$$

where $\Omega(x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)$ is homogeneous of degree one in ξ_1, ξ_2, ξ_3 and upon setting, to abbreviate the notation:

(13) $$\Omega = \Omega(x_1, x_2, x_3 ; dx_1, dx_2, dx_3).$$

One can consider these formulas as establishing a correspondence at each point (x_1, x_2, x_3) between the directions (dx_1, dx_2, dx_3) of the linear elements of that point and the orientations (p_1, p_2, p_3) of the contact elements of that plane, which reduces to orthogonality in the case of a homogeneous and isotropic medium. We call it *pseudo-orthogonality* and we say, in turn, that the rays that serve to transport the contact elements of a surface are the *pseudo-normals* to that surface. Moreover, it is the *pseudo-normals* to the ∞^1 surfaces that are the successive states of a wave that has the surface in question for its initial position. This is the generalization of *Malus's theorem*.

The set of such a family of ∞^1 surfaces – or *wave family* – is represented by an equation:

(14) $$f(x_1, x_2, x_3) = t,$$

where f is a solution of equation (2), when considered to be a partial differential equation; i.e., in which one has set:

(15) $$p_i = \frac{\partial t}{\partial x_i} \qquad (i = 1, 2, 3).$$

Conversely, any solution of that partial differential equation will provide the equation (14) of a family of waves.

2. For every contact element that is defined by x_1, x_2, x_3, and the ratios of the p_1, p_2, p_3, we adopt definite values for the p_1, p_2, p_3 that satisfy equation (2). This is possible because H is an invariant of the group (G), and consequently equation (2) remains invariant under that group. Let A be the point (x_1, x_2, x_3). Each contact element of that point will thus correspond to a vector N that is perpendicular to that element and has A for its origin and components that equal the values of p_1, p_2, p_3 that satisfy equation (2).

In order to interpret the vector N, imagine an arbitrary wave that contains the contact element (E) in question at the instant t. The values of p_1, p_2, p_3 are then given by formulas (15), so the wave belongs to the family of waves that is represented by equation (14) and the vector N is measured by the normal derivative dt / dn along the normal to the contact element (E); i.e., the inverse $N = 1 / V$ of the velocity $V = dn / dt$ of the normal displacement of the front to any wave that contains (E), when evaluated at the point A. One can call that vector N the *index of the medium relative to the contact element E*.

If one introduces the wave surface that has A for its origin then the tangent plane (P) that is parallel to that contact element (E) has precisely equation (1) for its equation, with the values that were adopted for p_1, p_2, p_3. If one lets M denote the contact point of that plane (P) and lets K denote the foot of the perpendicular that is based at A on (P) then the vector AK is the vector V, which is the velocity of the wave front, and that vector AM is the velocity vector v under the motion of the point A along the ray that is the pseudo-normal to the element (E). This gives, in particular, the geometric definition of pseudo-orthogonality ([1]).

One agrees to observe that the function H has, in general, only *positive* homogeneity. This amounts to saying that for each contact element (E), one must take into account the sense of propagation of the wave that one regards it as belonging to; the vectors V and N will be directed in that sense of propagation. One can say that one considers each contact element as doubling into two of them, each of which have a definite face, and correspond to the edge of the space towards which that element begins to displace.

3. The introduction of the *index vector* N permits one to interpret the integral invariants that are common to all homogeneous contact transformations from the viewpoint of the propagation of waves; i.e., all infinitesimal transformations of the form (3).

The simplest case of these invariants results from saying that these homogeneous contact transformations are defined entirely ([2]) by the property that they leave the Pfaff expression:

$$\omega = \sum_{i=1}^{3} p_i dx_i \tag{16}$$

invariant, which gives the integral invariant that is characteristic of the propagation of waves: It is the *work done by the index vector:*

$$J_1 = \int \omega = \int N \cos(N, ds)\, ds, \tag{17}$$

which applies to any one-dimensional continuum that is composed of contact elements $(x_1, x_2, x_3 ; p_1 : p_2 : p_3)$.

This continuum (K) is an arc of a curve CD that carries a contact element (E) at each of its points A. Each of these contact elements corresponds to a ray. If these rays all coincide then the arc CD is pseudo-orthogonal to all of the elements (E) considered, and one has, from equations (6) and (2):

$$\omega = \sum_{i=1}^{3} p_i dx_i = \sum_{i=1}^{3} p_i \frac{\partial H}{\partial p_i} dt = H\, dt = dt. \tag{18}$$

Upon supposing, for the sake of neatness, that ω remains positive when one displaces from C to D, J_1 will then be, in this case, the time that an arbitrary wave that is pseudo-orthogonal to CD at C takes to propagate to D. During its evolution, the wave will then sweep out the ray CD while constantly remaining pseudo-orthogonal to it.

As we will see, the general case gives an analogous interpretation. The rays that are pseudo-normal to the elements (E) of the continuum (K) generate a surface (s). At each

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([1]) This results from the fact that the parametric equations of the wave surface are: $X_i - x_i = \partial H / \partial p_i$ (i = 1, 2, 3). See Ann. Éc. Norm. sup. (3) **26** (1909) pp. 403 and 436.

([2]) S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, v. II, pp 260.

point M of that surface (s), is tangent plane contains a linear element that belongs to the contact element that is pseudo-orthogonal to the ray that passes through M. We say that the linear element is *pseudo-orthogonal* to the ray. Upon *uniting* these linear elements, one obtains a family of curves in (s) that are pseudo-orthogonal to the rays that generate (s), and consequently, to some *bands* of contact elements that have the rays considered for pseudo-normals. The bands, in turn, belong to an infinitude of wave families. In order to define one of these families, one can give an arbitrary initial wave, provided that it contains, for example, those of the bands of contact elements in question that contain the element (E) that is carried by the point C. We thus have a well-defined family (14).

At each of the points of the arc CD, the values (15) of the derivatives of the function f coincide with the values that are given at that point for the coordinates of the vector N, because these derivatives, like the given p_i, must satisfy the pseudo-orthogonality conditions for the rays considered and the condition equation (2). One thus has:

$$\omega = \sum_{i=1}^{3} \frac{\partial t}{\partial x_i} dx_i = dt. \tag{19}$$

Upon further supposing that ω remains positive when one displaces from C to D, one sees, moreover, that J_1 is the time that it takes for a wave to sweep out the curve CD from C to D, that wave being such that at each point A of CD it is tangent to the given contact element (E).

In a previous paper, we showed that this duration for the transmission of a disturbance from C to D is maximal when one requires the wave to remain constantly pseudo-orthogonal to the curve CD ([1]). This curve is then a sort of edge of regression for the surface (s) because it is the envelope of the rays that generate that surface.

In any case, that duration of transmission depends upon only the family of waves and points of C and D, and not upon the particular choice of curve CD.

4. The last remark exhibits an essential property of the invariant J_1 , and in turn, of the ones that we shall deduce later on. In order to arrive at it, it suffices to abstract from contact elements and consider only the rays. To that effect, one can consider each index vector N as being attached, not to the contact element (E) that it is referred to by its original definition, but to the linear element of the ray (dx_1, dx_2, dx_3) that is pseudo-orthogonal to that contact element (E). In other words, it will be formulas (11) that define the components of the index vector analytically. As a result, each ray (R) is attached to an index vector N at each of its points that is defined by the direction of the ray at that point.

In the foregoing, one can, moreover, start with ∞^1 rays (R) that generate a surface (s), and consider an arc of the curve CD that is traced on that surface. That arc suffices to define the integral J_1 if one agrees to takes the index vector N at each point of (s) to be the one that corresponds to the ray (R) that generates the surface (s) that passes through

([1]) Ann. Éc. Norm. sup. (3) **26** (1909), 440-443. – That maximum is what we have called the *duration of propagation along CD*; it assumes the concavity of the wave surfaces. Its analytic expression is the curvilinear integral $J = \int \Omega$, when it is taken along CD. This is the generalization of the *optical path length.*

that point. If ξ_1, ξ_2, ξ_3 denote the direction coefficients of that ray and Ω_ξ denotes the function $\Omega(x_1, x_2, x_3 ; \xi_1, \xi_2, \xi_3)$ then J_1 will be the following curvilinear integral, which is taken along a curve that is traced in (s):

(19) $$J_1 = \int \sum_{i=1}^{3} \frac{\partial \Omega_\xi}{\partial \xi_i} dx^i .$$

Having said that, the stated property is that under these conditions the integral J_1 will depend upon only the extremities of the integration arc, on the condition that the variation of that arc *CD* be done in a continuous manner and in a region of (s) such that one can apply the geometric construction of the preceding paragraph to it. It results immediately from that fact that the wave family that results from that construction depends upon only upon the surface (s), and not upon the curve *CD* on that surface that is being considered.

5. We obtain a more general result directly by calculating the variation of J_1 under the hypothesis that one varies the contact elements (x, p) along the trajectories of propagation, but while leaving the manner by which they describe these trajectories arbitrary. The variables $x_1, x_2, x_3, p_1, p_2, p_3$ are then subject to variations of the form:

(20) $$\delta x_i = \varphi \frac{\partial H}{\partial p_i} \delta u \qquad (i = 1, 2, 3),$$

(21) $$\delta p_i = - \varphi \frac{\partial H}{\partial x_i} \delta u \qquad (i = 1, 2, 3),$$

in which φ is an arbitrary factor. For example, if one wishes to consider the variation as an infinitesimal transformation then one supposes that it is an arbitrary function of $x_1, x_2, x_3, p_1, p_2, p_3$. Since it results from these formulas that δH is zero, the index vectors N will be exchanged amongst themselves, because the condition (1) remains invariant.

One then has, in turn:

(22) $$\begin{aligned} \delta\omega &= \delta \sum p_i \, dx_i = d \sum p_i \, \delta x_i + \sum \delta p_i \, dx_i - \sum \delta x_i \, dp_i , \\ &= d(\varphi H) \, \delta u - \varphi \, dH \, \delta u = H \, \delta u \, d\varphi . \end{aligned}$$

Therefore, if one takes condition (2) into account then one will obtain:

(23) $$\delta\omega = d\varphi \, \delta u ,$$

i.e., the desired formula:

(24) $$\delta J_1 = [\varphi]_C^D \, \delta u .$$

In particular, if one supposes that C and D remain fixed then one will fall within the condition of the preceding paragraph, and one indeed recovers the result that J_1 then remains constant ([1]).

6. On the other hand, δJ_1 is also zero when the points C and D coincide; i.e., if δJ_1 is taken along a closed contour.

In other words, J_1 *is a relative invariant for infinitesimal transformations* $\varphi(H, f)$, *where* φ *is arbitrary, on the condition that it be taken along a (closed) line that is traced on the multiplicity* H = const.

It is therefore an invariant from the standpoint of propagation for the closed cylinder that is composed of rays, which is then composed of the surface (s) that proves the closed continuum considered (no. 3).

In order to interpret this, we imagine a tube (s) of rays, and on that tube, the pseudo-orthogonal trajectories to the rays, and we remark that the integral J_1, when taken on (s) along an arc of such a trajectory, is always zero, since $\cos(N, ds)$ is then zero all along the arc. A pseudo-orthogonal trajectory that starts from a point A of one of the rays (R) that the tube is comprised of amounts to cutting the same ray at a point A' after having made a circuit of the tube, and one then returns from A' to A by following the ray (R). The integral J_1 , when taken along the closed path thus constituted, will reduce to the duration of the propagation from A' to A along the ray (R). It has the same value no matter what point A is considered, and it is equal to the integral J_1 , when taken on the tube along any closed curve that makes a single circuit of the tube ([2]).

It is zero if the pseudo-orthogonal trajectory considered is closed, and all of these trajectories are then closed at the same time. This is the case when the tube of rays is composed of all the rays that are pseudo-orthogonal to the same wave at various points of a simple closed contour that is taken on that wave.

That invariant will play the role of a *period* if one proposes to find the general value for the integral J_1 for all of the arcs that are traced on (s) between two given points, and one will get a simple generalization by taking (s) to be some closed cylinders that are composed of rays and have a more complicated nature than the simple tubes that were considered previously.

7. One deduces a series of invariant constructions from the linear differential form (16) that are given by multiple integral invariants of increasing order by the symbolic

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([1]) If one supposes that a given arc (CD) lies entirely within a region such that one and only one ray passes through two points of that region then one can construct a surface (s) such that one ray will be the ray [CD] that joins C to D, and the other rays join the points of the arc (CD) pair-wise. The preceding result then proves that the integral (19), when taken along the ray [CD] – i.e., the integral $J = \int_{[CD]} \Omega$ – is equal to the integral J_1, when taken along the arc (CD) on that surface (s). Now, this is less than the integral $J = \int_{(CD)} \Omega$, as we explained at the end of no. 3. One thus proves, in a very simple manner, the minimum property of rays relative to the integral $J = \int \Omega$. Cf., Ann. Éc. Norm. sup. (3) **26** (1909), 447-448.

([2]) Cartan has developed these considerations for ruled tubes and the orthogonal trajectories of generators, as well as the ones that result for normal congruences, in an article "Sur les intégrales de l'espace réglé," Bull. Soc. math. France **24** (1898), 160-165.

calculations that were employed by Cartan in his study of the Pfaff problem ([1]). They are, with the same Cartan notations:

(25) $$\omega' = \sum_{i=1}^{3} dp_i dx_i ,$$

(26) $$\omega'' = \omega\omega' = (p_3\, dp_2 - p_2\, dp_3)\, dx_2\, dx_3 + (p_1\, dp_3 - p_3\, dp_1)\, dx_3\, dx_1 + (p_2\, dp_1 - p_1\, dp_2)\, dx_1\, dx_2 ,$$

(27) $$\omega''' = \tfrac{1}{2}\omega'^2 = dp_3\, dp_2\, dx_2\, dx_3 + dp_1\, dp_3\, dx_3\, dx_1 + dp_2\, dp_1\, dx_1\, dx_2 ,$$

(28) $$\omega^{\mathrm{IV}} = \omega\omega''' = (p_1\, dp_3\, dp_2 + p_2\, dp_1\, dp_3 + p_3\, dp_1\, dp_2)\, dx_1\, dx_2\, dx_3 ,$$

(29) $$\omega^{\mathrm{V}} = \tfrac{1}{6}\omega'^3 = - dp_1\, dp_2\, dp_3\, dx_1\, dx_2\, dx_3 .$$

The last one is zero identically, since we have required the variables x_1, x_2, x_3, p_1, p_2, p_3 to verify the relation (2). The forms ω' and ω''' are the most interesting, because they are invariant under the transformations $\varphi(H, f)$, and as a result they correspond to geometric properties of systems of ∞^2 and ∞^4 rays. For ω', which is the bilinear covariant of ω, that invariance amounts to the fact that, by virtue of the Stokes formula, the integral:

(30) $$J_2 = \iint \omega'$$

will be equal to the integral J_1 when taken around a closed contour (no. 6). The equality $\omega''' = \frac{1}{2}\omega'^2$ then results from this.

The interpretation of J_2 is, in fact, immediate: One must associate each point A of a portion of the surface (σ) with a contact element (E) according to an arbitrary law. One can then consider the ∞^2 rays (R) that are pseudo-orthogonal to the contact element (E) considered, and the ∞^3 index vectors that are associated with these rays at their various points. J_2 is the vorticity flux of the vector field thus defined upon crossing the surface (σ), and its value is the same as J_1 when taken around the contour of (s) on the tube of rays that serves as the frontier of the closed continuum of ∞^2 rays (R) considered.

Conversely, one can give a congruence (C) of rays and remove an arbitrary brush that is limited by a tube of rays of that congruence, and cut that brush with an arbitrary surface that gives the area (σ) that was considered above.

In order for the integral J_2 to be zero, no matter what the brush was that was removed from (C) – i.e., in order for J_1 to be zero around a closed contour that encircles an

([1]) E. CARTAN, Ann. Éc. Norm. sup. (3) **16** (1899), 244-253. *See* the introduction of that article (pp. 241-242). *Compare*, as well, the article already cited by E. CARTAN, Bull. Soc. math. France **25** (1896), 140-177. A systematic exposition of the calculus of multilinear differential expressions is given, along with some less condensed notations, by De Donder in a recent publication: Bull. Acad. royale Belgique, cl. sciences, **12** (1913), 1043-1073. – *See* also the proofs that were given by De Donder in this volume (viz., XLII) of the Bulletin de la Société mathématique, pp. 91, relative to the Hadamard invariant.

arbitrary tube of rays that belongs to the congruence (C), it is necessary and sufficient that the components p_1, p_2, p_3 of the index vectors (N) that comprise the field that is associated with the congruence (C) be functions of x_1, x_2, x_3 such that $\omega = \sum_{i=1}^{3} p_i dx_i$ is the exact total differential $\omega = dt$ of a function $t = f(x_1, x_2, x_3)$.

This is equivalent to saying that there is a family of waves (14) whose pseudo-normals coincide with the rays of the congruence (C), since they satisfy the same differential system (10).

In other words, the condition that J_2 be zero for any brush (or that J_1 be zero for any tube) of the congruence is necessary and sufficient for that congruence to be a congruence of pseudo-normals ([1]).

8. In order to interpret the integral invariant:

$$J_4 = -\iiiint \omega''' , \tag{31}$$

one must introduce a continuum (γ) of ∞^1 contact elements, and in turn, a bounded four-dimensional portion of the system of ∞^4 rays in space. From the preceding, the continuum (γ) can be chosen arbitrarily, provided that it contains a contact element that is pseudo-orthogonal to each of the ∞^4 rays (R) that define the four-dimensional brush considered, one can suppose that it is defined by cutting that brush with an arbitrary auxiliary surface ([2]) on which it determines an area (σ): ∞^2 rays (R) of the brush pass through point A of that area, each of which corresponds to the pseudo-orthogonal contact element (E) and the index vector N that relates to that element (no. 2).

The coordinates x_1, x_2, x_3 are then functions of two variables u_1, u_2 that vary over a field (U), and p_1, p_2, p_3 are functions of four variables u_1, u_2, v_1, v_2, where the variables v_1, v_2 vary over a corresponding field $V(u)$ for each system of values u_1, u_2 , and one has:

$$\begin{aligned} -\omega'' = \; & \frac{\partial(x_2, x_3)}{\partial(u_1, u_2)} du_1 du_2 \frac{\partial(p_2, p_3)}{\partial(v_1, v_2)} dv_1 dv_2 \\ & + \frac{\partial(x_3, x_1)}{\partial(u_1, u_2)} du_1 du_2 \frac{\partial(p_3, p_1)}{\partial(v_1, v_2)} dv_1 dv_2 \\ & + \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} du_1 du_2 \frac{\partial(p_1, p_2)}{\partial(v_1, v_2)} dv_1 dv_2 ; \end{aligned} \tag{32}$$

([1]) Analogous facts have been pointed out by E. Cartan for rectilinear rays and normal congruences. [Bull. Soc. math. France **24** (1896), 162.]

([2]) Here, we apply the mode of interpretation that was imagined by Hadamard for rectilinear rays to the general case. [J. HADAMARD, C. R. Acad. Sc., 14 March 1898. – *Cf.*, the article of DONTOT, Bull. Soc. math. France **42** (1914), 72-78.]

i.e., the differential element of J_4 is the scalar geometric product of the two vectors that figure in the elementary areas of the surface (σ) at the point A and the *figuratrix* surface ([1]) that is swept out by the extremity of the vector N when A remains fixed.

Now, the normal to that figuratrix surface, whose equation is equation (2), in which one considers p_1, p_2, p_3 to be point-like coordinates, has the direction coefficients $\frac{\partial H}{\partial p_1}$, $\frac{\partial H}{\partial p_2}, \frac{\partial H}{\partial p_3}$. Its direction is therefore that of the ray (R) that the index vector N (no. 4) is attached to. Therefore, if one denotes the measure of the elementary area on the surface σ by $d\sigma$, that of the elementary area in the figuratrix surface by $d\nu$, the direction of the normal to the surface (σ) by n, and that of the ray R then one can write:

$$-\varpi''' = \cos(R, n)\, d\sigma\, d\nu. \tag{33}$$

However, one can also replace $d\nu$ with its value ([2]):

$$d\nu = N^2 \frac{d\alpha}{\cos(R, N)}, \tag{34}$$

upon replacing the solid angle $d\alpha$ that is swept out by the extremity of the direction vector of N, and one arrives at the definitive formula for J_4 :

$$J_4 = \iiiint N^2 \frac{\cos(R,n)}{\cos(R,N)}\, d\sigma\, d\alpha, \tag{35}$$

which generalizes the formula that was given by Hadamard ([3]) for the case of rectilinear propagation in an isotropic medium:

$$[N = \text{const.};\ \cos(R, N) = 1].$$

9. Since ϖ''' provides $\frac{1}{2}\varpi''$ by symbolic differentiation, the integral J_4 is equal to one-half the integral:

([1]) We employ the term that was used by Hadamard in his *Leçons sur le calcul des variations*, t. I, pp. 90.

([2]) This formula, which is often considered to be obvious, is easily established by the symbolic calculus. Upon denoting the direction cosines of N by α_1, α_2, α_3, one has:

$$p_i = N\,\alpha_i$$

and

$$\cos(R, N)\, d\nu = \alpha_1\, dp_2\, dp_3 + \ldots = \alpha_1\, dN\, \alpha_2\, dN\, \alpha_3 + ..$$
$$= \alpha_1\, [N\, dN\, (\alpha_2\, d\alpha_3 - \alpha_3\, d\alpha_2) + N^2\, d\alpha_2\, d\alpha_3] + \ldots = N^2\, (\alpha_1\, d\alpha_2\, d\alpha_3 + \ldots) = N^2\, d\alpha.$$

([3]) C. R. Acad. Sc., 14 March 1898.

(36) $$J_3 = -\iiint \omega'',$$

taken along the closed, three-dimensional multiplicity that serves as the frontier of the continuum (γ) considered in the preceding paragraph. Under these conditions, the integral J_3 will possess the same invariance character as J_4 . It thus corresponds to the closed continuum of ∞^3 rays.

At each point of the area s, one will have to consider index vectors N whose extremities will describe a closed curve (c) on the figuratrix surface.

Let $d\varepsilon$ denote the angle between one of these vectors and an infinitely close vector, and consider the vector whose components:

(37) $$p_2\,dp_3 - p_3\,dp_2, \qquad p_3\,dp_1 - p_1\,dp_3, \qquad p_1\,dp_2 - p_2\,dp_1$$

figure in ω''.

One can consider the characteristic d as being that of the displacement on the curve (c), because x_1, x_2, x_3 depend upon just two variables u_1, u_2 , while p_1, p_2, p_3 depend upon u_1, u_2 , and another variable v that corresponds to the displacement on (c). The length of that vector is $N^2\,d\varepsilon$, in such a way that one has the interpretation:

(38) $$-\omega'' = N^2 \cos(n', n)\,d\sigma\,d\varepsilon,$$

in which n' denotes the normal direction to both the index vector N and an infinitely close index vector.

10. As for the integral:

(39) $$J_5 = -\iiiint \omega^{\text{IV}},$$

one can consider it only when it is taken over an open multiplicity; otherwise, it would be zero, since it would be equal to $-\int^{\text{V}} \omega^{\text{V}}$ if it were taken over a closed multiplicity.

It is therefore invariant only for the propagation that is being considered, and it is no longer an invariant that is attached to a system of rays. Its element ω^{IV} is, up to a numerical factor, the product:

(40) $$N^3\,dw\,d\alpha,$$

in which the solid angle that is swept out by the index vector appears, along with the volume element dw of space, because the factor of $dx_1\,dx_2\,dx_3$ in ω^{IV} is three times the elementary volume that is swept out by the index vector.

This is the result that was found by Hadamard along a totally different path in the case of homogeneous, isotropic media ([1]).

([1]) C. R. Acad. Sc., 14 March 1898. – *Cf.*, DONTOT, Bull. Soc. math. France **42** (1914), 72-75.

II. – RAY TRANSFORMATIONS

11. The differential expressions ω, ω', ω'', ω''', ω^{IV}, ω^{V} are invariant for *any* homogeneous contact transformation, in such a way that the integral invariants that we studied previously *that relate to the mode of propagation that is defined by a particular function:*

$$H(x_1, x_2, x_3 ; p_1, p_2, p_3)$$

are also preserved by all of the homogeneous contact transformations that leave equation (2) unaltered. Since that equation is not homogeneous in p_1, p_2, p_3, these transformations are the ones that leave the function H itself invariant ([1]). They form a group (Γ) in which the group (G) of the propagation is invariant. Its infinitesimal transformations have characteristic functions that are the integrals, which are homogeneous of first degree in p_1, p_2, p_3, of the equation:

(41) $$(H\,f) = 0.$$

In order to study this group, we convert it into a canonical form by reducing H itself to the canonical form p_3 by using a homogeneous contact transformation.

We write one of these canonical transformations as:

(42) $$\begin{cases} y_i = G_i(x_1, x_2, x_3; p_1, p_2, p_3) & (i = 1, 2, 3), \\ q_i = H_i(x_1, x_2, x_3; p_1, p_2, p_3) & (i = 1, 2), \\ q_3 = H(x_1, x_2, x_3; p_1, p_2, p_3), \end{cases}$$

upon denoting the transformed variables by y_1, y_2, y_3 ; q_1, q_2, q_3 . Under it, the propagation group becomes the group (g) of translations:

(43) $$y_1 = y_1^0, \qquad y_2 = y_2^0, \qquad y_3 = y_3^0 + \theta, \quad q_i = q_i^0, \qquad (i = 1, 2, 3),$$

which is equivalent to the fact that the group (G) itself, upon taking condition (2) into account, has its ∞^4 trajectories defined by the equations:

(44) $$G_1 = y_1^0, \qquad G_2 = y_2^0, \qquad H_1 = z_1^0, \qquad H_2 = z_2^0, \qquad H = 1,$$

if one considers y_1^0, y_2^0, z_1^0, z_2^0 in these equations to be arbitrary constants. Upon eliminating p_1, p_2, p_3 from these equations, one will have the equations for ∞^4 rays, which are also characterized by the four arbitrary constants y_1^0, y_2^0, z_1^0, z_2^0.

([1]) A contact transformation that changes H into another function $\bar{H}$ must be regarded as changing the given medium M into another one $\bar{M}$. It conserves the integral invariants in question if one takes the modification of the medium into account in the expressions for these invariants. On the contrary, in this article, we suppose that one must always apply these invariants to the given medium M.

The canonical transformation (42) converts (Γ) into the group (γ) that is composed of the homogeneous contact transformations (y_1, y_2, y_3 ; q_1, q_2, q_3) that leave q_3 invariant. The finite equations of any one of the contact transformations:

(45) $$y_i' = Y_i(y_1, y_2, y_3 \,;\, q_1, q_2, q_3), \quad q_i' = Q_i(y_1, y_2, y_3 \,;\, q_1, q_2, q_3), \qquad (i = 1, 2, 3)$$

of that group (γ) thus contain the equation $q_3' = q_3$. From the relations of S. Lie ([1]):

(46) $$(Y_i\, Y_k) = (Q_i\, Y_k) = (Q_i\, Q_k) = 0, \qquad (Q_i\, Y_i) = 1 \qquad (i \neq k),$$

one then concludes, due to the fact that $Q_3 \equiv q_3$, that Y_1, Y_2, Q_1, Q_2 does not depend upon y_3 and that Y_3 is of the form:

(47) $$Y_3 = y_3 + \Phi(y_1, y_2 \,;\, q_1, q_2, q_3).$$

If one takes the degrees of homogeneity into account, and if one sets:

(48) $$\frac{q_1}{q_3} = z_1, \qquad \frac{q_2}{q_3} = z_2, \qquad \frac{q_1'}{q_3'} = z_1', \qquad \frac{q_2'}{q_3'} = z_2'$$

then the transformation (45) will be written:

(49) $$\begin{cases} y_i' = A_i(y_1, y_2; z_1, z_2) & (i = 1, 2), \\ z_i' = B_i(y_1, y_2; z_1, z_2) & (i = 1, 2), \end{cases}$$

with

(50) $$y_3' = y_3 - \psi(y_1, y_2 \,;\, z_1, z_2), \qquad q_3' = q_3,$$

and the identity:

$$\sum_{i=1}^{3} q_i\, dy_i = \sum_{i=1}^{3} q_i'\, dy_i'$$

will reduce to:

(51) $$B_1\, dA_1 + B_2\, dh_2 = z_1\, dy_1 + z_2\, dy_2 + d\psi.$$

One thus gets the group (γ) by starting with the group of transformations (49) that leave the Pfaff expression $z_1\, dy_1 + z_2\, dy_2$, *up to an additive total differential*, and by *extending* each of these transformations that are defined by formulas (50) and (48), where the function ψ is obtained – up to an additive constant – by means of the quadrature:

(52) $$d\psi = z_1'\, dy_1' + z_1'\, dy_2' - z_1\, dy_1 - z_2\, dy_2.$$

([1]) S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, v. II, pp. 137.

The group (γ_0) of transformations (49) that were considered was introduced by S. Lie ([1]). As Cartan remarked, one can also define it by the invariance of the bilinear expression ([2]):

$$dy_1\, dz_1 + dy_2\, dz_2\,.$$

It is a simple group, and its infinitesimal transformations are of the form ($K\,f$), where K is an arbitrary function of $y_1, y_2\,;\, z_1, z_2$ ([3]).

12. Each contact transformation of the group (Γ) then corresponds to a transformation (49) of the group (γ_0). Its significance results from the form (44) of the equations of the trajectories. Indeed, upon taking degrees of homogeneity into account and setting:

(53) $$\frac{p_1}{p_3} = l_1\,, \qquad \frac{p_2}{p_3} = l_2\,,$$

one can set:

(54) $$G_i = \mathcal{Y}_i(x_1, x_2, x_3\,;\, l_1, l_2), \qquad H_i = \mathcal{Z}_i(x_1, x_2, x_3\,;\, l_1, l_2) \qquad (i = 1, 2),$$

and the equations of the trajectories become:

(55) $$\mathcal{Y}_i = y_i^0\,, \qquad \mathcal{Z}_i = z_i^0 \qquad (i = 1, 2),$$

in which $y_1^0,\ y_2^0\,;\ z_1^0,\ z_2^0$ are the coordinates of any of these ∞^1 trajectories, or of the ∞^4 rays that support them.

An arbitrary transformation of (Γ) is defined by the elimination of the y and q between equations (42), when one writes the analogous equations with primed letters, and equations (48), (49), (50). It then implies the equations:

(56) $$\mathcal{Y}_i(x' \mid l') = A_i(\mathcal{Y} \mid \mathcal{Z}), \qquad \mathcal{Z}_i(x' \mid l') = B_i(\mathcal{Y} \mid \mathcal{Z}) \qquad (i = 1, 2);$$

i.e., it changes each trajectory (55) into the trajectory whose coordinates $\bar{y}_1^0,\ \bar{y}_2^0\,;\ \bar{z}_1^0,\ \bar{z}_2^0$ are given by the equations:

(57) $$\bar{y}_i^0 = A_i(y^0 \mid z^0), \qquad \bar{z}_i^0 = B_i(y^0 \mid z^0), \qquad (i = 1, 2),$$

which are the equations of the transformation (49).

Therefore, the transformation (49) gives the law of exchange for the ∞^4 trajectories under the ∞^1 transformations of the group (Γ) that they correspond to by means of the quadrature (52), and by the intermediary of the canonical transformation (42).

([1]) S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, v. II, pp. 128, 129, 232, 259, 260.
([2]) E. CARTAN, C. R. Acad. Sc., 21 May 1907.
([3]) S. LIE, Leipziger Berichte, 1895, pp. 292.

13. Since the transformations of the group (Γ) leave the partial differential equation (2) of the wave family invariant, they change any family of waves into another family of waves. They therefore exchange the ∞^2 trajectories that serve to propagate one family of waves into the ∞^2 trajectories that are likewise associated with the same family of waves.

On the other hand, one can say that they exchange rays of the propagation if one agrees to operate on a ray by operating on the contact element that it is pseudo-orthogonal to (no. 1).

Therefore, the transformations of the group (Γ) change the ∞^2 rays that are the pseudo-normals of one family of waves into ∞^2 rays that enjoy the same property, and in turn, the transformations (49) that give the law of exchange of the ∞^4 rays under the homologous transformations of (Γ) in the indicated sense change any congruence of pseudo-normals into a congruence of pseudo-normals. One can say that they are *Malus transformations.*

14. Conversely, imagine an arbitrary transformation (49) that operates on the coordinates of ∞^4 rays, and try to express the idea that it changes any congruence of pseudo-normals into a congruence of pseudo-normals.

In order to do that, one must examine how the coordinates y_1^0, y_2^0; z_1^0, z_2^0 of the trajectories (55) must be chosen as functions of the two parameters u_1, u_2 in order for them to be the trajectories of propagation of a family of waves. Because of the fact that the congruence of rays is given, the trajectories that these rays serve to support are deduced by the adjunction to each point of each ray of the contact element that is pseudo-normal at that point.

One passes from equations (55) to equations (44) directly, and upon adjoining the auxiliary equation:

(58) $$G_3 = u$$

to them, where u is a third parameter, one will define the coordinates x_1, x_2, x_3 ; p_1, p_2, p_3 of the contact elements that correspond to the congruence of rays being considered as functions of u, u_1, u_2 . Moreover, we know that the condition that expresses the idea that they belong to the ∞^1 waves of the same family is that the expression:

(16) $$\omega = \sum_{i=1}^{3} p_i \, dx_i$$

must be an exact total differential in u, u_1, u_2 (*cf.*, no. 7). Because this is true, the integral $t = f(x_1, x_2, x_3)$ of that differential is an integral of the partial differential equation (2) of the family of waves, due to the last of equation (44), and conversely, if the contact elements considered are those of a family of waves – viz., $t = f(x_1, x_2, x_3)$ – then the p_i are proportional to the derivatives $\partial t / \partial x_i$, and as a result they are equal to them, respectively, due to the same relation $H = 1$.

Now, the transformation (42) is a homogeneous contact transformation, and one has the identity:

(59) $$H_1\, dG_1 + H_2\, dG_2 + H_3\, dG_3 = \sum_{i=1}^{3} p_i\, dx_i\ ,$$

from which, one concludes the identity:

(60) $$\sum_{i=1}^{3} p_i\, dx_i = du + z_1^0\, dy_1^0 + z_2^0\, dy_2^0$$

for the functions u, u_1, u_2 considered, if one takes equations (44) and (58) into account.

The desired condition is therefore that the Pfaff expression:

(61) $$z_1^0\, dy_1^0 + z_2^0\, dy_2^0$$

must be an exact total differential in u_1, u_2 .

Therefore, in order for a transformation (49) to be a Malus transformation, it is necessary and sufficient that it must possess the following property: In order for $z_1'\, dy_1' + z_2'\, dy_2'$ to be a total differential, it is necessary and sufficient that $z_1\, dy_1 + z_2\, dy_2$ also be one. In other words, the symbolic equation:

(62) $$dz_1\, dy_1 + dz_2\, dy_2 = 0$$

must be invariant. The equations of the transformation must then lead to an identity:

(63) $$dz_1'\, dy_1' + dz_2'\, dy_2' = \rho\,(dz_1\, dy_1 + dz_2\, dy_2)\ .$$

Upon differentiating them symbolically, one finds the new identity:

(64) $$0 = d\rho\,(dz_1\, dy_1 + dz_2\, dy_2),$$

from which, one effortlessly concludes that ρ is a constant.

The Malus transformations are then the ones that multiply the differential form $dz_1\, dy_1 + dz_2\, dy_2$ by a constant; i.e., the ones that give rise to identities of the form:

(65) $$dz_1'\, dy_1' + dz_2'\, dy_2' - \rho\,(dz_1\, dy_1 + dz_2\, dy_2) = d\psi,$$

in which ρ denotes a constant, and ψ is a function of y_i, z_i, y_i', z_i'.

In other words, one is dealing with the transformations in (x, p) of S. Lie ([1]), and one falls back upon the previously-encountered transformations only in the case where the constant ρ is equal to unity.

([1]) S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, v. II, pp. 131.

15. One recovers the general Malus transformations by considering the group (Γ') that is formed from the homogeneous contact transformations that leave the group (G) invariant. Each of them multiplies the characteristic function H by a constant $1 / \rho$, which is different from 1, in general ([1]).

Operate on the canonical form of (G), as in no. 11. The only change is in the equation $q_3' = q_3$, which must be replaced by $q_3' = q_3 : \rho$ here. As a result, equation (47) is replaced with:

(66) $$y_3' = \rho\, y_3 + \Phi(y_1, y_2 ; q_1, q_2, q_3) ,$$

and equations (50) become:

(67) $$y_3' = \rho\, y_3 - \psi(y_1, y_2 ; z_1, z_2) , \qquad q_3' = \frac{1}{\rho} q_3 .$$

As a result, the identity $\sum q_i\, dp_i = \sum q_i'\, dy_i'$ then gives:

(68) $$B_1\, dA_1 + B_2\, dA_2 = \rho\, (x_1\, dy_1 + x_2\, dy_2) + d\psi,$$

which gives the property that is characterized by the identity (65) for the transformation (49) of the rays.

Conversely, any transformation (49) that belongs to the group (γ_0') is characterized by the form of the identity (65); i.e., any Malus transformation can be *extended* by means of formulas (67) and (48); the function y results from the integration of the total differential (65). It is thus effectively provided by one of the contact transformations that leave the group (G) invariant, and even a ∞^1 of such transformations.

However, if ρ is not equal to *one* then, in reality, these transformations will modify the elastic nature of the medium, since everything happens as if each wave surface were replaced by its homothetic transform with respect to its origin, with a homothety ratio that is constant and equal to ρ. Indeed, this is what results from changing H into $1/\rho\, H$ in the tangential equation (2) of the wave surface.

16. As far as integral invariants are concerned, the application of the canonical transformation (42) exhibits the fact that J_2 and J_1 are attached to systems of rays, because, upon taking the condition $q_3 = 1$ into account, one deduces the expressions:

(69) $$\omega' = dz_1\, dy_1 + dz_2\, dy_2 , \qquad \omega''' = - dz_1\, dz_2\, dy_1\, dy_2 ,$$

from $\sum p_i\, dx_i = \sum q_i\, dy_i$, into which only the constant coordinates of each ray enter.

Moreover, one verifies that since the transformations of the group (Γ) give rise to the identity (52), the invariance of ω' then results immediately from symbolic

([1]) *Ibid.*, pp. 264 and 267. – The infinitesimal transformations of the group (Γ) are the integrals of the equations $(H\, f) = c\, H$, where c is an arbitrary constant.

differentiation, and as a result, that of ω''' (which was already remarked on no. 11). The invariants J_2 and J_4 are therefore unaltered by these transformations.

On the contrary, the transformations of the group (Γ') multiply J_2 by ρ and J_4 by ρ_2, as symbolic differentiation of the identity (65) shows. The same is therefore true for the Malus transformations, which leave the invariants J_2 and J_4 invariant only in the special case of $\rho = 1$.

17. In the case of a homogeneous and isotropic medium, one has:

(70) $$H = \frac{1}{n}\sqrt{p_1^2 + p_2^2 + p_3^2}\,,$$

in which n is a constant that is the constant length of the index vector. Here, one can suppose that it is equal to unity.

One can then take the canonical transformation to be:

(71) $$\begin{cases} q_1 = p_1, & q_2 = p_2, & q_3 = \sqrt{p_1^2 + p_2^2 + p_3^2}, \\ y_1 = x_1 - x_3\dfrac{p_1}{p_3}, & y_2 = x_2 - x_3\dfrac{p_2}{p_3}, & y_3 = \sqrt{p_1^2 + p_2^2 + p_3^2}\,\dfrac{x_3}{p_3}. \end{cases}$$

The equations of the ray that is *normal* to an arbitrary contact element (x_1, x_2, x_3; p_1, p_1, p_1) are:

(72) $$\frac{X_1 - x_1}{p_1} = \frac{X_2 - x_2}{p_2} = \frac{X_3 - x_3}{p_3}$$

here, in which y_1 and y_2 are the coordinates, in the plane $X_3 = 0$, of the foot of the ray in that plane, and our other two coordinates:

(73) $$z_1 = \frac{q_1}{q_3} = \frac{p_1}{\sqrt{p_1^2 + p_2^2 + p_3^2}}\,, \qquad z_2 = \frac{q_2}{q_3} = \frac{p_2}{\sqrt{p_1^2 + p_2^2 + p_3^2}}$$

are the two direction cosines of the ray relative to the x_1-axis and the x_2-axis.

One recovers the variables that were introduced by H. Bruns in his theory of the eikonal ([1]).

The variable y_3 is, moreover, the distance from the foot of the ray to an arbitrary point of the ray.

One then sees that the success of the Bruns transformation amounts to the fact that it changes the group of dilatations into a group of translations.

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([1]) Abhandlungen der k.sächs, Gesellschaft. der Wiss, v. XXI, 1895. – *Cf.*, F. HANSDORFF, Leipziger Berichte, 1896.