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ORDINARY DIFFERENTIAL EQUATIONS.

ELEMENTARY INTEGRATION METHODS.

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For *linear* equations, in particular:

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For integration by *definite* integrals, one can cf., II B 3, for *algebraic* integration, and above all, the *function-theoretic methods*, cf., II B 5. For *approximation methods*, cf., II A 4 a (esp. no. **4**) and 7a, as well as the articles on astronomical mechanics in volumes IV and V. For the *form of curves that are defined by differential equations*, cf., III D 8. For *mechanical integration*, cf., II A 1, no. **61**.

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1. Fundamental problems. Definitions. – The problems that the theory of differential equations addresses were given before in II A 4a, no. **1**. Here, we shall add only:

One understands the *general integral* of an n^{th}-order system:

$$\frac{dx_i}{dx} = \lambda_i (x, x_1, \ldots, x_n) \tag{1}$$

to mean a solution:

$$x_i = \theta_i (x \mid a_1, \ldots, a_n) \qquad (i = 1, 2, \ldots, n) , \tag{2}$$

that depends upon n arbitrary constants $a_1, \ldots, a_n$ *essentially*. The functions θ_i are not defined completely, because one can replace the a_i in them with n arbitrary functions of n other arbitrary constants, i.e., one performs the most general point transformation in the a_i .

A. Cauchy ([1]) and *C. G. J. Jacobi* ([2]) showed that the problem that is thus posed is not separate from that of *integrating the homogeneous linear partial differential equation:*

$$L f \equiv \frac{\partial f}{\partial x} + \sum_{i=1}^{n} \lambda_i (x, x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0 \tag{3}$$

(cf., II A 4a, no. **6** and **16**; II A 5, no. **11**). One knows, in fact, that since *Lagrange* ([3]), the solutions to the equations (3) for the constants will give a *fundamental system of solutions of* (3), i.e., a system of n independent solutions to that equation. Conversely, one obtains the general integral of (1) when one sets n functions $z_1 (x, x_1, \ldots, x_n), \ldots, z_n (x, x_1, \ldots, x_n)$ that define a fundamental system of solutions to (3) equal to arbitrary constants. Moreover, one derives the most general system of integrals from a particular one by performing the most general point transformation on the z. Equation (3) is said to be *equivalent* to the system (1) or *associated* with it. Any equation that one obtains by setting a solution of (3) equal to an arbitrary constant a *first integral* or simply an *integral* of (1). For brevity, one also often calls such a thing a solution in its own right.

The n^{th}-order differential equation:

([1]) C. R. Acad. Sci. Paris **2** (1836), pp. 85; *Oeuvres* (1) **5**, pp. 236. He referred to (3) as the *characteristic equation*.
([2]) J. f. Math. **23** (1841), 1-104; *Ges. Werke* **4**, pp. 149.
([3]) Cf., II A 5, footnote 58.

(4) $$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right)$$

will be converted into an n^{th}-order system when one sets:

(5) $$y = x_1 , \qquad \frac{dy}{dx} = x_2, \qquad \ldots, \qquad \frac{d^{n-1} y}{dx^{n-1}} = x_n .$$

They then define only a special class of n^{th}-order systems for the theories of integration. A *first integral* of (5) is then a relation $\varphi\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right) = a$ that is fulfilled by any solution of (5) for a corresponding value of the constant a. In general, a relation:

(6) $$\varphi\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}} \,\middle|\, a_1, a_2, \ldots, a_k\right) = 0$$

that is fulfilled by any solution of (5) for corresponding values of the a will be called a k^{th} *integral* of k^{th}-order integral. Finally, the *general integral* of (5) is defined by a relation of the form $\varphi(x, y, a_1, \ldots, a_n) = 0$, and as a result, it will represent a family of ∞^n *planar integral curves*. Those definitions can be adapted immediately to arbitrary systems of higher order.

2. Historical overview. Formal theories of integration. – Originally, the goal that was set was that of integrating the system (1) by means of functions q that would contain a finite number of symbols for algebraic operations and "elementary transcendents" (i.e., "explicit integration"). In that form, except for entirely special cases, the problem is *insoluble*, which emerged from the investigations of *N. H. Abel* and *J. Liouville* into transcendents (II B 2) that are defined by equations of the form $dy / dx = f(x)$. However, the mathematicians of the Eighteenth Century were already convinced of that impossibility and then had to address a broader problem: the *integration by quadratures*, in which one demands only that the q can be expressed by a finite number of algebraic operations and *indefinite integrals*. The methods of *separation of variables* (no. **4**) and *the integrating factor* (no. **5**) relate to that problem. The fact that the new problem is not soluble, in general, was accepted for a long time, and was first proved by *J. Liouville* (nos. **8** and **37**) (cf., II B 5). In fact, the goal of the *classical integration theories* was only the reduction of the system (1) of order n to a simpler problem of the same kind in well-defined cases. Thus, in the classical cases, the *reduction of order* of the problem led to the integration of an equation of lower order and to quadratures. With the method of *Euler multipliers for higher-order equations*, one lowers the order by one unit as soon as the multiplier is known. The method of *Jacobi multipliers* allows one to finalize the integration by quadratures under certain assumptions.

It was only in the last few years that anyone considered the *nature of the auxiliary system* to which one reduces the given system to be an essential element of the simplifications to be achieved.

That new viewpoint is the result of the ongoing discovery of numerous properties of various *classes of equations* whose investigation gradually became an important part of the general theory. Namely, the *linear equations* (II B 4), the *canonical systems of dynamics* (IV), and finally, the *Ricatti equation* (viz., the simplest first-order equation that cannot be integrated by quadratures).

The general theory of *transformation groups* (II A 6) that *S. Lie* addressed and its appendix, the theory of *differential invariants*, have decisively required the general theories of integration, as well as the investigation of special classes of equations. *Lie*'s theory of the integration of *systems that admit known transformation group* (no. **13**) not only allows one to reduce most classical methods to *one* principle, but also to infer precise consequences on the nature of the auxiliary system that suffices for the integration of the given system from the structure of the group that one assumes to be known. At the same time, *Lie systems* or *systems with fundamental solutions* (no. **29**) were introduced, of which the linear systems and the Ricatti equations are only very special cases, and for which there exist actual methods of integration. Finally, the canonical systems of dynamics are also special cases of very general systems whose investigation is connected with that of the *infinite* groups.

Finally, the totality of those *formal integration theories* emerges from the following two problems:

1. Derive the greatest possible benefit from previously-known general properties of a system of differential equations.

2. Define special classes of differential equations and investigate their properties.

3. Introducing new variables. Equivalence problems. Rational integration theories. – The oldest and most general integration method is the *method of introducing new variables* or the *transformation of the differential system.* It consists of the search for a transformation of $(n + 1)$-dimensional space:

$$(7) \qquad \bar{x} = F(x, x_1, \ldots, x_n), \quad \bar{x}_i = F_i(x, x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

that will convert the system (1) into a system of a special form that has already been integrated. The method, when understood in that way, is so vague that it hardly deserves the name of a method: Two arbitrary systems (1) can be converted into each other by infinitely-many transformations (7). Nevertheless, it has not only yielded numerous special results, but one can say, with *S. Lie*, that it defines the common basis for all formal integration theories that only have the goal of making it more precise in the individual cases. In fact, each of those theories examines a special integration problem: It follows from the special circumstances of the problem that the given system can be converted into a system that has already been integrated by transformation that belong to a special set of such things. The properties of that set of transformations determines the nature of the integrations that are required for the solution of the given problem. *Lie* could undertake a

systematic investigation of the present integration problem by starting from the general principle that *any integration problem is equivalent to a transformation problem* ([4]).

The transformation method actually leads to precisely-posed questions then when one needs to decide whether two differential systems of the same class can be transformed into each other with the help of a transformation in a (finite or infinite) group whose transformations convert all systems of that class into systems of the same class. The solution of that *equivalence problem* is closely connected with the theory of *differential invariants* (no. **33**).

Finally, let the term *rational integration theories* ([5]) refer to certain new theories whose goal is to specify the nature of the simplifications that permit the integration of a given *special* system in comparison to that of the most general system of the same class. More precisely, one must answer the following three questions:

1. When is a system to be regarded as a *special* system of the given class, and what sort of different categories of special systems include that class?

2. What simplifications suggest themselves for the integration of each of those categories?

3. How can one methodically decide whether a given system is special, and indeed, which category it belongs to?

The second question falls within the scope of the formal integration theories. We shall pass over the third one here, since, on the one hand, it presently admits no definitive answer in most cases, while on the other hand, it requires the tools of the theory of functions of complex variables.

First-order equations.

4. Method of separation of variables. – The simplest system (1) is the one that consists of a first-order equation ($n = 1$):

$$\frac{dy}{dx} - \lambda(x, y) = 0 \tag{8}$$

or more symmetrically:

$$\frac{dx}{\alpha(x, y)} = \frac{dy}{\beta(x, y)}. \tag{8.a}$$

The general integral is $z(x, y) = a$ when z is the integral of the equivalent equation:

$$L(f) \equiv \frac{\partial f}{\partial x} + \lambda(x, y)\frac{\partial f}{\partial y} = 0, \tag{9}$$

or

([4]) Leipziger Berichte **47** (1895), pp. 269; *ibid.* **48** (1896), pp. 390.
([5]) **J. Drach** (Paris thesis 1898, pp. 30) used the expression *logical integration* in an analogous sense.

(9.a) $$\mathsf{A}(f) \equiv \alpha(x,y)\frac{\partial f}{\partial x} + \beta(x,y)\frac{\partial f}{\partial y} = 0\,,$$

resp. *Leibniz* and his immediate school have investigated the case in which λ is a product of a function of x and a function of y. The equation can then be written as:

(10) $$P(x)\,dx + Q(y)\,dy = 0\,,$$

and its general integral is:

(11) $$\int P(x)\,dx + \int Q(y)\,dy = \text{const.},$$

in which only the given quadratures are to be performed. In this case, one says that *the variables* in equation (8) *have been separated*.

The *method of separation of variables*, which Leibniz's school based upon that remark, consists of effecting the conversion of the given equation into another one in which the variables are separated by the introduction of new variables. It is achieved, *inter alia*, for the *homogeneous* equation:

(12) $$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$$

by means of the substitution $y = u\,x$ (*Joh. I Bernoulli*) and for the *linear* equation:

(13) $$\frac{dy}{dx} = A(x)\,y + B(x)$$

by means of the substitution $y = u\exp\int A(x)\,dx$ (*Jac. I Bernoulli*). One generalizes the method when one seeks to reduce the given equation into a case that has been resolved already [e.g., (12) or (13)] by the introduction of new variables. Mention should be made of the *equation of Jac. I Bernoulli:*

(14) $$\frac{dy}{dx} = A(x)\,y + B(y)\;y^{-m+1},$$

which is reduced to a linear equation by the substitution $u = y^m$ ([6])

([6]) For the bibliography of this subsection, one might cf., e.g., **Lacroix**, *Traité* **2** or **M. Cantor**, *Geschichte der Mathematik* **3**, Leipzig 1898, Chap. 100.

5. Method of Euler multipliers. – A second method ([7]) is based upon the fact that the integral z is defined by an identity of the form:

$$dz = M(x, y)\,[\alpha(x, y)\,dy - \beta(x, y)\,dx]\,. \tag{15}$$

Hence, if one knows the *multiplier* or *integrating factor* $M(x, y)$ then one will get z by a quadrature. The general determination of a multiplier is no easier than that of an integral, because it will require the integration of the equation:

$$\frac{\partial(M\alpha)}{\partial x} + \frac{\partial(M\beta)}{\partial y} = 0\,. \tag{16}$$

Euler showed that its general solution is $M = M_0\,\varphi(z_0)$, if M_0 means any multiplier, z_0 is a first integral, and φ is an arbitrary function. It then follows that the quotient of two multipliers will be an integral when it is not a constant.

In certain special cases, the form of the equation can lead one to look for multipliers of a special form that can actually be determined then. For example, if the linear equation (13) has a multiplier then the function will be one of only x and can then be determined by a quadrature: $M = \exp -\int A\,dx$. In that way, *Euler* could successfully apply his method to not only those examples that his predecessors had treated by separation of variables, but also to many other ones. For example, he found various integrable cases for the equation ([8]):

$$y\,dy + [A(x)\,y + B(x)]\,dx = 0\,. \tag{17}$$

However, *Euler*'s most fruitful idea was to treat the *inverse problem*, i.e., look for equations that have multipliers of a given form. He then obtained numerous special results that were, of course, almost all very specialized ([9]).

6. Lie's method. – *S. Lie* remarked ([10]) that most of the equations that could be integrated by the foregoing methods could be converted into themselves by a one-parameter group of transformations, and therefore by an infinitesimal transformation. Thus, equation (12) admits the group: $\bar{x} = m\,x$, $\bar{y} = m\,y$, and (13) admits the group $\bar{y} = y + \exp\int A\,dx$. He was then led to the following problem: *One knows a one-parameter group that leaves equation* (8) *invariant (in terms of its finite equations). What benefit can one derive from that fact for its integration? Lie* showed

([7]) **L. Euler**, N. Comm. Petr. **8** (1760), pp. 3 and *ibid.* **17** (1772), pp. 105, as well as *Inst. calc. int.* **2.** – The idea of a multiplier seems to go back to **A. Clairaut**, Paris Hist. 1739 and 1740.

([8]) Equation (17) has been examined many times since then. For instance, **N. H. Abel** (*Oeuvres* **2**, pp. 26); **Minding**, Petersbourg Mem. (7) **5** (1862), pp. 1; **V. Z. Elliot**, Ann. éc. norm. sup. (3) **7** (1890), pp. 101; **Korkine**, Math. Ann. **48** (1897), pp. 317.

([9]) Here, let us list: **N. Alexeyev**, *Intégration des équations différentielles*, Moscow 1878; **A. Winckler**, Wiener Ber. **99** (1890), pps. 475, 875.

([10]) For all of the concepts from the theory of continuous transformation groups that appear here, one might confer II A 6.

that it would suffice to put the group into its canonical form $\bar{X} = X$, $\bar{Y} = Y + t$. In terms of those new variables, equation (8) will have the integrable form $dY / dX = \lambda (X)$. The reduction of the group to its canonical form generally requires a quadrature ([11]).

Later ([12]), *Lie* took up the problem again under the assumption that only one infinitesimal transformation in the group that takes equation (8) or (8.a) into itself:

$$(18) \qquad \mathsf{T}(f) \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

is known. He found that in this case, the reciprocal value of $\Delta = \alpha\,\eta - \beta\,\xi$ is a multiplier of $\alpha\,dy - \beta\,dx$. A quadrature will then suffice to integrate the equation. The most general infinitesimal transformation that leaves (8.a) [(9.a), resp.] invariant is defined by an identity of the form ([13]):

$$(19) \qquad (\mathsf{A}\,f, \mathsf{T}\,f) \equiv \rho(x, y) \cdot \mathsf{A}(f)$$

that produces a system of partial differential equations for the determination of ξ, η. One can simplify that investigation by considering only transformations of the form $\mathsf{T}\,f \equiv \eta\,(x, y)\,\partial f / \partial y$, which is not an essential restriction. However, the condition that $1 / \eta$ must be a multiplier of (8) will then be such a thing. *Lie*'s method is then precisely equivalent to that of *Euler* (no. **5**). However, it is generally easier to apply. *Lie* derived a geometric interpretation of the multiplier from it ([14]).

Finally, *Lie* presented his method in yet another form ([15]): The identity (19) shows that the equations:

$$(20) \qquad \mathsf{A}\,f = 0\,, \qquad \mathsf{T}\,f = 1$$

have a common solution. It will be obtained by a quadrature that will also lead to the multiplier $1/\Delta$. Equations (20) are the analytical translation of *Lie*'s first method, which systematized the method of separation of variables (no. **4**). One will then have the connecting link between the two classical methods, as well.

Following *Euler*'s example, *Lie* also treated the inverse problem: *Determine all first-order equations that admit a given one-parameter group*. One can give the general solution here, namely: $\Phi(J_0, J_1) = 0$, if Φ means an arbitrary function, J_0 is the invariant of the group, and J_1 is its first-order differential invariant. Those two invariants can be calculated by elimination as soon as one knows the finite equations of the group ([16]).

([11]) **F. Klein** and **S. Lie**, Math. Ann. **4** (1871), pp. 50.
([12]) Christ. Förh. (1874), pp. 242.
([13]) For the definition of the Jacobi bracket expression, one might confer II A 6, no. **5**.
([14]) *Loc. cit.*, Cf., **Lie-Scheffers**, *Differentialgeichungen*, Chap. 9, where there are also various applications.
([15]) Cf., e.g., **Lie-Scheffers**, p. 96.
([16]) **Lie-Scheffers**, p. 138.

7. Discussion. Comparing transcendents. Algebraic integration. – As one sees, the three basically-equivalent methods that were discussed have a certain character of indeterminacy in common: It lies in the foregoing search, whether it be for the transformation that leaves the equation invariant, or the search for a multiplier, or the search for an infinitesimal transformation that leaves the equation invariant. They have, in fact, produced only very few results. Their main uses are in the definition of types of integrable equations, the most interesting of which were found already in *Euler*'s integral calculus.

On the other hand, in regard to these methods, it should be remarked that they frequently mask the true nature of the integral, since they aspire to an integration by quadratures directly. Thus, the method of separation of variables gives the integrals of the equations:

(21) $$\frac{dx}{P(x)} + \frac{dy}{P(y)} = 0$$ (P is a polynomial degree 1 or 2),

(22) $$\frac{dx}{\sqrt{P(x)}} + \frac{dy}{\sqrt{P(y)}} = 0$$ (P is a polynomial degree 1, 2, 3, or 4)

in transcendental form, while those integrals are actually algebraic. That will follow from the theory of exponential and circular functions for (21) and (22) when P has degree 1 or 2. When P has degree 3 or 4, *Euler* had proved that for (22) when P has degree 3 or 4, and that equation then bears his name. He developed a theory from that, which goes by the name of the *comparing transcendents*, and which includes the germ of the theory of elliptic functions (II B 6 a). Various processes for integrating Euler's equation were given later. The true basis for the aforementioned theorem lies in *Abel's theorem* (II B 2), which also leads to generalizations.

Furthermore, one can confer the *algebraic integration* of first-order equations in Article II B 5.

8. Jacobi and Ricatti equations. – Among the specialized investigations that do not fall within the purview of the general methods, the most important ones are:

The equation of C. G. J. Jacobi ([17]):

(23) $$P(x, y)\,dx + Q(x, y)\,dy + R(x, y)\,(x\,dy - y\,dx) = 0$$

(P, Q, R are first-degree polynomials) has the integral:

(24) $$U^{\lambda_2-\lambda_1} V^{\lambda_2-\lambda_1} W^{\lambda_2-\lambda_1} = \text{const.},$$

([17]) J. f. Math. **24** (1842), pp. 1; *Ges. Werke* **4**, pp. 256.

in which *U, V, W* are first-degree polynomials, and λ_1, λ_2, λ_3 mean the roots of a third-degree equation. *J. A. Serret* ([18]) gave the form of the integral for the cases in which that auxiliary equation had equal roots with the help of *d'Alembert*'s method (no. **21**). The investigations of *G. Darboux* ([19]) into algebraic integrals of first-order equations are connected with that.

The Ricatti equation:

$$\frac{dy}{dx} = A(x)\,y^2 + B(x)y + C(x) \tag{25}$$

gets its name from the special case:

$$\frac{dy}{dx} + a\,y^2 = b\,x^m \qquad (a, b \text{ constants}) \tag{26}$$

that was examined by *Count J. Ricatti* ([20]). The latter can be integrated by elementary functions when $m = -\frac{4k}{2k \pm 1}$ (k is a positive whole number) that have a close relationship to *Bessel functions* (II B 4 b, no. **44**), moreover. The general Ricatti equation cannot be integrated by quadratures. That fact was exhibited for the special equation (26) by *J. Liouville* ([20.a]) and also follows from the corresponding theorems on linear equations (no. **37**), as well as *Maximovich's* ([21]) determination by quadratures of integrable classes of equations of the form: $dy / dx = F(x, y, \eta(x), \ldots, \eta_n(x))$ ($\eta, \ldots, \eta_n$ are undetermined functions, and F is a well-defined function of its arguments). The general integral of the Riccati equation has the form:

$$y = \frac{c\,\varphi(x) + \psi(x)}{c\,\chi(x) + \omega(x)}\,, \tag{27}$$

when c means the integration constant. In other words, the double ratio of any four particular integrals is constant. That property, which is characteristic of the Ricatti equation, seems to have first been emphasized only in recent years ([22]). However, it is implied immediately from fact that *Euler* already knew that one can convert the equation into a linear one by the substitution $y = y_1 + u^{-1}$ when one knows a particular integral y_1 .

([18]) *Calc. diff. et int.* **2**, pp. 431.
([19]) Bull. soc. math. (1878), pp. 72.
([20]) Acta erud. (1722). **D'Alembert**, Berl. Hist. (1763) [70], pp. 242, had already used the name in the more general sense that is now customary.
([20.a]) J. de math. **6** (1841), pp. 1.
([21]) C. R. Acad. Sci. Paris **101** (1885), pp. 809.
([22]) **E. Picard**, Ann. éc. norm. (1) **6** (1877), pp. 341; **Ed. Weyr**, Prague Abh. (6) **8** (1875) [77], pp. 30. The importance of Ricatti's equations was already implied by **O. Bonnet**'s studies of ruled surfaces. Cf., **G. Darboux**, *Théorie des surfaces* **1**, Chap. 2 and **E. Picard**, *Traité d' analyse* **2**, pp. 329.

9. Unsolved equation. Integration by differentiation. – If a first-order differential equation is given in unsolved form:

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \tag{28}$$

then if one is to be able to apply the general methods, one must first solve it for $dy / dx = y'$, in general. That solution might be impracticable. One often avoids it by an extension of the transformation method by which one applies a transformation that also includes y'. One calls that *integration by differentiation*. The oldest of those transformations ([23]) consists of introducing y' in place of one of the old variables, e.g., *y*. One must then eliminate *y* and *dy* from (28) and the equations:

$$dy - y'\,dx = 0\,, \qquad dF \equiv \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y'}dy' = 0\,. \tag{29}$$

One will then get a first-order differential equation in *x* and y'. One seeks its general integral $V(x, y', a) = 0$, and one will then need only to eliminate y' from $F = 0$ and $V = 0$. That transformation achieves the goal for the *d'Alembert equation:*

$$x\varphi(y') + y\psi(y') + \chi(y') = 0\,, \tag{30}$$

since it converts that equation to one of the linear type. A special case of that is the *Clairaut equation* ([24]):

$$y - a\,x + \chi(a) = 0\,. \tag{31}$$

Above all, one always applies it when $F = 0$ can be solved for *y* (or *x*, *mutatis mutandis*).

More generally, when *F*, *G*, *H* are three independent functions, one can set:

$$X = G(x, y, y')\,, \qquad Y = H(x, y, y')\,. \tag{32}$$

One eliminates the equations $dX = dG$, $dY = dH$ from those equations (which are derived from them) and *x*, *y*, y', *dx*, *dy*, dy' from (28) and (29). If $V(X, Y, a) = 0$ is the general integral of the transformed equation in *X*, *Y* then one must still eliminate y' from $F = 0$ and $V = 0$. Such a transformation can be applied, e.g., when $F(x, y, y') = 0$ represents a rational surface in *x*, *y*, y' ([25]).

([23]) **J. d'Alembert**, Berl. Hist. (1748).

([24]) **A. Clairaut**, Paris Hist. (1734). A generalization of it is in **E. Goursat**, Bull. soc. math. **23** (1895), pp. 88; An investigation of the transformation into y' is in **L. Raffy**, *ibid.*, pp. 50.

([25]) **Halphen**, C. R. Acad. Sci. Paris **87** (1878), pp. 241. – $F(y, y') = 0$ is treated similarly when $F = 0$ represents a rational curve.

In particular, one can apply *contact transformations* (III D 7), i.e., the most general transformations that take any first-order equation into another first-order equation, and at the same time, take any solution of the former into a solution of the latter. In order to avoid any complications in that, one must understand a solution of $F = 0$ to mean any *element-manifold* M_1 (II A 5, no. **9**) whose first-order elements all fulfill the equation $F = 0$ ([26]). One can then conceptualize the integration problem as follows: Find a contact transformation that reduces $F = 0$ and $Y = 0$. For the calculations, that will emerge from the search for a *first integral* of (29) that is different from F, i.e., a solution of the equation ([27]):

$$[F, f] = 0 , \tag{33}$$

which is equivalent to that system (no. **1**). That is because when $U(x, y, y')$ is that solution, the desired transformation will be:

$$X = U , \qquad X = F , \qquad Y' = \frac{\partial F}{\partial y'} : \frac{\partial U}{\partial y'} , \tag{34}$$

and the general integral of $F = 0$ is defined by $F = 0$, $U = a$.

The method of *S. Lie* ([28]) is given in that form. However, for some time now, the process of integration by differentiation has been in practice, in which one replaces the equation $F = 0$ with the second-order system (29). Since one already knows a first integral of that, namely F, one is close to an application of the method of Jacobi multipliers (no. **12**) to it. The investigations of *J. Liouville* ([29]), *Malmstén* ([30]), *E. Laguerre* ([31]) relate to that and yield integrable types that include arbitrary function by quadratures. *Lie*'s theories (on equations that admit a given infinitesimal contact transformation) give analogous types ([31]). *A. Mayer* ([32]) determined the condition for an equation that includes an arbitrary function to be capable of being converted into one of the linear type by a contact transformation that is independent of that arbitrary function.

Equations of the form:

$$\varphi(U(x, y, y'), V(x, y, y')) = 0$$

are immediately integrable when U, V are two first integrals of a second-order equation ([33]). Their general integral is defined by the equations $U = a$, $V = b$, $\varphi(a, b) = 0$, between which one must eliminate y'. The condition that U, V must satisfy is $[U, V] = 0$, such that the result can be derived from the foregoing method of *Lie* ([34]).

([26]) Cf., e.g., **Lie-Engel**, *Transformationsgruppen* **2**, Chap. 1, § **8**.
([27]) For the definition of the *Poisson bracket expression* [ ], cf., II A 5, no. **24**.
([28]) J. de math. **20** (1855), pp. 143.
([29]) J. de math. (2) **7** (1862), pp. 314.
([30]) Bull. soc. math. **6** (1878), pp. 224 (*Oeuvres* 1, pp. 409).
([31]) Cf., e.g., **Lie-Scheffers**, *Berührungstransformationen* **1**, pp. 111.
([32]) Leipziger Ber. (1890), pp. 491.
([33]) **J. Lagrange**, *Leçons sur le calcul des fonctions*, Leçon 16; *Oeuvres* 10, pp. 220.
([34]) Cf., e.g., **Lie-Engel**, *Transformationsgruppen* **2**, Chap. 1.

10. Geometric interpretations. Use of homogeneous coordinates. – it is often advantageous to replace a first-order equation with a system of equations. Even for the normal form (8.a), one often replaces it with the system ([35]):

$$\frac{dx}{dt} = \alpha(x, y), \qquad \frac{dy}{dt} = \beta(x, y), \tag{34}$$

and from the Lie standpoint, that amounts to the same thing as determining the finite equations of the one-parameter group that is generated by the infinitesimal transformation (9.a). Another example of that is given by the use of *homogeneous coordinates* ([36]): The first-order differential equation expresses a relationship between an arbitrary point on an integral curve and its tangent at that point, and as a result, it can be brought into the doubly-homogeneous form:

$$\Phi(x, y, z \mid u, v, w) = 0, \tag{35}$$

in which x, y, z mean the homogeneous coordinates of the point, and u, v, w are those of the tangent. The "solved" equation corresponds to the case in which Φ has the form:

$$A(x, y, z)\, u + B(x, y, z)\, v + C(x, y, z)\, w, \tag{36}$$

in which A, B, C are homogeneous and have the same degree m ([37]). One can replace that equation with the system:

$$dx : dy : dz = A : B : C, \tag{37}$$

i.e., with the associated equation:

$$A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} + C\frac{\partial f}{\partial z} = 0. \tag{38}$$

One must seek a homogeneous integral of order 0 of that. To that end, one can determine a multiplier μ that satisfies the **Jacobi** relation:

$$\frac{\partial(\mu A)}{\partial x} + \frac{\partial(\mu B)}{\partial y} + \frac{\partial(\mu C)}{\partial z} = 0.$$

If one is careful about ensuring that it is homogeneous of order $-(m + 2)$ then the differential:

([35]) Cf., e.g., the methods of integrating the **Euler** equation (no. **7**) that **L. Euler** (*Inst. calc. int.* 4, pp. 481), **J. Lagrange** (*Oeuvres* **2**, pp. 6), and **G. Darboux** employed.

([36]) **A. Clebsch**, Göttinger Abh. (1872) and Math. Ann. **5** (1872), pp. 427 (cf., **Clebsch-Lindemann**, *Geometrie* 1, Leipzig 1876, pp. 963), **G. Darboux**, Bull. soc. math. **6** (1878), pp. 68; cf., also **Fouret**, C. R. Acad. Sci. Paris **78** (1874), pp. 837.

([37]) $m = 1$ gives the **Jacobi** equation (no. **8**); cf., **A. Clebsch** and **P. Gordan**, Math. Ann. **1** (1869), pp. 359.

$$\mu \begin{vmatrix} A & B & C \\ x & y & z \\ dx & dy & dz \end{vmatrix}$$

will be an exact differential (II A 2, no. **43**). One will get the desired general integral when one integrates that differential.

For the relationships between that and *Clebsch*'s theory of connections, one can cf., III C 10.

S. Lie gave another geometric interpretation ([38]): He considered $F(x, y, y') = 0$ to be the equation of a surface on which the curves that belong to the linear complex $dy - y'\,dx = 0$ are to be determined. One can develop the theory of *singular integrals* (II A 4a, no. **20**) synthetically by starting from that remark.

Systems of first-order equations. General theories.

11. Systems of multipliers. – The classical process of integrating the general system (1) consists of successively lowering the order of the system. To that end, one seeks a first integral $z(x, x_1, x_2, \ldots, x_n) = a$ and uses it to eliminate one of the unknowns (e.g., x_n) from the system (1). One will thus obtain a system of order $(n - 1)$ in x, x_1, …, x_{n-1} that one further treats in the same way. When regarded in that way, the integration will decompose into n successive operations, namely: Determine *one* integral of an n^{th}-order system, *one* integral of an $(n - 1)^{\text{th}}$-order system, etc., and finally, an integral of a first-order equation. – More generally: If one knows k first integrals of (1) then one can reduce it by elimination to order $n - k$.

The search for a first integral z can take place with the help of a *system of multipliers*, because since it is defined by an identity of the form:

$$dz = \sum_{i=1}^{n} \mu_i [dx_i - \lambda_i\, dx] \ , \tag{39}$$

it can be obtained by a quadrature as soon as the multipliers μ_i are determined in such a way that the right-hand side of (39) becomes a complete differential. *C. G. J. Jacobi* ([39]) has investigated those multipliers [which are an immediate generalization of the *Euler* multipliers (no. **5**)]. Their general determination is no easier than the integration of the given system. *Jacobi* made an interesting application to the *n^{th}-order linear system:*

$$dx_i - \left[\sum_{h=1}^{n} a_{ik}(x)\, x_h + b_i(x) \right] dx = 0 \qquad (i = 1, 2, \ldots, n) \ . \tag{40}$$

([38]) Cf., e.g., **Lie-Scheffers**, *Berührungtransformationen* 1, pp. 182. In a somewhat-modified form, cf., **L. Autonne**'s "Untersuchungen über algebraische Integration," C. R. Acad. Sci. Paris **105**, Nov. 1887.

([39]) J. f. Math. **23** (1842), 1-104; *Ges. Werke* **4**, pp. 236; **Euler** worked with such multipliers for $n = 2$ and $n = 3$, but without asserting their existence in general.

Here, one can look for the μ_i as functions of only x, because they are themselves determined by a likewise-linear system, namely, the *adjoint* ([40]) system to the given one, namely:

$$d\mu_i + \sum_{j=1}^{n} \mu_j \, a_{ji}(x)\, dx = 0 \,. \tag{41}$$

If the system (40) is *homogeneous*, i.e., all $b_i(x)$ are identically zero, then the two systems (40) and (41) are mutually reciprocal, so each of them is the adjoint of the other.

Ordinarily, one restricts oneself to a choice of the μ_i such that the linear homogeneous combination of equations (1):

$$\sum_{i=1}^{n} \mu_i \,(dx_i - \lambda_i\, dx) = 0 \tag{42}$$

becomes an "exact equation" (II A 5, no. **19**). One sees from this that their left-hand sides depend upon only two functions u, v of the variables x, x_1, …, x_n, and their differentials du, dv. One thus obtains a first-order equation between u and v whose integration will yield a first integral of the system (1). Since one deals with only the ratios of the μ_i in that, one can choose one of them arbitrarily. *J. d'Alembert* ([41]) applied that method to the linear system (40). When he set $\mu_n = 1$ and looked for μ_1, …, μ_{n-1} as functions of x alone, he brought the combination (42) into the form:

$$\frac{du}{dx} - u\left\{\sum_{i=1}^{n-1} \mu_i\, a_{in} + a_{nn}\right\} - \left\{\sum_{i=1}^{n-1} \mu_i\, b_{in} + b_{nn}\right\} - \sum_{h=1}^{n-1} x_h\, \Phi_h = 0 \,, \tag{43}$$

with:

$$\left\{\begin{array}{c} u = \displaystyle\sum_{i=1}^{n-1} \mu_i\, x_{n-1} + x_n \,, \\ \Phi_h = \dfrac{d\mu_h}{dx} + \displaystyle\sum_{i=1}^{n-1} \mu_i\, a_{ih} + a_{nh} - \mu_h \left\{\sum_{i=1}^{n-1} \mu_i\, a_{i,n} + a_{n,n}\right\} \\ (h = 1, 2, \ldots, n-1). \end{array}\right. \tag{44}$$

The combination (43) will become an exact equation when one determines μ_1, …, μ_{n-1} from the equations $\Phi_h = 0$ ($h = 1, 2, \ldots, n-1$), which define a system of order $n - 1$ ([42]). If it is integrated then what will remain is a linear equation in u, and the integration will be completed by two quadratures.

One can further reduce a recently-proposed method of *A. Guldberg* ([43]) for integrating an n^{th}-order ordinary differential equation to this method.

([40]) **Jacobi**, J. f. Math. **27** (1844), pp. 199 and **29** (1845), pp. 213 and 333; *Ges. Werke* 4, pp. 319.
([41]) Berl. Hist. (1748) [50], pp. 283; cf., also Paris Hist. (1768) and (1769).
([42]) For the properties of such systems, cf., no. **29**.
([43]) C. R. Acad. Sci. Paris **121** (1895), pp. 49.

In the general case, the partial differential equations that μ_i must satisfy will be complicated. However, it will suffice to know a particular (or even singular) solution of them. Thus, in the case where the a_{ik} are constant, the method of *d'Alembert* will lead to the goal when one takes constant values for the μ_i that satisfy the equations $\Phi_h = 0$. That remark is true for the *Euler* multiplier (no. **5**).

12. The Jacobi multiplier ([44]) is likewise a generalization of *Euler*'s, but from a whole different viewpoint. Let the system (1) be given in the symmetric form:

$$\frac{dx_0}{\alpha_0(x_0, x_1, \ldots, x_n)} = \frac{dx_1}{\alpha_1(x_0, x_1, \ldots, x_n)} = \ldots = \frac{dx_n}{\alpha_n(x_0, x_1, \ldots, x_n)}, \tag{45}$$

such that the associated equation (3) will be:

$$\mathsf{A} f \equiv \sum_{i=0}^{n} \alpha_0(x_0, x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0 . \tag{46}$$

When $z_1, z_2, \ldots, z_n$ define a fundamental system of solutions of $\mathsf{A} f = 0$, there will exist (II A 5, no. 15) a multiplier M that satisfies the identity ([45]):

$$\mathsf{M} \cdot \mathsf{A} f = \frac{\partial(f, z_1, \ldots, z_n)}{\partial(x_0, x_1, \ldots, x_n)} , \tag{47}$$

which might also be f. Conversely, when $\mathsf{M}, z_1, z_2, \ldots, z_n$ satisfies such an identity, they will define a fundamental system of solutions of $\mathsf{A} f = 0$. That identity is then the *definition of Jacobi multiplier* of system (45). It is equivalent to the linear partial differential equation:

$$\sum_{i=0}^{n} \frac{\partial(M \alpha_i)}{\partial x_i} = 0 . \tag{48}$$

(For $n = 1$, one gets back to the *Euler* multiplier.) Here as well, the quotient of two multipliers is an integral, and the *singular* solutions of the system satisfy the equation $M^{-1} = 0$. If one introduces new variables $y_0, y_1, \ldots, y_n$ into the system in place of the $x_0, x_1, \ldots, x_n$ then:

$$N = M \frac{\partial(x_0, x_1, \ldots, x_n)}{\partial(y_0, y_1, \ldots, y_n)} \tag{49}$$

([44]) The theory is developed in the reference that was cited in footnote 39, along with most of the applications.
([45]) For the notation of functional determinants, one might cf., I B 1 b, no. **19**.

will be a multiplier of the transformed system. If k of the y_i are first integrals of the given system then the transformed system will reduce to one of order $n - k$, and N will be a multiplier of the reduced system. For $k = n - 1$, that result will lead to the *theorem of the last multiplier:* When one knows $n - 1$ integrals and a multiplier, the last integral can be determined by a quadrature. If one assumes, in addition, that one of the variables does not appear explicitly in the equations of the system then either the penultimate integral can be determined by a quadrature or the last one can be determined without a quadrature. This last result implies the *principle of the last multiplier* for dynamics.

Jacobi applied that theory to the linear system (48), which admits the multiplier:

$$\mathsf{M} = \exp\left[-\int \sum_{h=1}^{n} a_{hh}\, dx\right]. \tag{50}$$

He further derived from it that the integration of the equation:

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x, y) = 0 \tag{51}$$

requires only one quadrature as long as one knows a first integral. However, the most important applications relate to the *canonical equations of W. R. Hamilton:*

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2, \ldots, n), \tag{52}$$

in which H is a function of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, and t. It admits a constant as a multiplier. It includes the equations of mechanics (IV 7) as special cases, as well as those of the calculus of variations (II A **9**), and the equations of the characteristics of first-order partial differential equations (II A 5, no. **28**, *et seq.*).

Malmstén ([46]) obtained a formally more general result by a combination of the theory of multipliers and the method of permutation of variables and carried out numerous applications, namely, to equations of orders 3 and 2, as well as to first-order equations with the use of the method of differentiation (no. **9**). For example, the equations $\frac{d\varphi(y, y')}{dx} = \psi(y, y')$ and $\frac{d\varphi(y, y')}{dx} = y''\psi(y, y')$ can be integrated by quadratures as soon as one knows a first integral.

13. Lie method: Integrating systems with known transformation groups. – If a system (45) is given then it can happen that as a result of the nature of the problem that leads to that system or as a result of the form of its equations, one knows transformations that take the system to itself and

([46]) J. de math. (2) **7** (1862), pp. 257. For the applications of the Jacobi method, one might also cf. **Andreyevski**, C. R. Acad. Sci. Paris **68** (1869), pp. 716 and **A. Winckler**, Wiener Ber. **80** (1879), pp. 948.

depend upon arbitrary constants. One derives infinitesimal transformations that leave the system invariant from them by differentiation. Thus arose the problem that *S. Lie* ([47]) posed: *Integrate the system* (45) *when one knows q infinitesimal transformations:*

$$(53) \qquad \mathsf{X}_h f \equiv \sum_{i=0}^{n} \xi_{hi}(x_0, x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} \qquad (q = 1, 2, \ldots, h)$$

that leave the system invariant. The condition for the system to admit the infinitesimal transformation $\mathsf{X} f$ is: As long as z is an integral of $\mathsf{A} f = 0$, $\mathsf{X} f$ must also be an integral, and as a result, an identity of the form:

$$(54) \qquad (\mathsf{X} f, \mathsf{A} f) \equiv \rho (x_0, x_1, \ldots, x_n) \cdot \mathsf{A} f$$

must exist. When the system admits $\mathsf{X}_1 f$ and $\mathsf{X}_2 f$, it will also admit the infinitesimal transformation $(\mathsf{X}_1 f, \mathsf{X}_2 f)$. Finally, when an identity of the form:

$$(55) \qquad \mathsf{X}_q (f) \equiv \sum_{h=0}^{q-1} u_h(x_0, x_1, \ldots, x_n) \mathsf{X}_h(f) + \sigma(x_0, x_1, \ldots, x_n) \mathsf{A} f$$

exists, each of the functions u_h will be either a constant or an integral of $\mathsf{A} f = 0$. On the basis of those theorems, one can, in certain cases, increase the number q of known infinitesimal transformations that leave the system invariant and include a certain number of first integrals without integration, with the help of which one can lower the order of the system. Furthermore: When $\mathsf{X} f$ leaves the system invariant, $\mathsf{X} f + \tau (x_0, x_1, \ldots, x_n) \mathsf{A} f$ will have the same property, no matter what τ might be. As a result of that, one can assume that the term with $\partial f / \partial x_0$ is missing $\mathsf{X}_1 f, \ldots, \mathsf{X}_q f$. After making all of those simplifications, one will be dealing with an analogue of the original problem in which, however, the infinitesimal transformations define a finite group G whose order is equal to at most the order of the system. We can then assume that the same thing is true for the $\mathsf{X}_k f$, and that the equations $\mathsf{A} f = 0, \mathsf{X}_1 f = 0, \ldots, \mathsf{X}_q f = 0$ are mutually independent. Therefore, if $q < n$ then one can determine the $n - q$ integrals of the complete system $\mathsf{A} f = 0, \mathsf{X}_1 f = 0, \ldots, \mathsf{X}_q f = 0$, which requires the prior integration of an ordinary system of order $n - q$. When one then lowers the order of $\mathsf{A} f = 0$ with the help of those first integrals, one will come back to the case in which the order of the group is equal to that of the system. We shall then assume that $q = n$.

In that case, one will next have the general theorem that the determinant:

([47]) Christ. Forh. (1874), pp. 255; Math. Ann. **11** (1877), pp. 464 and esp. *ibid.*, **25** (1885), pp. 71; cf., **Lie-Scheffers**, *Differentialgleichungen*, Chaps. 20 and 24.

(56) $$\Delta \equiv \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \xi_{10} & \xi_{11} & \cdots & \xi_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n0} & \xi_{n1} & \cdots & \xi_{nn} \end{vmatrix},$$

which is not identically zero, is the inverse value of a *Jacobi multiplier* (no. **12**) of the system. However, once more:

When the group G is not simple (II A 6, no. **17**), *one can arrive at a series of problems of the same type in which systems of lower order and simple groups will appear. When the group G is simple, the integration of the system will be equivalent to that of an auxiliary system that belongs to a special class that is defined by the structure of the group*.

Lie first proved that result under the assumption that one knows the finite equations of the group *G*. To that end, let G_1 be a maximal invariant subgroup of *G* of order n_1 . One introduces $\nu_1 = n - n_1$ independent invariants of that group G_1 as new variables. They are defined as functions of the independent variables (e.g., x_0) by a system of order ν_1 for a simple group that one can denote by G / G_1 ([48]). If that system has been integrated then one can use its first integrals to lower the order of (45). In that way, the system will reduce to a system of order n_1 with a group that is holohedrally isomorphic to G_1, and which can once more be treated similarly. If one then has a normal decomposition $G, G_1, G_2, \ldots, G_m$ of *G*, then if n_k is the order of G_k and $n_{k-1} - n_k = \nu_k$, one will, as a result, have to integrate a series of auxiliary systems of orders $\nu_1, \nu_2, \ldots, \nu_k$, resp. The k^{th} one of them ($k = 1, 2, \ldots, m$) admits a group of order ν_k and the structure of G_{k-1} / G_k [for which the determinant that is analogous to (56) is non-zero]. All normal decompositions of *G* deliver the same numbers ν_k and the same structures for their simple auxiliary groups, apart from their sequence.

The nature of the auxiliary system is easier to understand than another method that goes back to *E. Vessiot* ([49]), in principle, and it has the additional advantage that it employs only the infinitesimal transformations of *G*. Let $\mathsf{X}_{\nu_1+1} f$, $\mathsf{X}_{\nu_1+2} f$, …, $\mathsf{X}_n f$ be those of G_1 . One will then have:

(57) $$(\mathsf{X}_i , \mathsf{X}_k) = \sum_{s=1}^{\nu_1} c_{iks} \mathsf{X}_s + \sum_{j=1}^{n_1} c'_{ikj} \mathsf{X}_{\nu_1+j} \qquad (i, k = 1, 2, \ldots, \nu_1)$$

identically, in which the constants c_{iks} define the structure of the simple group (G / G_1). Let:

([48]) One can reduce the order of that auxiliary system when one introduces the invariants of a subgroup of *G* that includes G_1 in place of the invariants of G_1 . One derives all of the first integrals that yield the auxiliary system in the text from the one thus-determined by differentiation [**Lie**, Math. Ann. **25** (1885), pp. 71, cf., *infra* no. **37**.]

([49]) Toulouse Ann. **8** (1894), H, pp. 29; cf., **Lie-Scheffers**, *Differentialgleichungen*, pp. 554 and 568.

(58) $$\mathsf{Y}_k f = \sum_{l=1}^{p} \eta_{kl}(y_1,\ldots,y_p)\frac{\partial f}{\partial y_i} \qquad (k = 1, 2, \ldots, \nu_1)$$

be any group γ_1 with that structure, so $(\mathsf{Y}_i, \mathsf{Y}_k) = \sum c_{iks}\mathsf{Y}_s$. If one then integrates the complete system (II A 5, no. **13**):

(59) $$\mathsf{A} f = 0, \qquad \mathsf{X}_1 f = \mathsf{Y}_1 f, \ldots, \mathsf{X}_{\nu_1} f = \mathsf{Y}_{\nu_1} f, \mathsf{X}_{\nu_1+1} f = 0, \ldots, \mathsf{X}_n = 0,$$

one will get p integrals of $\mathsf{A} f = 0$. Since they include the arbitrary y, they will give ν_1 independent first integrals of $\mathsf{A} f = 0$ for every p. When one introduces them as new variables, one lowers the order of (45), and at the same time, succeeds in making the group G_1, just like that of the transformed system, include only the n_1 preserved variables. The first reduction is completed with that, and one can continue in the same way. Furthermore, if one integrates the system (59) using the method of *A. Mayer* (II A 5, no. **17**) then one will arrive at *one* equation of the form:

(60) $$\frac{\partial f}{\partial t} + \sum_{k=1}^{\nu_1} \Theta_k(t)\mathsf{Y}_k f = 0,$$

which is equivalent to a system:

(61) $$\frac{dy_l}{dt} = \sum_{k=1}^{\nu_1} \Theta_k(t)\eta_{kl}(y_1,\ldots,y_n) \qquad (l = 1, 2, \ldots, p),$$

which is the asserted form for the auxiliary system. The order of that system will be as small as possible when one chooses γ_1 to be a group with the given structure in the fewest-possible variables. One will then obtain a canonical type of auxiliary system that corresponds to the structure G / G_1. From a different viewpoint, one can always choose γ_1 to be a *linear* group, e.g., the adjoint group to the structure G / G_1 (II A 6, no. **15**). *The application of Lie's theory can always be done in such a way that all of the auxiliary systems are linear.* If $\nu_1 = \nu_2 = \ldots = \nu_m = 1$, i.e., the group is integrable, then the integration will be carried out by mere quadratures ([50]).

E. Cartan ([51]) had treated the group-theoretic problems upon whose solution the application of *Lie*'s method depends: The determination and properties of the normal series of subgroups, the reduction of the structure to a group in canonical form, and the determination of the simplest auxiliary systems.

14. Integrating systems for which one knows differential or integral invariants. – In order to simplify the system (45), one can also benefit from some other facts. For example, one can assume that one knows any system of partial differential equations that can be satisfied by the

([50]) **Lie**, Math. Ann. **25** (1885), pp. 71. One will also find the stated reducibility to linear systems for a large number of cases there. – **Lie** had repeatedly assured us [e.g., Leipziger Ber. **47** (1895), pps. 262 and 506] that his theory would yield *all* simplifications of the integration that would follow from the assumptions. However, he had never published a complete proof of that.

([51]) **E. Cartan**, Paris thesis 1894 and Am. J. Math. **18** (1896), pp. 1.

integrals of a suitably-chosen fundamental system of $\mathsf{A}\, f = 0$. *E. Vessiot* ([52]) treated that case on the basis of *Lie*'s theories: The assumption is equivalent to that of *knowing certain differential invariants* of the infinitesimal transformation $\mathsf{A}\, f$, from which one can generally derive new ones. The integration can be reduced to that of a system Σ_G of partial differential equations whose most general solution can be derived from a particular one by the most general transformation of a certain finite or infinite group G (no. **31**). *Vessiot* also showed that the assumption is equivalent to the following one: *One knows the differential equations of a finite or infinite group G that leaves* $\mathsf{A}\, f = 0$ *invariant*. One can next seek to determine that group then, and one will then be addressing the problem in no. **13**.

One can further assume that one knows *integral invariants* of $\mathsf{A}\, f$. One understands that to mean simple or multiple integrals of the form ([53]):

$$\mathfrak{I} = \iint\cdots\int \Phi\left(x_0, x_1, \ldots, x_n, \frac{\partial x_k}{\partial x_0}, \ldots, \frac{\partial x_n}{\partial x_{k-1}}, \frac{\partial^2 x_k}{\partial x_0^2}, \ldots\right) dx_0\, dx_1 \cdots dx_{k-1} \tag{62}$$

that remain unchanged under all transformations of the one-parameter group that is generated by $\mathsf{A}\, f$. The meaning of those integral invariants for the integration of $\mathsf{A}\, f = 0$ was clarified by *H. Poincaré* ([54]). He mainly examined first-order integral invariants that are entire functions of the derivatives and made some application to the equation of celestial mechanics. Of his results, one should mention: If $\mathfrak{I} = \int \sum_{k=0}^{n} \beta_k(x_0, x_1, \ldots, x_n)\, dx_k$ is an integral invariant then $\sum_{k=0}^{n} \alpha_k\, \beta_k$ will be a first integral. In order for $\mathfrak{I} = \int^{n+1} M(x_0, x_1, \ldots, x_n)\, dx_0 \cdot dx_1 \cdots dx_n$ to be an integral invariant, it is necessary and sufficient that M should be a *Jacobi* multiplier (no. **12**) for $A(f) = 0$. *G. Koenigs* ([55]) investigated the relationship between the existence of integral invariants of the form $\int \sum \beta_k\, dx_k$ and the reducibility of the system (45) to a canonical system by means of a point-transformation. His results were obtained previously in a different formulation by *S. Lie* ([56]), who has since then also shown ([57]) how his theory of *differential* invariants gives the key to the general determination of *integral* invariants. The difference between the two problems is that there do not necessarily exist integral invariants of each multiplicity and order, such that one must distinguish different *classes* of equations here. Each problem that is obtained in that way can be treated systematically using the methods if *Lie*. Basically, one will always be led to the determination of a group from its differential equations. Among the cases the *Lie* treated in detail, one should mention the case of a

([52]) C. R. Acad. Sci. Paris **128** (1899), pp. 544. One can also link the problem with the theories of **J. Drach** (no. **38**).

([53]) If Φ includes the derivatives of x_k, x_{k+1}, …, x_n with respect to x_0, x_1, …, x_{k-1} up to any order m then $\mathfrak{I}$ will be a k-fold integral invariant of order m.

([54]) Acta math. **13** (1890), pp. 46 and *Méthodes nouvelles de la mécanique céleste* **3**, Paris 1899, pp. 1.

([55]) C. R. Acad. Sci. Paris **121** (1895), pp. 875.

([56]) Norw. Arch. **2** (1877), pp. 10.

([57]) Leipziger Ber. **49** (1897), pps. 342 and 369. – Other investigations of integral invariants: **C. Żorawski**, Krakow. Ber. (1895); **E. Cartan**, Bull. soc. math. **24** (1896), pp. 140; **A. Hurwitz**, Göttinger Nachr. (1867), pp. 71.

second-order system for which one knows a double first-order integral invariant: One will then get either a *Jacobi* multiplier, or a multiplier and the determination of a first integral by a first-order equation, or a multiplier and an integral.

15. Systems of variation. – *Poincaré* ([58]) coupled the determination of the integral invariants of a system (1) with the study of the *system of variation* that it is associated with it, i.e., of the linear system:

$$\frac{d\varepsilon_k}{dx} = \sum_{i=1}^{m} \frac{\partial \lambda_k}{\partial x_i} \varepsilon_i \qquad (k = 1, 2, \ldots, n), \tag{63}$$

that defines the first variations $\delta x_1 = \varepsilon_1 (x)\, \delta t$, …, $\delta x_n = \varepsilon_n (x)\, \delta t$, which one can attach to the particular integral $x_1 = x_1 (x)$, …, $x_n = x_n (x)$ when one assumes that $\delta x = 0$, to simplify. *C. G. J. Jacobi* ([59]) had already given the theorem for that system: When one replaces x_1, …, x_n in (63) with the general solution of (1), $x_1 = \theta_1 (x \mid a_1, \ldots, a_n)$, the general solution of (63) will be:

$$\varepsilon_k = \sum_{h=1}^{n} c_h \frac{\partial \theta_k}{\partial a_h} \qquad (k = 1, 2, \ldots, n), \tag{64}$$

in which $c_1, c_2, \ldots, c_n$ are arbitrary integration constants.

That system of variation, which mainly finds its applications in mechanics, is closely linked with the determination of the infinitesimal transformations of the form:

$$\mathsf{X} f = \sum_{k=1}^{n} \xi_k (x, x_1, \ldots, x_n) \frac{\partial f}{\partial x_k} \tag{65}$$

that leave (1) invariant. In fact, any such infinitesimal transformation will imply a corresponding one for (63) from any solution of (1) when one sets $\varepsilon_k = \xi_k (x, x_1 (x), \ldots, x_n (x))$ $(k = 1, 2, \ldots, n)$. On the other hand, if one has n first integrals of the system that is composed of (1) and (63) that are independent functions of $\varepsilon_1, \ldots, \varepsilon_n$, and one sets them equal to arbitrary constants $\omega_1, \ldots, \omega_n$ then one will get the equations:

$$\varepsilon_k = \xi_k (x, x_1, \ldots, x_n \mid \omega_1, \ldots, \omega_n) , \tag{66}$$

whose right-hand sides will give the most general value of $\mathsf{X} f$ when one replaces $\omega_1, \omega_2, \ldots, \omega_n$ in them with n arbitrary integrals of the system (1) (which can also be constants).

([58]) *Méthodes nouv.* (footnote 55) **1**, Chaps. 4 and 3, Chap. 22.
([59]) *Vorlesungen über Dynamik* 12 (*Ges. Werke*, Suppl.-Bd., pp. 90). Cf., **Serret** *Cours* **2**, pp. 568.

Special theories for n^{th}-order equations.

16. Method of Euler multipliers. – The most general method for integrating an n^{th}-order equation:

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right) \tag{67}$$

consists of the search for a first integral that one then treats similarly, etc. To that end, *L. Euler* ([60]) generalized the concept of a *multiplier:* His theorem was based upon the determination of the conditions under which an *n^{th}-order differential function* $V\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right)$ would be the *total* derivative of an $(n-1)^{\text{th}}$-order differential function $W\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right)$. They consist of saying that one must have:

$$\frac{\partial V}{\partial y} - \frac{d}{dx}\left(\frac{dV}{dy'}\right) + - + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{dV}{dy^{(n)}}\right) = 0 \qquad \left(y^{(k)} = \frac{d^k y}{dx^k}\right) \tag{68}$$

identically. They were obtained by *Euler* using the calculus of variations and directly by many authors since then ([61]). If they are fulfilled then one will obtain W by a quadrature. One of them says that V must be of first degree in $y^{(n)}$.

Now let (67) be given in the somewhat more general form:

$$F(x, y, y', \ldots, y^{(n)}) \equiv A(x, y, y', \ldots, y^{(n-1)})\, y^{(n)} + B(x, y, y', \ldots, y^{(n-1)}) = 0\,. \tag{69}$$

Euler's method then consists of the search for a multiplier $\mathsf{M}(x, y, y', \ldots, y^{(n-1)})$ such that $V = \mathsf{M} F$ satisfies the conditions (68). If one has $V = dW / dx$ then $W = a$ will be the desired first integral.

The partial differential equations that M must satisfy are complicated. However, *Euler* had made some applications to equations of orders 2 and 3. The most interesting of them relates to *linear* equations. When generalized to such an n^{th}-order equation, it will lead to the concept of the *adjoint equation* ([62]). In fact, if we seek an *Euler* multiplier z for the linear equation:

$$p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n(x)\, y = 0 \tag{70}$$

([60]) *Inst. calc. int.* **2**, pp. 97.

([61]) **Euler**, *ibid.*, **3**, pp. 425; **Condorcet**, *Du calcul integral*; **Lexell**, Petrop. N. Comm. **15** and **16**; **Lagrange**, *Calcul des fonctions* (*Oeuvres* 10, pp. 364); **J. Bertrand**, J. éc. poly., cah. **28** (1841), pp. 364; **J. Raabe**, J. f. Math. **31** (1846), pp. 181; **F. Joachimsthal**, *ibid.* **33** (1847), pp. 95; **Stoffel** and **Bach**, J. de math. (2) **7** (1862), pp. 49; **A. Winckler**, Wiener Ber. **88** (1883), pp. 820.

([62]) **J. Lagrange**, Misc. Taur. **3** (1762/65), pp. 179 (*Oeuvres* **1**, pp. 471).

that depends upon only x then it must likewise satisfy the linear equation:

$$p_n(x)\,z - \frac{d}{dx}(p_{n-1}\,z) + \cdots + (-1)^n p_n(x)\,y = 0\,. \tag{71}$$

It is called the "Lagrange adjoint" of (70).

Moreover, *Euler*'s method is only a special application of the method of systems of multipliers (no. **11**). Namely, if $\mathrm{V} = dV/dx$, and one sets:

$$d\,\mathrm{V} = X\,dx + Y\,dy + Y'\,dy + \cdots + Y^{(n)}\,dy^{(n)}$$

then the expression:

$$\mathrm{V}\,dx + Y^{(n)}(dy^{(n-1)} - y^{(n)}\,dx) + \left[Y^{(n-1)} - \frac{dY^{(n)}}{dx}\right](dy^{(n-2)} - y^{(n-1)}\,dx)$$
$$+ \cdots + \left[Y' - \frac{dY''}{dx} + \frac{d^2Y'''}{dx^2} - + \cdots\right](dy - y'\,dx)$$

will be an exact differential of a function of the quantities x, y, y', …, $y^{(n-1)}$ ([63]), which are considered to be independent variables. It will then follow that any Euler multiplier of $F = 0$ is equivalent to a system of multipliers of the equivalent system:

$$A\,dy^{(n-1)} + B\,dx = 0\,, \quad dy^{(n-2)} - y^{(n-1)}\,dx = 0\,, \ldots, \qquad dy - y'\,dx = 0\,. \tag{72}$$

17. Cases in which one lowers the degree. – Some cases in which the degree of equation (67) can be lowered have been known for a long time ([64]). The main ones are:

1. The equation has the binomial form $\frac{d^n y}{dx^n} = f(x)$. It can then be integrated by n overlapping quadratures that one can also replace with successive ones with the help of partial integration.

2. One of the variables does not appear explicitly, e.g., y. One lowers the order to $(n-1)$ when one introduces y' as an auxiliary unknown.

3. y and its derivatives up to $y^{(k-1)}$, inclusive, do not appear explicitly. If one introduces $y^{(k)}$ as an auxiliary unknown then one will have an equation of order $n-k$. What will then remain is a binomial equation of order k.

([63]) Cf., e.g., the treatises of **Joachimsthal**, **Stoffel**, and **Bach** that were cited in footnote 61.
([64]) Cf., the textbooks, as well as **Allan Cunningham** Mess. **17** (1887), pp. 118 and *ibid*. **18** (1888), pp. 122.

4. The equation is homogeneous in y and its derivatives: The substitution $x = e^z$ will lead to case 2.

5. The equation is homogeneous in x, y, dx, dy, d^2y, …, d^ny. The substitution $x = e^u$, $y = ve^u$ will likewise lead to case 2.

S. Lie remarked ([65]) that in each of those cases the equation remains invariant under the transformations of a finite group, namely, $\bar{x} = x$, $\bar{y} = y + a$ in case 2. $\bar{x} = x$, $\bar{y} = y + a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}$ in case 3, and for $k = n$, in case 1. In case 4, one has $\bar{x} = x$, $\bar{y} = a y$, and in case 5, $\bar{x} = a x$, $\bar{y} = a y$. In that way, he was led to the *theory of higher-degree equations that admit groups of point-transformations in x and y.*

18. Lie's theory. Equations that admit groups of point transformations. Generalizations. – An n^{th}-order differential equation ($n > 1$) generally admits *no* continuous group of point-transformations, but if it does then the group will be *finite*, and when $n = 2$, it will contain at most eight parameters ([66]). Equations D, with a group G of transformations, decompose into *classes* according to the *type* of G (II A 6, no. **17**). The equations of each class can be converted into each other by point-transformations. Since, *Lie* had determined the different types of point-transformations, that implies the determination of the associated equations D immediately from the theory of invariant functions and equations. In addition to certain exceptional equations that can be obtained setting certain determinants equal to zero, the equations D will have the form:

$$\Phi\left(J, \mathfrak{J}, \frac{d\mathfrak{J}}{dJ}, \frac{d^2\mathfrak{J}}{dJ^2}, \ldots, \frac{d^k\mathfrak{J}}{dJ^k}\right) = 0\,, \tag{73}$$

in which J, $\mathfrak{J}$ mean the two differential invariants of lowest order for G, and Φ is an arbitrary function. *Lie* gave an integration procedure for each of the types of equation that is obtained in that way: One first lowers the order by k units by introducing J and $\mathfrak{J}$ as variables. What will then remain to be integrated is an equation of the form $F(J, \mathfrak{J}) = 0$, for which there are certain specialized simplifications (integration by quadrature or reduction to linear auxiliary equations). As special types, one gets the *linear* equations and the ones that admit the general projective group. The latter were also introduced and investigated by *Halphen* and *Sylvester* in their work on differential invariants (of the projective group) and reciprocants (I B 2, no. **20**).

Another remarkable type, namely:

([65]) Göttinger Nachr. (1874); Norw. Arch. **8** (1883), pp. 167, 249, 371; *ibid.* **9** (1884), pp. 431. The first two treatises in Norw. Arch. **8** were reprinted in Math. Ann. **32** (1888), pp. 213.

([66]) In regard to this question and more general analogous ones, one might cf., **Lie**, Leipziger Ber. **46** (1894), pp. 322 and **Fr. Engel**, *ibid.*, pp. 297.

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = \omega(x) ,$$

was investigated by *H. A. Schwarz* ([66.a]).

If any differential equation is given then one can decide whether it is an equation D by differentiations and eliminations, and indeed when the defining equations define its group G and determine the type to which it belongs. That is an *equivalence* problem, since one deals with deciding whether the equation can be converted into a special equation by a point-transformation. However, it can be solved explicitly with the help of differential invariants only in a few cases (nos. **33-35**). In order to integrate the given equation, one can seek the transformation that converts the group G into its canonical form ([67]). One can also derive the infinitesimal transformations of G from its defining equations, which amounts to the integration of a linear auxiliary equation, and then apply *Lie*'s general method for the integration of differential systems with known groups (no. **13**), of which the one that is discussed here is basically only one specialized application.

As a result, that theory can be generalized. Thus, *Lie* ([68]) had examined the n^{th}-order equations that admit a group of *contact transformations* in the same way. The problem is meaningful for only $n > 2$, because two second-order equations can always be taken to each other by infinitely-many contact transformations. Their integration requires that of an n^{th}-order linear auxiliary equation (except that for $n = 3$, it will be fourth-order).

Above all, *Lie*'s theory makes it possible for one to discuss the equations that are invariant under any group G in arbitrarily-many variables or higher-order system of equations.

19. Unsolved equations. Types of integrable equations. – The methods for treating first-order equations that are not solved (no. **9**) can be easily adapted to n^{th}-order equations. One can then set:

$$(74) \qquad X = \xi(x, y, y', \ldots, y^{(n)}) , \qquad Y = \eta(x, y, y', \ldots, y^{(n)}) ,$$

in which one understands ξ, η to mean arbitrary functions, and then eliminate x, y, y', …, $y^{(n)}$ from those equations, the given equations, and the ones that are derived from it by differentiation.

In particular, one has a class of equations of the form ([69]) $\Phi(U, V) = 0$, in which $U(x, y, y', \ldots, y^{(n)}) = a$ and $V(x, y, y', \ldots, y^{(n)}) = b$ are two first integrals of an $(n+1)^{\text{th}}$-order equation, that are analogous to the *Lagrange* equations (no. **9**). One will obtain a first integral of such an equation when one eliminates $y^{(n)}$ and one of the constants from $U = a$, $V = b$ and $\Phi(a, b)$

([66.a]) J. f. Math. **75** (1872), pp. 292 (*Ges. Abh.* **2**, pp. 220).
([67]) **S. Lie**, in the places that were cited in footnote 65, as well as Math. Ann. **25** (1885), pp. 120.
([68]) Christ. Förh. 1881, 1882, 1883.
([69]) Cf., the Note III (by **J. A. Serret**) in **Lagrange**'s *Leçons sur le calcul des fonctions* and **J. A. Serret**, J. de math. **18** (1853), pp. 1.

= 0. One can also generalize an idea of *G. Monge* ([70]) and consider equations of the form $\Phi(U_1, U_2, \ldots, U_{n+1}) = 0$, when $U_1 = a_1, \ldots, U_{n+1} = a_{n+1}$ are $n + 1$ first integrals of the same $(n + 1)^{\text{th}}$-order equation. One will get the general integral here when one eliminates y', y'', …, $y^{(n)}$, and one of the constants from $U_1 = a_1, \ldots, U_{n+1} = a_{n+1}$ and $\Phi(a_1, a_2, \ldots, a_{n+1}) = 0$.

Special classes of equations and systems of equations.

a) The linear n^{th}-order equation.

20. General concepts. Fundamental systems of solutions. – Among the special classes of differential equations, the earliest one to be examined was that of the *linear equations*. The properties of the *n^{th}-order linear homogeneous equations:*

$$P(y) \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n(x)\, y = 0 \tag{75}$$

present the greatest analogy to those of the n^{th}-order algebraic equation:

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0\,. \tag{76}$$

Just as the properties of the latter are linked with those of the polynomial $f(x)$, so do the properties of the *differential expression* $P(y)$ depend upon the properties of the polynomial $P(y)$, in which y is treated as an undetermined function ([71]). Hence, *Taylor*'s formula (I B 1 a, no. **6**) corresponds to the formula:

$$P(y, z) = z\,P(y) + \frac{1}{1}\frac{dz}{dx}P'(y) + \frac{1}{1\cdot 2}\frac{d^2 z}{dx^2}P''(y) + \cdots + \frac{1}{n!}\frac{d^n z}{dx^n}P^{(n)}(y) \tag{77}$$

here, in which $P'(y)$, $P''(y)$, … are derived from $P(y)$ by the same algorithm as the one by which $f'(x)$, $f''(x)$, … are derived from $f(x)$. That formula makes it possible to lower the degree of (75) by one unit as soon as one knows a particular solution y_1, because if one sets $y = y_1 z$ then the equation for z will no longer include z explicitly ([72]). If y_1 likewise satisfies the equations $P(y) = 0$, $P'(y) = 0$, …, $P^{(n)}(y) = 0$, in other words, if $y_1, x\,y_1, , \ldots, x^{k-1}y_1$ are simultaneous solutions of (75), then the degree will be lowered by k units: That case corresponds to a k-fold root of (76) ([73]). One can then successively lower the order by k units when one knows k particular solutions.

([70]) Paris Hist. (1783), pp. 719.

([71]) The first idea for the symbolic representation of such a differential operation goes back to **B. Brisson**, J. éc. polyt. cah. **14** (1808), pp. 197 [and that journal (1804)]. **A. Cauchy** has explained and extended it [*Exerc. de math.* (1827), *Oeuvres* (2) **7**, pp. 198] He used the notation $f(D)\,y$, in which D stands for d / dx. Cf., also the course by **Sturm**, and for more recent studies, the books of **Boole**.

([72]) **J. d'Alembert**, Misc. Taur. **3** (1762/65) [66], pp. 381 in the second pagination.

([73]) **Brasinne**, in a note to **Sturm**'s course.

However, one must assume in so doing that those solutions are *linearly independent*, i.e., they cannot be coupled by any homogeneous linear relation with constant coefficients. The condition for that is that the determinant:

$$\Delta(y_1, y_2, \ldots, y_k) = \begin{vmatrix} y_1 & \cdots & y_k \\ y_1' & \cdots & y_k' \\ \vdots & \vdots & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix} \tag{78}$$

must not be identically zero ([74]). With the help of the general methods of **A. Cauchy** (II A 4 a, nos. **3**, **9**, **11**) or the specialized procedure of **L. Fuchs** (II B 4), one can prove that under certain conditions that we will assume are always fulfilled here, equation (75) will have systems of n linearly-independent solutions; one calls them *fundamental systems* ([75]). Any fundamental system can be brought into the form:

$$y_1 = v_1\,, \qquad y_1 = v_1 \int v_2\, dx\,, \ \ldots,\ y_n = v_1 \int v_2\, dx \int \cdots \int v_n\, dx\,, \tag{79}$$

in which none of the functions $v_1, v_2, \ldots, v_n$ is identically zero.

J. Lagrange proved that ([76]) that:

$$y = \sum_{k=1}^{n} c_k\, y_k \tag{80}$$

is the general solution, when $y_1, y_2, \ldots, y_n$ define a fundamental system of particular solutions, and $c_1, c_2, \ldots, c_n$ are arbitrary constants. That form for the general integral is characteristic of the linear homogeneous equations, and it is, in fact, identical to saying that equation (75) [except for the infinitesimal transformation $y \dfrac{\partial f}{\partial y}$] admits a group of transformations of the form $\eta_k(x) \dfrac{\partial f}{\partial y}$ (k = 1, 2, …, n). The linear equation then belongs to equations *D* of no. **18** ([77]).

21. Equations with constants coefficients. Lagrange equations. D'Alembert's method. – If the ratios of $p_0, p_1, \ldots, p_n$ are constant then equation (74) can be written:

$$P(y) \equiv \frac{d^n y}{dx^n} + b_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_n\, y = 0\,, \tag{81}$$

([74]) **O. Hesse**, J. f. Math. **54** (1857), pp. 227; **E. Christoffel**, *ibid.* **55** (1858), pp. 293.
([75]) The term seems to go back to **L. Fuchs** [J. f. Math. **66** (1866), pp. 121].
([76]) Misc. Taur. **3** (1762/66) [65], pp. 181 with the second pagination. *Oeuvres* **1**, pp. 473.
([77]) Cf., e.g., **Lie-Scheffers**, *Differentialgleichungen*, Chap. 16.

in which $b_1, b_2, \ldots, b_n$ are constants. One then calls it an *equation with constant coefficients.* It can be integrated by elementary functions. That is because if one sets:

(82) $$f(x) = x^n + b_1 x^{n-1} + \cdots + b_n$$

and lets ω denote a constant then one will have:

(83) $$P(e^{\omega x}) = e^{\omega x} f(\omega), \qquad P'(e^{\omega x}) = e^{\omega x} f'(\omega), \quad \ldots$$

Therefore, any root of the *characteristic equation* $f(\omega) = 0$ will give a solution $y = e^{\omega x}$, and any k-fold root will give k solutions $e^{\omega x}$, $x e^{\omega x}$, …, $x^{k-1} e^{\omega x}$. One then obtains n solutions and proves that they are linearly-independent. One has *L. Euler* ([78]) to thank for that result. *J. d'Alembert* ([79]) treated the case of equal roots by a method that bears his name that is useful in many analogous cases (e.g., no. **8**). He started from the Euler form of the general integral $y = \sum_{k=1}^{n} c_k e^{\omega_k x}$ with unequal ω and sought to derive limiting forms for that in the case where the roots coincide. The next-simplest case is the one in which only two roots ω_1 and ω_2 are allowed to coincide. One must then replace the c with functions of the ω such that y will be indeterminate for $c_1 = c_2$, and one will then have to determine its "true values" (II A 1, no. **13**). To that end, one writes:

(84) $$y = c_1' e^{\omega_1 x} + c_2' \frac{e^{\omega_2 x} - e^{\omega_1 x}}{\omega_2 - \omega_1} + c_3 \frac{e^{\omega_2 x} - e^{\omega_1 x}}{\omega_2 - \omega_1} + \cdots + c_n e^{\omega_n x},$$

in which $c_1' = c_1 + c_2$ and $c_2' = (\omega_2 - \omega_1) c_2$ can be regarded as the new arbitrary constants. The passage to the limit for $\omega_2 = \omega_1$ poses no difficulty. One likewise passes from the case of a double root to that of a triple root, etc.

If the constant coefficients are *real* then one can replace the $2k$ particular solutions that correspond to two conjugate-complex k-fold roots $\omega = \alpha \pm \beta i$ of the equation $f(\omega) = 0$ with solutions of the real form $x^h e^{\alpha x} \cos \beta x$, $x^h e^{\alpha x} \sin \beta x$ $(h = 0, 1, 2, \ldots, k-1)$.

If $p_k = A_k (x + b)^{n-k}$, in which one understands b and the A_k to mean constants ([80]), then the substitution $x + b = e^t$ will take the equation to one with constant coefficients.

G. H. Halphen ([81]) gave another elementary class of linear equations that can be integrated by elementary functions.

([78]) Misc. Berol. **7** (1743), pp. 193 and *Inst. calc. int.* **2**, sect. 2.

([79]) Cf., e.g., the article *Tangentes* in the Dictionnaire des math. der Encyclopédie méthodique. – The process can be justified by the theorem of **Poincaré** (II A 4 a, no. **15**).

([80]) **J. Lagrange**, Misc. Taur. **3** (1762/65) [66], pp. 190 with the second pagination; *Oeuvres* 1, pp. 481.

([81]) C. R. Acad. Sci. Paris **92** (1881), pp. 779.

22. Equations with a right-hand side. Method of variation of constants. – The general form of the *inhomogeneous* linear equation (viz., an equation "with a right-hand side") is:

(85) $$P(z) = q(x) .$$

If one knows a particular integral z_0 and sets $z = z_0 + y$ then one will get back to $P(y) = 0$. Thus, one will get the general integral:

$$z = z_0 + \sum_{k=1}^{n} c_k y_k$$

when one adds the general integral to the corresponding equation "without a right-hand side" to z_0 ([82]). That form for the integral is characteristic of the present class. It corresponds to the situation in which (85) admits a group of transformations of the form $\eta_k(x)\frac{\partial f}{\partial y}$.

Conversely: If one knows the general integral of $P(y) = 0$ then one will get that from (85) by quadratures, as *J. Lagrange* had derived from his *method of the variation of constants* ([83]). That is a special form of the method of introducing new variables. One initially employs it in order to derive a more general solution of an arbitrary system (1) that includes p arbitrary constants $a_1, a_2, \ldots, a_p$ from a particular solution. To that end, one replaces p of the x_i (e.g., $x_1, x_2, \ldots, x_p$) with the functions $u_1, u_2, \ldots, u_p$ that one obtains when one replaces the a_i in p of the equations that define the known particular solution with the new variables u_i. In that form, *Lagrange* had employed the method in order to lower the order of $P(y) = 0$ when one knows p particular integrals y_k, so one knows the solution $y = \sum_{k=1}^{n} a_k y_k$. Another case in which this method is applicable is: The equations of a system (1) include certain parameters $m_1, m_2, \ldots, m_r$. One knows the general integral for special values $m_1^0, m_2^0, \ldots, m_r^0$ of that parameter. One then introduces new unknown functions that take the form of ones that are defined by the general integral equations of that special case when one regards the integration constants in them as new variables. In order to apply that method here, one forms the equation $P(z) = m\,q(x)$. For $m = 0$, one knows the general integral $z = \sum_{k=1}^{n} c_k y_k$.

It will also be $z^{(i)} = \sum_{k=1}^{n} c_k y_k^{(i)}$ for $i = 1, 2, \ldots, n-1$ then. The substitution:

(86) $$z = \sum_{k=1}^{n} u_k y_k , \qquad z^{(i)} = \sum_{k=1}^{n} u_k y_k^{(i)}$$

leads to the system of equations:

(87) $$\sum_k y_k \frac{du_k}{dx} = 0 , \ldots, \quad \sum y_k^{(n-2)} \frac{du_k}{dx} = 0 , \quad \sum y_k^{(n-1)} \frac{du_k}{dx} = m\,q(x)$$

([82]) **J. d'Alembert**, Misc. Taur. **3** (1762/65) [66], pp. 381 in the second pagination.
([83]) Berl. nouv. mém. (1774) and (1781) (*Oeuvres* **4**, pp. 111; *ibid.*, **5**, pp. 123).

(in which one can set $m = 1$). That system can be integrated by n simultaneous quadratures, and by a lower number of them in special cases.

That result is also true for the binomial equation $d^n z / dx^n = q(x)$, in particular. *A. Cauchy* ([84]) showed that it comes down to the form:

$$z = \int_{\alpha}^{x} \varphi(\alpha, x)\, d\alpha , \tag{88}$$

if $\varphi(\alpha, x)$ is the integral of $P(y) = 0$ that satisfies the initial conditions $y = 0,\ y' = 0, \ldots,\ y^{(n-2)} = 0,\ y^{(n-1)} = q(\alpha)$ for $x = \alpha$. As an application of his theory of residues, he also gave ([85]) the general integral for the case of constant $p_0, p_1, \ldots, p_n$ in an explicit form that one can also obtain directly, moreover.

Since the integral of $P(z) = m\, q(x)$ is given by that of (85) (one needs only to replace the z in it with z / m), one can replace the equation with a right-hand side with the $(n + 1)^{\text{th}}$-order homogeneous equation:

$$q(x)\frac{dP(z)}{dx} - q'(x)\,P(z) = 0 \tag{89}$$

that one obtains by eliminating m. All n solutions of $P(z) = 0$ satisfy it. One will then get its $(n + 1)^{\text{th}}$ solution by one quadrature. The two given forms for the method of variations of constants can be converted into each other in a completely general way by a similar argument (vg., also II A 4 a, no. **15**).

One can also find that result by applying *Lie*'s method ([86]).

23. Lowering the order of the equation. Common solutions to two linear equations. – One can also lower the order of equation $P(y) = 0$ whenever one knows another linear equation that has solutions in common with the latter. That is implied by a theory ([87]) that is analogous to the theory of the division and greatest common divisors of polynomials (I B 1 a, no. 12). Let $A(y)$, $B(y)$ be two expressions with the same form as $P(y)$, the second of which is not of higher order than the first. If one assumes, to simplify, that the first coefficients of all of those expressions reduce to 1 then one can determine two analogous expressions $Q(y)$ and $R(y)$, the second of which has lower order than $B(y)$, by rational operations, in such a way that one has:

$$A(y) \equiv Q(B(y)) + \rho(x) \cdot R(y) \tag{90}$$

identically. Every common solution to $A = 0$ and $B = 0$ will also be a solution to $R = 0$ then, and conversely. In that way, one will get a linear equation $C(y) = 0$ whose solutions are all of the

([84]) *Oeuvres* (2) **7**, pps. 40, 198; C. R. Acad. Sci. Paris **8** (1839), pps. 827, 845, 889, 931. Cf., e.g., **H. Laurent**, *Traité d'analyse* **5**, pp. 141.

([85]) *Oeuvres* (2) **6**, pp. 253, 316 and (2) **7**, pp. 40. Cf., the method of **Laplace** (II B 4).

([86]) Cf., e.g., **Lie-Scheffers**, *Differentialgleichungen*, Chap. 16.

([87]) **G. Libri**, J. f. Math. **10** (1833), pp. 135; **Brassine**, in a note in **Sturm**'s course.

common solutions to $A = 0$ and $B = 0$ by a process that is analogous to the determination of the greatest common divisor. The assumption can then be replaced by the assumption that $P(y) = 0$ admits *all* solutions to a k^{th}-order linear equation $S(y) = 0$, and that, as a result, an identity of the form $P(y) = T(S(y))$ exists, in which $T(y)$ means a linear differential expression of order $n - k$. One determines $n - k$ linearly-independent solutions $u_1, u_2, \ldots, u_{n-k}$ of $T(u) = 0$ and then has to integrate the equations with right-hand sides $S(y) = u_h$ ($h = 1, 2, \ldots, n - k$), which requires the integration of $S(y) = 0$ and quadratures ([88]).

One can also derive a process that is analogous to Descartes's process of ridding an algebraic equation of multiple roots (I B 1 b, no. 14) from that. Other methods of eliminating y from $A(y) = 0$ and $B(y) = 0$ were given by *G. von Escherich* ([88.a]).

24. Equation with a given fundamental system. Symbolic methods. – The linear equation that possess a given fundamental system of solutions $y_1, y_2, \ldots, y_n$ is:

$$P(y) \equiv \Delta(y_1, y_2, \ldots, y_n) = 0 ,$$

with the meaning for Δ that was given in (78) ([89]). One will then obtain expressions for the ratios $p_i : p_0$ in in terms of $y_1, y_2, \ldots, y_n$ and their derivatives from that in the form of quotients of determinants that correspond to the expressions for the coefficients of an algebraic equation in terms of their roots (I B 3 b, no. **1**). In particular, one has the formula of *J. Liouville* ([90]):

$$\frac{p_1}{p_0} = -\frac{d \log \Delta(y_1, y_2, \ldots y_n)}{dx} . \tag{91}$$

The equation $P(y) = 0$ can be brought into yet another form. For $p_0 = 1$ and $\Delta(y_1, y_2, \ldots, y_n) = \Delta_k$, one has:

$$P(y) \equiv \frac{\Delta_n}{\Delta_{n-1}} \frac{d}{dx} \frac{\Delta_{n-1}^2}{\Delta_n \Delta_{n-1}} \frac{d}{dx} \cdots \frac{d}{dx} \frac{\Delta_1^2}{\Delta_2 \Delta_3} \frac{d}{dx} \frac{y}{\Delta_1} . \tag{92}$$

If one takes the fundamental system in the form (79) then one will have ([91]):

$$P(y) \equiv \frac{d}{dx} v_n^{-1} \frac{d}{dx} v_{n-1}^{-1} \frac{d}{dx} \cdots \frac{d}{dx} v_1^{-1} y . \tag{93}$$

([88]) Cf., also **L. W. Thomé,** J. f. Math. **76** (1873), pp. 273 and **G. Frobenius**, *ibid*., pp. 256; *ibid.*, **80** (1875), pp. 321, and *ibid.* **85** (1878), pp. 185.

([88.a]) Wien. Denkschr. **46** (1882), pp. 61 and *ibid.* **47** (1883), pp. 1.

([89]) **G. Libri**, J. f. Math. **10** (1833), pp. 185.

([90]) J. de math. (1) **3** (1838), pp. 349, cf., **Brassine**, *loc. cit.*; **E. Christoffel**, J. f. Math. **55** (1858), pp. 296. The theorem had already been found by **N. H. Abel**, J. f. Math. **2**, pp. 2 (*Oeuvres* 1, pp. 251) for the case of a second-order equation.

([91]) **G. Frobenius**, J. f. Math. **76** (1873), pp. 256 and *ibid.* **77** (1873), pp. 245.

As *G. Floquet* showed ([92]), it follows from these results that one can write symbolically:

$$P\,(y) = A_n\, A_{n-1} \ldots A_1\,(y) \tag{94}$$

when one sets $A\,(B\,(y))$ equal to $AB\,(y)$. In so doing, one has:

$$A_k\,(y) = \frac{dy}{dx} - a_k\, y\,, \qquad a_k = \frac{d \log (v_1\, v_2 \cdots v_k)}{dx} = \frac{d}{dx} \log \frac{\Delta_k}{\Delta_{k-1}}\,. \tag{95}$$

Formula (94) corresponds to the decomposition of a polynomial into linear factors. For the equation with constant coefficients (no. **21**), the a_k are the roots of the characteristic equation. *Floquet* reduced the general case of commuting A_k (I A 6, no. **1**) to that of constant coefficients.

Those symbolic decompositions give new methods ([92.a]) for integrating equations with a right-hand side (no. **22**). In the case of constant coefficients, one has to solve:

$$\left(\frac{d}{dx} - \omega_n\right)\left(\frac{d}{dx} - \omega_{n-1}\right) \cdots \left(\frac{d}{dx} - \omega_1\right) z = q\,(x)\,. \tag{96}$$

If one sets:

$$\left(\frac{d}{dx} - a\right)^{-1} u = e^{ax} \int e^{-ax} u\, dx\,, \tag{97}$$

by definition, then one will get:

$$z = \left(\frac{d}{dx} - \omega_1\right)^{-1} \left(\frac{d}{dx} - \omega_2\right)^{-1} \cdots \left(\frac{d}{dx} - \omega_n\right)^{-1} q(x)\,. \tag{98}$$

One proves that that successive quadratures can be separated by means of a formula that is analogous to the decomposition of rational functions into partial fractions (I B 1 a, no. **2**; II A 2, no. **26**):

$$z = \sum_{k=1}^{n} f'(\omega_k)^{-1} \left(\frac{d}{dx} - \omega_k\right)^{-1} q\,(x)\,. \tag{99}$$

Just as for rational functions, that result can be extended to multiple roots of the characteristic equation.

25. Rational differential functions of solutions of a fundamental system. Invariant functions. Transformation. – Even the theorem of the rational functions of the roots of an

([92]) Ann. éc. norm. sup. (2) **8** (1879), Suppl. (Paris thesis), pp. 49. Cf., also, **E. Grünfeld**, J. f. Math. **98** (1885), pp. 333.

([92.a]) Cf., Footnote 71.

algebraic equation (I B 3 b) has its analogue in the theory of differential equations. The most general fundamental system $\bar{y}_1$, $\bar{y}_2$, …, $\bar{y}_n$ to equation (75) can be derived from a special y_1, …, y_n by means of the equations:

$$\bar{y}_i = \sum_{k=1}^{n} a_{ik}\, y_k \qquad (i = 1, 2, \ldots, n), \tag{100}$$

which define the *general homogeneous linear group* Γ (II A 6, np. **20**). That group then plays the same role here that the group of permutations of n symbols plays for an n^{th}-order algebraic equation. As functions of the y and their derivatives, the p_k / p_0 are differential invariants of the group Γ, and every differential invariant of Γ that is constructed from y and its derivatives is a function of only the p_k / p_0 and its derivatives. Moreover, if it is a rational function of the y, its derivatives, and the x then its expression in terms of the p_k / p_0 , their derivatives, and x will be rational. Any *relative* differential invariant of Γ is the product of a rational function of the p_k / p_0 and their derivatives with $\exp -\int p_1 / p_0\, dx$ ([93]).

More generally, let R be a differential function that is formed rationally from y_1, …, y_n, their derivatives, and x. Its properties are coupled with those of *its group* ([94]), i.e., that subgroup G of Γ under which the form of that function remains unchanged. The function that is derived from R by an arbitrary transformation of Γ (one calls it the "general value" of Γ) is the general solution of a rational differential equation $\Psi(R) = 0$ whose coefficients are rational functions of the p_k / p_0 , their derivatives, and x. One calls $\Psi(R) = 0$ a *transform* or *resolvent* of $P(y) = 0$. It has order $n^2 - r$, if r is the number of parameters of G.

Let S be another differential function of the same nature as R, and let H be its group. If H is contained in G completely then R can be expressed rationally in terms of S, the p_k / p_0, their derivatives, and x. In general, R satisfies a rational differential equation whose coefficients are rational functions of S, the p_k / p_0, their derivatives, and x, and its order is equal to the difference between the number of parameters of H and the greatest common subgroup of G and H ([94]).

If S does not admit any transformation of Γ besides the identity transformation then any function R (so each of the integrals y_1, …, y_n, as well) can be expressed rationally in terms of S, the p_k / p_0, their derivatives, and x. In particular, that is true for the function $S = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$, in which the u_i mean undetermined functions of x. It satisfies a linear homogeneous equation of order n^2 (*general resolvent*) ([95]).

If one takes:

$$z = R(y) = u_1(x)\frac{d^{n-1}y}{dx^{n-1}} + u_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + u_n(x)\, y, \tag{101}$$

([93]) These theorems go back to **P. Appell**, Ann. éc. norm. sup. (2) **10** (1881), pp. 400.

([94]) **E. Vessiot**, Paris thesis 1892 [Ann. éc. norm. sup. (3) **9**, pp. 197]. – The theorems can be adapted to an arbitrary domain of rationality (no. **36**).

([95]) **E. Picard**, C. R. Acad. Sci. Paris **96** (1883), pp. 1131 and Toul. Ann. **1** (1887), A.

in which one understands u_1, u_2, …, u_n to mean rational functions of x, the p_k / p_0, and their derivatives, then one will get a *linear resolvent* $Q(z)$, which has order n, in general, like $P(y)$. The solutions of $P(y) = 0$ and $Q(z) = 0$ correspond to each other by way of the relation $z = R(y)$, and also by way of a inverse relation of the same form $y = S(z)$. One then has the analogue of the Tschirnhausen transformation here (I B 2, no. **19**). Two linear differential equations $P(y) = 0$ and $Q(z) = 0$ are said to have the same *type* ([96]) when they are connected to each other in that way.

26. Associated equations. Adjoint equation. – From the aforementioned, the simplest resolvents are the ones that are satisfied by the minors of the determinant $\Delta(y_1, \ldots, y_n)$. That determinant itself satisfies the first-order equation (91). All of those same minors that are constructed from m rows of Δ satisfy one and the same equation and defines a fundamental system of solutions for it. If one replaces the m rows with m other ones then one will get equations of the same type (no. **25**). For every m, one can then confine oneself to the study of the equation that $\Delta(y_1, y_2, \ldots, y_n)$ satisfies. It is called the $(n - m)^{\text{th}}$ *associate* of $P(y) = 0$. The solutions to a fundamental system for it are (except for $m = 1$ and $m = n - 1$) coupled by entire homogeneous relations with constant coefficients. The properties of those associates have been dealt with many times ([97]). Among other things, they facilitate the investigation of the reducibility of linear equations (nos. **36** and II B 4).

One ordinarily replaces the $(n - 1)^{\text{th}}$ associate with the (easily-derived) equation for which a system of fundamental solutions is:

$$z_k = (-1)^{n+k} \frac{\Delta(y_1, \ldots, y_{n-1}, y_{k+1}, \ldots, y_k)}{\Delta(y_1, y_2, \ldots, y_k)} \quad . \tag{102}$$

C. G. J. Jacobi ([98]) showed that it is identical to the *Lagrange adjoint* ([99]) $P'(z) = 0$, where:

$$P'(z) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k (p_{n-k}\, z)}{dx^k} \tag{103}$$

is called the expression *adjoint* to $P(y)$. Its properties are derived from the identity:

$$z\,P(y) - y\,P'(z) \equiv \frac{dP(y,z)}{dz}. \tag{104}$$

([96]) **H. Poincaré**, Acta math. **5** (1884), pp. 212. Cf., **L. Schlesinger**, *Lineare Diff.-Gl.* **2**, pp. 118.

([97]) **L. Fuchs**, Berl. Ber. (1888), pp. 1115; **A. B. Forsyth**, Trans. London Math. Soc. **179** (1888), pp. 420; **J. Cels**, Ann. éc. norm. sup. (3) **8** (1891), pp. 341; **E. Borel**, *ibid.* **9** (1892), p. 63; **E. Grünfeld**, J. f. Math. **115** (1895), pp. 328; **A. Gutzmer**, Habilitationsschrift, Halle, 1896.

([98]) J. f. Math. **92** (1846), pp. 189 (*Ges. Werke* **2**, pp. 127).

([99]) Cf., no. **16** and **N. H. Abel**, *Oeuvres* **2**, pp. 47.

The adjoint to the adjoint is once more the original equation. With the help of the fundamental system (102), the solution of the equation with a right-hand side $P(y) = q(x)$ can be represented in the explicit form:

$$z = \sum_{k=1}^{n} y_k \int q(x)\, z_k \, dx \,. \tag{106}$$

The general form of the even-order expressions that are identical to their adjoints is ([100]):

$$P(y) \equiv Q' Q(y) \,. \tag{107}$$

Those of the odd-order equations that are equal and opposite to their adjoints are ([101]):

$$P(y) \equiv Q' \frac{dQ(y)}{dx} \,. \tag{108}$$

If $P(y) = M_1 M_2 \dots M_k(y)$, and M_1', M_2', …, M_k' are the adjoints of M_1, M_2, … M_k, resp., then $P'(z) = M_k' M_{k-1}' \cdots M_1'(z)$ ([102]).

Those relationships admit a *geometric representation* ([103]). One considers the equations $x_i = y_i(x)$ $(i = 1, 2, \dots, n)$, in which one understands $y_1, \dots, y_n$ to mean a fundamental system of solutions $P(y) = 0$, to be equations of a curve in $(n-1)$-dimensional space in homogeneous coordinates. The transition from one fundamental system to another can be interpreted as a transformation of the homogeneous coordinates. One can then say that from the projective viewpoint, the equation $P(y) = 0$ defines a certain *integral curve*. The adjoint equation gives that curve dualistically. The associated equations give it in the various types of tangential coordinates that exist in R_{n-1}. That representation is also useful for other questions ([104]). Furthermore, it should be observed that knowing an integral curve will give only the ratios of the integrals. In order to determine the integrals themselves will require yet another quadrature.

27. Second-order equations. – The second-order equation:

$$P_2(y) \equiv p_0 \frac{d^2 y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0 \tag{109}$$

([100]) **C. G. J. Jacobi**, J. f. Math. **17** (1837), pp. 68 (*Ges. Werke* **4**, pp. 39). Cf., footnote 98. Cf., **G. Frobenius**, J. f. Math. **85** (1877), pp. 192; **G. Darboux**, *Théorie des surfaces* **2**, pp. 109. The self-adjoint equations are important in the calculus of variations (cf., II A 9).

([101]) **G. Darboux**, *Théorie des surfaces* **2**, Paris, 1889, pp. 121.

([102]) **L. W. Thomé**, J. f. Math. **76** (1873), pp. 277; **G. Frobenius**, *ibid*., pp. 263; **E. Grünfeld**, *ibid.* **98** (1885), pp. 333.

([103]) **E. Borel**, Ann. éc. norm. sup. (3) **9** (1892).

([104]) **G. H. Halphen**, Paris sav. [étr.] (2) **28** (1884), pp. 115; **S. Lie**, Christ. Förh. (1883), no. 12.

has special properties. Its transform in $z = d \log y / dx$ is a *Ricatti* equation. If it is integrated then that will require the quadrature $\exp -\int p_1 / p_0 \, dx$. Conversely, if one sets $z = -\frac{1}{C\,y}\frac{dy}{dx}$ in the *Ricatti* equation $dz / dx = A + B\,z + C\,z^2$ then one will obtain an equation $P_2(y) = 0$ for y ([105]). Those theorems can be adapted to n^{th}-order linear equations and their transforms in $d \log y / dx$. *J. Liouville* used that transformation to prove that for $\lambda(x) = n(n+1)\,x^{-2}$, the equations:

(109.a)
$$\frac{d^2 z}{dx^2} + 2h\frac{dz}{dx} + \lambda(x)\,z = 0$$

can be integrated by elementary functions when h is a constant and n is a whole number ([105.a]). Most of the known cases can be reduced to that integrability case ([105.b]).

Above and beyond that, *Moutard* ([106]) exhibited all equations of the form (109.a) that have an integral that takes the form of an entire rational function of the parameter h. Its solution is based upon the consideration of certain consequences of equations $P_2(z) = 0$ that are such that as soon as one knows a particular integral of one of them, all of the foregoing ones can be successively integrated ([106.a]). That method is related to the *Laplace* method for second-order linear *partial* differential equations (II A 5, no. **59**). It is analogous to a method that *J. Cel* ([107]) used for the n^{th}-order linear equation by which the results that belong to $P(y) = 0$ will be obtained by alternately forming the *Lagrange* adjoint of each equation and its *first-row adjoint* [i.e., the transform in $\Delta\left(\frac{dy_1}{dx},\ldots,\frac{dy_{n-1}}{dx}\right) : \Delta(y_1,\ldots y_n)$].

For the *behavior of the integrals of second-order linear differential equation in the real domain*, the locations of their zeroes, etc., one might cf., II A 7 a. For *special classes of such equations and functions that satisfy them*, cf., II B 4 and 4 b.

b) Linear systems.

28. Extension of the foregoing theories to systems of linear equations. – The linear systems ([108]) whose general form is:

([105]) Cf., e.g., **G. Darboux**, *Théorie des surfaces* **1**, Paris, 1887, pp. 23.

([105.a]) J. de math. **6** (1841), pp. 1. Cf., also **L. Euler**, Misc. Taur. **3** (1762/65), II pp. 60.

([105.b]) For such cases, cf., *inter alia*, **Hargreave**, Trans. London Math. Soc. (1848), pp. 31, as well as the book by **Boole**.

([106]) C. R. Acad. Sci. Paris **80** (1875), pp. 729.

([106.a]) Cf., **L. Euler**, Misc. Taur. **3** (1762/65), pp. 88.

([107]) Ann. éc. norm. sup. (3) **8** (1891), pp. 341; C. R. Acad. Sci. Paris **115** (1892), pp.1057; *ibid.* **116** (1893), pp. 176.

([108]) The investigation goes back to **J. d'Alembert** [footnote 42]. Cf., **Natani**, *Die höhere Analysis*. **L. Königsberger**, *Lehrbuch*, Chap. III. **L. Sauvage**, Toul. Ann. **8** and **9** (1894/95).

(110) $$\frac{dx_i}{dx} - \sum_{k=1}^{n} a_{ik}(x) \cdot x_k = b_i(x)$$

have properties that are similar to those of the linear equations. We first consider the *homogeneous* system:

(111) $$\frac{dx_i}{dx} - \sum_{k=1}^{n} a_{ik}(x) \cdot x_k = 0 .$$

It possesses a *fundamental system* that is constructed from n solutions $x_i = x_{hi}(x)$ (h, $i = 1, 2, \ldots, n$) whose determinant $\Delta = |x_{hi}|$ is not identically zero. The general solution is:

(112) $$x_i = \sum_{h=1}^{n} c_h x_h \qquad (i = 1, 2, \ldots, n),$$

with n arbitrary constants c_h . That form for the general solution is characteristic of the linear systems. If one knows one or more particular solutions then one can lower the order of the system, e.g., with the method of the variation of constants (no. **22**). m solutions ($m \leq n$) are linearly independent when the m^{th}-order determinants that are constructed from them are not all identically zero.

If the a_{ik} are constant then there will be solutions of the form $x_i = m_i e^{\omega x}$ ($i = 1, 2, \ldots, n$), in which the ω are roots of the *characteristic equation:*

(113) $$f(\omega) \equiv \begin{vmatrix} a_{11} - \omega & a_{11} & \cdots & a_{1n} \\ a_{16} & a_{22} - \omega & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \omega \end{vmatrix},$$

and the m_i are constants whose ratios can be calculated from first-degree equations. If $f(\omega)$ has only simple roots then the integration will be complete with that. The case of multiple roots basically depends upon *Weierstrass*'s theory of elementary divisors (I B 2, no. **3**) ([109]). One can also use *d'Alembert*'s multiplier method (no. **11**) and its method of passing to the limit (no. **23**). The method of *Laplace* ([110]) and the residue calculus of *Cauchy* ([111]) give other forms for the solution. Furthermore, the question is closely related to the question of the canonical forms for linear transformations (whether finite or infinitesimal) ([112]).

([109]) **L. Sauvage**, *loc. cit.* and J. de math. (3) **10** (1884), pp. 387.
([110]) Cf., e.g., **E. Picard**, *Traité d'analyse*, **3**, pp. 395.
([111]) **A. Cauchy**, C. R. Acad. Sci. Paris **8** (1839), pp. 827, 845, 889, 930. Cf., e.g., **H. Laurent**, *Traité d'analyse* **5**, pp. 294; another presentation can be found in **J. Collet**, Grenoble Ann. **6** (1894), pp. 309.
([112]) **Lie-Engel**, **1**, pp. 585.

Vaschy ([113]) succeeded in treating the case of an unsolved system of n homogeneous equations with constant coefficients and n unknown functions with the use of symbolic notation. The symbolic form of the results was the same as in the theory of first-degree equations.

The inhomogeneous system (110) can be integrated by quadratures as soon as one has integrated the corresponding homogeneous one (111) ([113.a]). One can prove that by the method of variation of constants (no. **22**), *Cauchy*'s method (II B 4), or by applying the adjoint system (no. **11**).

One can also couple those results with *Lie*'s theory (no. **13**). The system (110) admits a group of the form:

$$\mathsf{X}_k(f) = \sum_{i=1}^{n} x_{ki}(x)\frac{\partial f}{\partial x_i} \qquad (k = 1, 2, \ldots, n), \tag{114}$$

while the system (111) admits $\mathsf{X} f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$, in addition. (Cf., no. **29**).

One can define the linear system that has n given n-tuples of functions of x, $x_i = x_{ki}(x)$ ($k, i = 1, 2, \ldots, n$) for a system of fundamental solutions. It reads:

$$\begin{vmatrix} \frac{dx_k}{dx} & \frac{dx_{1k}}{dx} & \cdots & \frac{dx_{nk}}{dx} \\ x_1 & x_{11} & \cdots & x_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n1} & \cdots & x_{nn} \end{vmatrix} = 0 \qquad (k = 1, 2, \ldots, n). \tag{115}$$

The expressions for the a_{ki} in terms of the x_{ki} and their derivatives can be inferred from that. They are invariants of the homogeneous linear group of order n^2:

$$\bar{x}_{kj} = \sum_{h=1}^{n} c_{kh}\, x_{hj} \qquad (k, j = 1, 2, \ldots, n),$$

which plays the role of the group Γ in no. **22** here. All of the theories in no. **22** can be adapted with almost no modification ([114]). Among the transformed systems, one will find *Jacobi's adjoint systems* (no. **11**):

$$\frac{dz_i}{dx} + \sum_{k=1}^{n} a_{ki}\, z_k = 0 \qquad (i = 1, 2, \ldots, n). \tag{116}$$

A fundamental system of solutions for it is given by:

([113]) C. R. Acad. Sci. Paris **116** (1893), pp. 491; cf., **C. Jordan**, *Cours d'analyse* **3**.
([113.a]) **E. Vessiot**, Toul. Ann. **8** (1894), H pp. 29.
([114]) **C. G. J. Jacobi**, *Werke* 4, pp. 403.

(117) $$z_i = z_{hi} = \frac{1}{\Delta}\frac{\partial \Delta}{\partial x_{hi}} \qquad (h, i = 1, 2, \ldots, n).$$

Liouville's formula (91) corresponds to ([114.a]):

(118) $$\frac{1}{\Delta}\frac{d\Delta}{dx} = -\sum_{i=1}^{n} a_{ii}$$

here.

c) Lie systems and generalizations.

29. Lie systems. Their various definitions. Their integration theory. – The linear systems are a special case of *Lie systems* ([115]), whose general form is:

(119) $$\frac{dx_i}{dt} = \sum_{k=1}^{r} \theta_k(t)\,\xi_{ki}(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n).$$

In that way, the r independent infinitesimal transformations:

(120) $$\mathsf{X}_k f = \frac{\partial f}{\partial t} + \sum_{k=1}^{r} \theta_k(t)\, X_k\ f = 0$$

will define "the group G that is associated with the system (119)." The equivalent partial differential equation (no. **1**) is:

(121) $$\Lambda f \equiv \frac{\partial f}{\partial t} + \sum_{k=1}^{r} \theta_k(t)\, X_k\ f = 0\ .$$

The system (199) will be linear when G is included in the general linear group and linear and homogeneous when G is included in the homogeneous linear group. It will reduce to a *Ricatti* equation (no. **8**) when G is the projective group in one variable, and to one of the systems $\Phi_k = 0$ (no. **11**) to which the application of *d'Alembert*'s multiplier method to systems of linear equations will lead when G is the projective group in $n - 1$ variables. In their general forms, such systems will appear as auxiliary systems in *Lie*'s theory of the integration of systems with known groups (no. **13**).

If the θ_k (t) are constants then equations (119) will define the finite equations of a transformation of G in canonical form. *Lie* showed ([116]) that the determination of those canonical equations will require at most quadratures when one knows the finite equations of G in any form.

([114.a]) **C. G. J. Jacobi**, *Werke* 4, pp. 403.
([115]) Math. Ann. **25** (1885), pp. 124.
([116]) Cf., e.g., **Lie-Engel** 3, pp. 624.

That is a generalization of the theorem on linear equations with constant coefficients. Another special case of it is the integrability of the Jacobi equation (no. **8**).

Lie has also ([116]) emphasized the meaning of those equations for his theory of integration and gave some suggestions for the use of knowing first integrals or particular integrals or relations between them.

Such systems will once more appear as the *most general systems with fundamental solutions*, i.e., ones whose general integral can be written in the form:

$$(122) \qquad x_i = \Phi_i\,(x_{11}, \ldots, x_{1n}\,;\, x_{21}, \ldots, x_{2n}\,;\, \ldots\,;\, x_{p1}, \ldots, x_{pn} \mid c_1, c_2, \ldots, c_n) \qquad (i = 1, 2, \ldots, n).$$

in which the $x_{11}, \ldots, x_{1n}\,;\, \ldots\,;\, x_{p1}, \ldots, x_{pn}$ mean p arbitrary (but subject to inequalities) solutions, that define the integral, which one calls a fundamental system. Furthermore, $c_1, \ldots, c_n$ are the integration constants, and the functions Φ *do not include t explicitly* and do not depend upon the choice of fundamental system. Those systems were determined for $n = 1$ ([117]) under the assumption that the Φ are algebraic by *L. Königsberger* ([118]), and without that restriction by *E. Vessiot* ([119]). The general case was treated by *A. Guldberg* ([120]), *E. Vessiot* ([121]), and *S. Lie* ([122]).

G. Bohlmann ([123]) had also found the Lie systems again by asking the question of what *classes of systems admit the integration methods*, i.e., systems of the form:

$$(123) \qquad \frac{dx_i}{dt} = H_i\,(x_1, \ldots, x_n \mid \theta_1\,(t), \ldots, \theta_r\,(t)) \qquad (i = 1, 2, \ldots, n)$$

(in which the θ_k mean undetermined functions of their arguments, but the H_i are well-defined ones) that have general integrals of the form:

$$(124) \qquad \Omega_i\,(x_1, \ldots, x_n \mid u_1\,(t), \ldots, u_r\,(t)) = c_i \qquad (i = 1, 2, \ldots, n),$$

in which the nature of the functions Ω depends upon only that of the H, while the u_i will be determined analytically as soon as one assigns special forms to the θ (e.g., by means of differential equations whose coefficients are defined by the θ_k and their derivatives).

Finally, yet another characteristic property of the system (119) is that one can bring its general integral into a form such that *the integration constants in it appear in the same way as the variables in the finite equations of the group G*. That is, if those finite equations are:

$$(125) \qquad x_i' = f_i\,(x_1, \ldots, x_n \mid a_1, \ldots, a_r) \qquad (i = 1, 2, \ldots, n)$$

([117]) **Ed. Weyr** had already given some results for this case. Cf., Prague Abh. (6) **8** (1875/76).
([118]) Acta math. **3** (1883), pp. 1.
([119]) Ann. éc. norm. sup. (3) **10** (1893), pp. 53.
([120]) C. R. Acad. Sci. Paris **116** (1893), pp. 964; J. f. Math. **115** (1895), pp. 111.
([121]) C. R. Acad. Sci. Paris **116** (1893), pp. 1112.
([122]) *Ibid.*, pp. 1233; Leipziger Ber. **45** (1893), pp. 341; *ibid.* **48** (1896), pp. 394. Cf., also **Lie-Scheffers**, *Transformationsgruppen*, Chap. 24.
([123]) J. f. Math. **43** (1894), pp. 207; *ibid.* **115** (1895), pp. 89.

then the general integral of (119) will have the form:

$$x_i = f_i(x_1^0, \ldots, x_n^0 \mid a_1(t), \ldots, a_r(t)) \qquad (i = 1, 2, \ldots, n), \tag{126}$$

in which the $x_1^0, \ldots, x_n^0$ are arbitrary constants, and the $a_k(t)$ are defined by the *Lie* auxiliary system:

$$\frac{da_k}{dt} = \sum_{j=1}^{r} \theta_j(t)\, \alpha_{jk}(a_1, \ldots, a_r) \qquad (k = 1, 2, \ldots, r), \tag{127}$$

for which the associated group is the first parameter group of G (II A 6, no. **14**). *E. Vessiot* ([124]) derived a theory of the integration of Lie systems from that remark, which one can regard as an application of the method of variation of constants, and under the assumption that the finite equations of G are known: Two Lie systems are equivalent, i.e., integrating the one will imply that of the other, when the associated groups are holohedrally isomorphic and the functions $\theta_k(t)$ are the same in both systems. The integration of the system (119) comes about with the help of the *adjoint linear system:*

$$\frac{de_s}{dt} = \sum_{k=1}^{r} \theta_k(t) \sum_{i=1}^{r} c_{iks}\, e_i \qquad (s = 1, 2, \ldots, r), \tag{128}$$

for which the associated group is the *adjoint group* (II A 6, no. 15) to G. After that auxiliary integration, one will have to perform at most some possible quadratures, namely, when G includes distinguished infinitesimal transformations. If G is not simple then one can also replace the integration of (119) by a sequence of *Lie* systems with simple groups.

The theorems of no. **25** concerning linear systems can be adapted to all *Lie* systems.

If one assumes that the finite equations of G are not known then the integration of (119) will include the determination of those finite equations as a special case, which is a problem for which *Lie* ([125]) gave some essential results. *E. Vessiot* ([126]) reduced the general case to the *Lie* theory that was presented in no. **13** by first determining the group of infinitesimal transformations of the form $\sum \vartheta_k(t)\, \mathsf{X}_k f$ that leave the system (119) invariant (cf., no. **28**). In particular, one concludes from this that the integration of (119) can always be reduced to that of auxiliary linear systems when G is transitive (II A 6, no. **11**), and also when G is intransitive, as long as one has calculate their invariants.

30. Most general system with fundamental solutions. Higher-order equations with fundamental systems of first integrals. Generalization of Lie systems. – One can extend the concept of *differential systems with fundamental solutions* when one allows the independent

([124]) Toul. Ann. **8** (1894), H.
([125]) Cf., e.g., **Lie-Engel 3**, pp. 518, 798, etc.
([126]) Toul. Ann. **10** (1896), G.

variable to appear explicitly in the expression for the general solutions in terms of particular solutions. However, one must then introduce an additional hypothesis, except for $n = 1$. A system:

$$\frac{dx_i}{dt} = H_i\,(x_1, \ldots, x_n, t) \qquad (i = 1, 2, \ldots, n) \tag{129}$$

is a system with fundamental solutions when one *knows* relations of the form:

$$x_i = \Phi_i\,(x_{11}, \ldots, x_{1n}\,;\,\ldots\,;\,x_{p1}, \ldots, x_{pn}\,;\,t \mid c_1, \ldots, c_n) \qquad (i = 1, 2, \ldots, n) \tag{130}$$

that will give the general integral as soon as one replaces $x_{11}, \ldots, x_{1n}\,;\,\ldots\,;\,x_{p1}, \ldots, x_{pn}$ in them with p particular solutions, and are arranged in such a way that the equations:

$$\overline{x}_{kj} = \Phi_j\,(x_{11}, \ldots, x_{1n}\,;\,\ldots\,;\,x_{p1}, \ldots, x_{pn}\,;\,t \mid a_{k1}, \ldots, a_{kn}) \qquad (k = 1, 2, \ldots, p\,;\,j = 1, 2, \ldots, n) \tag{131}$$

define a group whose parametric equations are independent of t. All transforms of the linear equations ([127]) and Lie systems fall within the scope of that definition. Such systems can always be integrated by means of auxiliary *Lie* systems ([128]).

A. Guldberg ([129]) generalized Lie systems in a different way by looking for higher-order systems of equations that admit fundamental systems of first integral. They are the systems:

$$\frac{d^p x_i}{dt^p} = H_i\left(t, x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{d^{p-1}x_n}{dt^{p-1}}\right) \qquad (i = 1, 2, \ldots, n) \tag{132}$$

whose general can be put into the form:

$$\frac{d^{p-1}x_n}{dt^{p-1}} = \Phi_i\,(\psi_{11}, \ldots, \psi_{1n}\,;\,\ldots\,;\,\psi_{s1}, \ldots, \psi_{sn}\,;\,t \mid c_1, \ldots, c_n) \quad (i = 1, 2, \ldots, n), \tag{133}$$

while the equations:

$$\frac{d^{p-1}x_n}{dt^{p-1}} = \psi_{ki}\left(t, x_1, \ldots, x_n, \frac{dx_1}{dt}, \ldots, \frac{d^{p-2}x_n}{dt^{p-2}}\right) \quad (i = 1, 2, \ldots, n\,;\,k = 1, 2, \ldots, s), \tag{134}$$

represent s particular first integrals. The determination and integration of such systems is likewise linked with the theory of transformation groups.

([127]) **Vessiot**, Thesis, pp. 31.
([128]) **Vessiot**, Toul. Ann. **8**, H, pp. 30; cf., a theorem of **T. Levi-Civita**, Lomb. Rend. (1895).
([129]) C. R. Acad. Sci. Paris **117** (1893), pp. 215.

A third generalization of *Lie* systems is provided by infinite groups, namely, systems of the form ([130]):

$$\frac{dx_i}{dt} = \xi_i (x_1, \ldots, x_n \mid t) \qquad (i = 1, 2, \ldots, n) \tag{135}$$

where the family of infinitesimal transformation that depend upon the parameter t:

$$\mathsf{X}_i f = \sum_{i=1}^{n} \xi_i (x_1, \ldots, x_n \mid t) \frac{\partial f}{\partial x_i} \tag{136}$$

is contained completely within an infinite group G (that is different from all points-transformations). The canonical systems of mechanics (no. **12**) belong to that class, *inter alia*. The arbitrary constants also appear in the general integral here in that same way that the variables appear in the finite equations of the group G. One can also say: The equivalent equation:

$$L f \equiv \frac{\partial f}{\partial t} + \mathsf{X}_i f = 0 \tag{137}$$

has n solutions that satisfy the defining equations for the finite transformations of G (cf., no. **14**).

31. Systems of partial differential equations with fundamental solutions. Applications. – From a more-general standpoint, a system of fundamental equations is any system Σ_G of partial differential equations:

$$F_k \left(x_1, \ldots, x_m, z_1, \ldots, z_n, \frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial z_n}{\partial x_m}, \frac{\partial^2 z_1}{\partial x_1^2}, \ldots \right) = 0 \tag{138}$$

whose general solution emerges from a particular one by the general transformation $z_i = \varphi_i (z_1^0, \ldots, z_n^0)$ of a (finite or infinite) group G ([131]). The system of defining equation for the finite transformations of G are a special case of that. In general, a system Σ_G is nothing but a system that admits the group G and whose solutions can be permuted transitively by G. The construction of all such systems thus depends upon determining the types of groups G and the general theory of differential invariants. Except for certain exceptional systems, they will be obtained when one sets a system of suitably-chosen differential invariants equal to given functions of $x_1, \ldots, x_m$.

If the group G is *finite* then the integration of Σ_G will fall within the scope of the problems that were treated before. If the group G is *infinite* and *not simple* then the integration of Σ_G can be reduced to a sequence of systems of the same type that correspond to simple group. Those simple groups can also be obtained from the *normal decomposition* of G into a sequence of groups, each of which is a distinguished maximal subgroup of the previous one (II A 6, nos. **6**, **17**). The number

([130]) **E. Vessiot**, C. R. Acad. Sci. Paris **125** (1897), pp. 1019.
([131]) **S. Lie**, Leipziger Ber. (1895), pp. 261 and **J. Drach**, Paris thesis 1898, pp. 90.

and structure of them are independent of the decomposition that one applies, up to the ordering of the sequence ([132]). If G is *infinite* and *simple* then one seeks to reduce the number of unknown functions $z_1, z_2, \ldots, z_m$ to a minimum. In that way, one will then come to systems Σ_G that one can regard as not further reducible and as the structure of the canonical system that corresponds to G ([133]).

We already encountered the system Σ_G in no. **14**. *Lie* ([131]) had further reduced the following general problem to the integration of such systems: *Integrate the equation* $\mathsf{X} f = 0$ *when one knows the defining equations of a group G and the general form of its infinitesimal transformations and* $\mathsf{X} f$ *is any infinitesimal transformation of G*. That problem subsumes, as special cases, the general theory of *Lie* systems (no. **29**), *Lie*'s theory of no. **13**, the system at the conclusion of no. **27**, and the first-order partial differential equations (II A 5, **IV**). The theory of Jacobi multipliers (no. **12**) is also subordinate to it: Knowing a multiplier M of the system (45) will give the equivalent equation:

$$\mathsf{X} f \equiv \mathsf{M} \sum_i \alpha_i \frac{\partial f}{\partial x_i} = 0 , \tag{139}$$

in which $\mathsf{X} f$ is an infinitesimal transformation of the group of point-transformations that leaves the volume unchanged. In that way, *Lie* could prove that the theory of last multiplier gives the *complete* simplification that results from knowing a multiplier M for the integration of the system. He had also once more carried out the integration of an n^{th}-order system when one knows a multiplier and an infinitesimal transformation from the same viewpoint ([134]): Either one derives one or more first integrals from it, or one completes the integration by two quadratures once one has determined $n - 2$ first integrals, or one obtains the last integral with no quadrature once one has determined $n - 1$ first integrals.

32. Various classes of equations. – The foregoing classes of differential systems were all obtained when one sought to generalize one or the other of the following two main properties of linear equations:

1. They possess fundamental systems of solutions, i.e., their solutions are not mutually-independent transcendents, in a certain sense.

2. One knows how the arbitrary constant are included in the general integral.

The closer consideration of the second property has led to the examination of other classes of equations: For example, second-order equations whose general integral is a linear fractional

([132]) **J. Drach**, Paris thesis 1898, pp. 104.
([133]) **Lie** discussed this latter point very unclearly.
([134]) Math. Ann. **11** (1877), pp. 508; Norw. Arch. **9** (1884), pp. 441.

function of the two integration constants ([135]) and the higher-order equations with general integrals of the form:

$$\frac{\sum a_k\, u_k(x)}{\sum b_k\, v_k(x)}, \tag{140}$$

in which the constants a_k, b_k are coupled by a suitable number of homogeneous relations ([136]). **P. Painlevé** showed ([137]) that any system (1) whose general solution depends upon the arbitrary constants *rationally* can be integrated by auxiliary linear systems. More generally, he also examined the systems (1) whose general solution depends upon the constants algebraically (II B 5).

One also seeks to generalize the peculiar form of the general integral to the *Jacobi* equation (no. **8**), e.g., one treats equations whose general integral has the form ([138]):

$$M(x)\prod_{i=1}^{q}[x_i-\theta_i(t)]^{m_i} = \text{const.} \tag{141}$$

Other classes shall be spoken of in what follows.

Equivalence problems.

33. Formulation of the problem. Introduction of differential invariants. General methods. – Let a class C of n^{th}-order differential systems (1) be defined by the general form of its systems, in which the right-hand side of the general system of the class depends upon a certain number of arbitrary elements (functions or parameters) in a well-defined way. It further gives a (finite or infinite) transformation group G of a type such that each of its transformations transforms each system in C into a system of the same class in such a way that any system in C can be taken to any other system by a transformation of G. Two systems in C are then called *equivalent* (under G) when there is at least one transformation of G that will take the one to the other. The *equivalence problem* will require the search for necessary and sufficient conditions for two systems in C to be equivalent ([139]). The solution of that problem depends upon the determination of the *differential invariants of class C* under the group G. Any transformation of G corresponds to a transformation of the arbitrary elements upon which the general system of the class depends. Those

([135]) **Hazzidakis**, J. f. Math. **90** (1881), pp. 174; **E. Vessiot**, Toul. Ann. **9** (1895), F.

([136]) **P. Appell**, C. R. Acad. Sci. Paris **12** and **19**/11 (1888); *ibid.* **24**/3 (1890); J. de Math. (2) **5** (1889), pp. 361; **P. Rivereau**, Paris thesis 1890.

([137]) C. R. Acad. Sci. Paris **116** (1893), pp. 173 and *leçons de Stockholm*, pp. 392.

([138]) **Minding**, Petersbourg Mém. (7) **5** (1862), pp. 1; **Elliot**, Ann. éc. norm. sup. (3) **7** (1890), pp. 101; **Koyalovich**, Diss. Petersbourg (1894); **Sonin**, Petersbourg Bull, (1895); **Korkin**, Math. Ann. **48** (1897), pp. 317 and C. R. Acad. Sci. Paris **122**, **123** (1896); **P. Painlevé**, *ibid.* and *leçons de Stockholm*. Cf., also **E. Haentzschel**, J. f. Math. **112** (1893), pp. 148; **W. Heymann**, *ibid.* **113** (1894), pp. 94.

([139]) The introduction of the *extended group* (II A **6**, no. **13**) shows that one must only deal with a special case of the general problem of the equivalence of two manifolds under a group. Cf., **S. Lie**, Christ. Förh. (1883), (1884); **Lie-Scheffers**, *Kontinuierliche Gruppen*, Chap. 23, § 4.

transformations define a group γ that is isomorphic to G, and one deals with the differential invariants of that group γ ([140]). They can all be derived by differentiation from a finite number of *fundamental invariants* by means of certain *differential parameters*. The equivalence of two general systems of a class is expressed by setting a finite number of invariants of the one system equal to the corresponding invariants of the other ([141]).

That method, which is very convenient for the discussion of equivalence conditions, has the disadvantage that it will yield the invariants in complicated forms that are usually irrational or even transcendental. For that reason, one likewise seeks systems of rational invariants (if they exist) by applying the other methods or special processes that are appropriate to each case. In that way, the *absolute* invariants (which are the only ones that have been spoken of up to now) take the form of quotients of powers of *relative* invariants, in general, i.e., ones that are reproduced by the general transformation of G but multiplied by a fact that depends upon only the arbitrary elements of that transformation.

Those equivalence theorems are useful for integration insofar as for each special system of the class C that one has integrated, they will produce a category of likewise-integrable systems, namely, the ones that are equivalent to it.

34. Invariants of linear equations. – The oldest example of the foregoing theory is defined by the *linear* equations:

$$\frac{d^n y}{dx^n} + \frac{n}{1} p_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \frac{n(n-1)}{1 \cdot 2} p_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n(x)\, y = 0 . \tag{142}$$

One will then obtain a new analogy with the theory of algebraic equations (the invariant theory of binary forms, II B 2). The arbitrary elements are the coefficients $p_1, p_2, \ldots, p_n$ here. The group G is the infinite group:

$$\bar{y} = y \cdot \eta(x) , \qquad \bar{x} = \xi(x) \qquad (\xi, \eta \text{ arbitrary}). \tag{148}$$

One must assume that $n \geq 3$, because two equations (142) will always be equivalent under that group for $n = 1$ and $n = 2$. It is the enveloping group of point (and also contact) transformations in x, y that leave classes of equations (142) invariant ([142]).

The relative invariants that *E. Laguerre* introduced ([143]) are reproduced but multiplied by a power of $d\bar{x} / dx$ whose exponent is called the *index* of the invariant. *Fr. Brioschi* laid the

([140]) For the concept of differential invariants and its history, one might cf., I B 2, no. **20**. – The general concept is due to **S. Lie**, Math. Ann. **24** (1884), pp. 537.

([141]) The fact that a *finite* number of fundamental invariants will always suffice was proved by **A. Tresse**, Paris thesis 1893, pp. 42 [Acta math. **18** (1894), pp. 1].

([142]) **Lie**, Christ. Förh. (1881).

([143]) C. R. Acad. Sci. Paris **88** (1879), pp. 116 and pp. 224. Cf., also **Cockle**, Quart. J. of Math. (1876).

foundation for their determination ([144]). *G. H. Halphen* ([145]) undertook their general investigation by connecting them with his earlier investigations of projective differential invariants of curves in the plane and space. Two equations (142) that are equivalent under the group (143) have, in fact, the same integral curves (no. **26**), and they are defined only up to a projective transformation. The projective-invariant properties of the integral curves are then expressed by invariant properties of the differential equation under the group. *Halphen* gave a method for defining invariants by means of a relation between the equations of order two and n, the concept of the *weight* and its relation to the index, the general form of the invariants of weight 3, θ_3 , and later ([146]) the general form of the invariants for $n = 4$. For the applications to the integration, *Halphen* used a reduced form with $p_1 = 0$, $\theta_3 = 1$. He investigated the integration of those equations that can be reduced to equations with constant coefficients by transformations (143) or to ones whose general integral is a single-valued function (that is expressible by rational or elliptic functions).

A. R. Forsythe ([147]) applied another normal form (which was used already by *Cockle* for $n = 3$ and by *Laguerre* for the general case). It is defined by $p_1 = 0$, $p_2 = 0$ and will be obtained by integrating second-order auxiliary equations and a quadrature. In that reduced form, one will obtain explicit expressions for $n - 2$ invariants $\theta_3, \theta_4, \ldots, \theta_n$, whose weights, which are given by the index, are linear in the p_k and their derivatives. Those expressions are independent of n. They are $\equiv 0$ when the general integral of (142) is the general n^{th}-order binary form of two independent solutions of a second-order equation. (142) can then be integrated by means of *one* second-order auxiliary equation. The invariants $\theta_3, \ldots, \theta_n$ are rational and entire in the coefficients of (142). In the general case, they consist of a linear part that was determined by *Fr. Brioschi* ([148]) and a part whose terms all have p_2 or one of its derivatives as a factor.

Equations (142), which admit transformations of the (143), define a special category. They were treated by *P. Appell* ([149]). Their determination is implied by *Lie*'s general theorems (no. **18**). One can reduce them to equations with constant coefficients ([150]).

35. Invariants of various classes of equations. – *R. Liouville* ([151]) investigated the classes of equations:

$$\frac{dy}{dx} = c_0 + 3c_1 y + 3c_2 y^2 + c_3 y^3 \tag{144}$$

(c_0, c_1, c_2, c_3 are arbitrary functions of x) under the group:

([144]) Bull. soc. math. **7** (1879), pp. 105.
([145]) Paris sav. [étr.] (2) **28** (1884), pp. 1.
([146]) Acta math, **3** (1883), pp. 325.
([147]) Trans. London Math. Soc. **179** (1888), pp. 377.
([148]) Acta math. **14** (1890/91), pp. 233.
([149]) Acta math. **15** (1891), pp. 281.
([150]) For the application of the invariants of linear equations, one can also cf., **L. Berzolari**, Ann. di mat. (2) **25** (1897).
([151]) C. R. Acad. Sci. Paris **6/9** (1886) and **12/9** (1887); Amer. J. Math. **10** (1888), pp. 283.

(145) $$\bar{x} = \xi(x), \qquad \bar{y} = y \cdot \eta(x) + \zeta(x)$$

and gave the law for defining a sequence of entire rational relative invariants. *P. Appell* ([152]) undertook that investigation again with the help of the reduced form $dY / DX = Y^3 + \mathfrak{I}(X)$, in which X and $\mathfrak{I}$ are absolute, but irrational, invariants. One and the same equation $F(\mathfrak{I}, d\mathfrak{I} / dx) = 0$ is true for all mutually-equivalent equations of the class, and it can be put into rational form with the help of the invariants of *R. Liouville*. Both authors gave different cases of integrability, namely, reducibility to constant coefficients and to special equations that can be solved by quadratures or a *Riccati* equation. *Elliot* ([153]) studied the *singular* case in which the polynomial $c_0 + 3\,c_1\,y + 3c_2\,y^2 + c_3\,y^3$ has a double root y. It led to the equation $y\,dy = dx + x\,X(x)$ (no. **5**).

More generally, *P. Appell* ([152]) examined the classes of equations:

(146) $$\frac{dy}{dx} = \frac{P_n(y)}{Q_p(y)} = \frac{a_0 + a_1 y + \cdots + a_n y^n}{b_0 + b_1 y + \cdots + b_p y^p}$$

(the a_k, b_k are arbitrary functions of x and $p \leq n - 2$). He employed the reduced form that is defined by $a_{n-1} = a_{p+1} = 0$, $a_n = b_p = 1$. For $p = n - 2$, another reduced form is required. *Appell* mainly applied that result to equations that are reducible to ones with constant coefficients.

P. Painlevé ([154]) studied the class of equations (146) under the group:

(147) $$\bar{x} = \xi(x), \qquad \bar{y} = \frac{y\,\eta(x) + \zeta(x)}{y\,\theta(x) + \omega(x)}$$

(ξ, η, ζ, θ, ω are arbitrary), which is a problem that is not essentially different from the previous one. He first used a canonical form that is analogous to Appell's in order to determine the equations of the class that admit subgroups of (147). Those equations can be integrated by quadratures, except for the Riccati equations. Since that canonical form has the disadvantage that there are infinitely-many equivalent reduced forms, *Painlevé* gave the means for constructing other ones that did not have that disadvantage. The solution of the equivalence problem is made easier by that. As *Painlevé* showed, that theory can be extended to the first-order equations that are rational in y and dy / dx and have undetermined coefficients that depend upon x.

P. Appell ([155]) and *P. Rivereau* ([156]) examined the higher-order homogeneous equations with undetermined coefficients that are entire rational in y and its derivatives under the group (143) using the method of reduced forms. The main applications related to equations that are reducible

([152]) J. de math. (4) **5** (1889), pp. 361.
([153]) Ann. éc. norm. sup. (3) **7** (1890), p. 101.
([154]) C. R. Acad. Sci. Paris **110** (1890), pp. 840; *Mém. sur les équations diff. du 1er ordre*, Paris, 1890 [Ann. éc. norm. sup. (1892)].
([155]) J. de math. (4) **5** (1889), pp. 361.
([156]) Paris thesis 1890.

to such things with constants coefficients, equations that admit an Euler multiplier that depends upon x alone, and finally, equations with general integral of the form (140).

The behavior of the equation:

$$y'' = a_0 y'^3 - a_1 y'^2 + b_1 y' - b_0 \tag{148}$$

(a_0, a_1, b_0, b_1 are undetermined functions of x and y) under the group of all point-transformations in x and y was investigated. *S. Lie* ([157]) gave the conditions for it to be equivalent to $y'' = 0$ and reduced the integration in that case to that of a third-order linear equation. *R. Liouville* ([158]) gave the law for constructing the rational relative invariants of that equation, various reduced forms for them, several cases of integrability, and finally applications to the theory of geodetic lines. *A. Tresse* ([159]) rediscovered those results with *Lie*'s methods.

E. Vessiot ([160]) studied the invariants and the reduced forms of the equations:

$$a(y y'' - y'^2) + b y'' + c y y' + d y' + p x^3 + q x^2 + r x + s = 0 \tag{149}$$

under the group (147).

C. Żorawski ([161]) determined the invariants for the general first-order equation under the infinite subgroups of the group of all point-transformations in x and y.

Finally, *A. Tresse* ([162]) gave very complete results on the invariants of the general equation $y'' = F(x, y, y')$ under the group of all point-transformations. He introduced relative invariants that were reproduced but multiplied by powers of two well-defined factors. One can construct an absolute from those three invariants. Those invariants are rational. One can construct all of them in succession with the help of four of them and three differential parameters. In addition to the general equivalence conditions, *Tresse* derived the conditions for a second- order equation to admit 1, 2, 3, or 8 independent infinitesimal transformations from that. Equation (148) is a special case. However, only the case of equations that are reducible to $y'' = 0$ is actually singular since all invariants are equal to zero for them.

Rational theories of integration.

36. Domain of rationality. Irreducibility. – The *rational theories of integration* are duplicates of *Galois's theory of equations* (I B 3 c). Their common principle is the replacement of the search

([157]) Norw. Arch. **8** (1883), pp. 372.
([158]) C. R. Acad. Sci. Paris **28/11** (1887) and J. éc. polyt. cah. **59** (1889), pp. 7.
([159]) Paris thesis 1893 [Acta math. **18** (1894), pp. 1].
([160]) Toul. Ann. **9** (1895), F. The same equation was examined from a different viewpoint by **E. Picard**, J. de math. (4) **5** (1889), pp. 277 and **G. Mittag-Leffler**, Acta math. **18** (1894), pp. 233.
([161]) Krakow. Ber. **26**.
([162]) *Preisschriften der Jablonowski'schen Gesellschaft*, Leipzig 1896.

for *one* solution in isolation with the search for a fundamental system of solutions (no. **30**). The success of that is then connected with the existence of such systems.

The definition of a special equation rests upon the concept of the *rationality domain* ([163]) (I B 3 c, no. **1**, and footnote 2). Such a domain [*R*] is defined by a certain number of *elements* that define its *basis*, and a certain number of *fundamental operations*. The elements here are: *all constants, certain independent or dependent variables, and certain well-defined or undetermined functions of those variables.* When no other operations are given, the operations here are *the rational algebraic operations and differentiation*. Any function that can be derived from the elements of its basis by a finite number of fundamental operations belongs to the rationality domain [*R*]. If the basis includes no function, but only independent variables, then [*R*] is the *absolute rationality domain*. If one derives another domain [*R*′] from a domain [*R*] in such a way that one adds a certain number of functions to the basis for [*R*] then one says that: One has *adjoined* those functions to the domain [*R*] (I B 3 c, no. **7**).

A rational ([164]) differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ is called *irreducible* in a domain *R* (that includes *x* and *y*) when it is irreducible in the algebraic sense (I B 1 b, no. **5**) and when it has no common integral with any likewise-rational equation of lower order; otherwise, it is *reducible*. The original idea of extending the concept of irreducibility to differential equations belongs to *G. Frobenius* ([165]), while the foregoing form of the definition belongs to *L. Königsberger* ([166]). Among the results of the latter, let us mention the following two:

When a rational equation has *one* solution in common with an irreducible equation, *all* solutions of the latter will satisfy the former.

When two rational equations have common solutions, there will be an irreducible equation that is satisfied by all of those common solutions, and only them. One obtains it by an algorithm that is analogous to that of the greatest common divisor.

One occasionally restricts the concept of irreducibility by imposing other conditions than rationality on the equations in question, e.g., that they must be linear and homogeneous. In the latter case, and upon restricting oneself to the absolute rationality domain, one will, in fact, be in a position to decide whether a given equation is or is not reducible ([167]).

37. Rational theories of integrating linear equations. – The n^{th}-order linear equation was the first one for which a rational theory of integration was developed, which is a new analogy between that equation and the n^{th}-order algebraic equation. We assume that its coefficients $p_1, p_2, \ldots, p_n$ are given functions of *x*. By definition, they belong to the rationality domain in question [*R*]. In addition, that domain includes the variable *x* and the undetermined functions $y_1, y_2, \ldots, y_n$, and can include other given functions of *x*, as well. With that, we have defined what we understand to mean a *rational differential function V of the y*. If we replace $y_1, y_2, \ldots, y_n$ in such a thing with the

([163]) **E. Vessiot**, Paris thesis 1892.
([164]) That is, *F* shall belong to the domain *R*.
([165]) J. f. Math. **76** (1873), pp. 234; **80** (1875), pp. 183.
([166]) J. f. Math. **91** (1881), pp. 199; **92** (1882), pp. 291. Cf., also his textbook, pps. 61 and 155.
([167]) **J. Bendixson**, Stockholm Öfv. **49** (1892), pp. 279; **E. Beke**, Math. Ann. **45** (1894), pp. 278.

integrals $y_1^0(x), \ldots, y_n^0(x)$ of a fundamental system that is chosen once and for all then V will go to a function $V_0(x)$ that one calls the *numerical value of* V. It might or might not be rational in $[R]$.

One now performs a linear homogeneous substitution with constant coefficients in V :

$$y_i = \sum_{k=1}^{n} c_{ik}\, y_k \qquad (i = 1, 2, \ldots, n). \tag{150}$$

V goes to a new rational differential function $\bar{V}$ that has a numerical value of $\bar{V}_0$. When $\bar{V}$ is identical to V as a function of all the indeterminates that appear in it, one says: V is *formally invariant* under (150). When $\bar{V}_0$ is identical to V_0 as a function of x alone, one says: V is *numerically invariant* under (150). V can be numerically invariant under (150) without being formally invariant ([168]).

An equation is said to be *special* in $[R]$ when any rational differential function V has a rational numerical value. The various *types of special cases* that are possible by that definition are obtained from the fundamental double theorem: In regard to its relationship to the rationality domain R, the equation corresponds to a subgroup of the general homogeneous linear group in n variables with the following two properties:

1. *Any rational differential function V whose numerical values is rational admits all transformations of G numerically.*

2. *Any function V that admits all transformations of G numerically has a rational numerical value.*

That group is called the *transformation group* or *rationality group* of the equation ([169]). Any special equation G is characterized by such a group. Moreover, since the fundamental system $y_1^0(x), \ldots, y_n^0(x)$ can be chosen arbitrarily, G is defined only up to a linear homogeneous transformation, in other words, only its *type* is defined. One can also say: Any special equation is characterized by a known ([170]) relation of the form:

$$\Omega\left(x, y_1, \ldots, y_n, \frac{dy_1}{dx}, \ldots\right) = \rho(x)$$

([168]) **F. Klein** stressed the importance of that distinction in *autogr. Vorlesungen über höhere Geometrie* **2** (1893), pp. 299. Cf., also **E. Beke**, Math. Ann. **49** (1897), pp. 573.

([169]) The concept of the transformation group of a linear equation is due to **E. Picard** [C. R. Acad. Sci. Paris **96** (1883), pp. 1131 and Toul. Ann. **1** (1887), A]. He was led to it by investigating the reducibility of the general resolvent (no. 25). **Picard**'s first formulation of the fundamental theorem was completed by **E. Vessiot** (Paris thesis 1892). Since its analysis also gave rise to certain difficulties, **Picard** [C. R. Acad. Sci. Paris **121** (1895), pp. 789; *Traité* **3,** pp. 536] completed his proof in such a way that he arrived at the formulation in the text.

([170]) For the actual determination of the rationality group, one might cf., **F. Marotte**, C. R. Acad. Sci. Paris **124** (1897), pp. 608; *ibid.*, **126** (1898), pp. 715, and Paris thesis 1898, pp. 44.

that is fulfilled by the solutions of a fundamental system, in which Ω is a function of V that (formally and numerically) admits all transformations of G and no other ones, and $\rho(x)$ belongs to the domain $[R]$. Ω is called a *characteristic invariant of G.*

The *integration method* ([171]) that arises from this consists of successively reducing the group G by adjoining new functions of x that are defined by auxiliary equations to $[R]$. Let G_1 be a distinguished maximal subgroup of G, and let Ω_1 be a characteristic invariant of G_1 . Ω_1, as a function of Ω, satisfies a rational differential equation. If it has been integrated then the adjunction of *one* of its integrals will suffice to reduce the rationality group to G_1 . One will arrive at the same result when one adjoins *all* integrals of the equation that is satisfied by a characteristic invariant of any subgroup of G that includes G_1 as a function of Ω. That fact allows one to use lower-order equations. Those auxiliary equations have fundamental systems of solutions. One can then replace their integration (no. **30**) with that of a linear auxiliary equation whose transformation group is simple and isomorphic to G / G_1 ([172]).

If the group reduces to G_1 then one proceeds likewise.

The significance of this method comes from the following theorem:

If the complete integration of a rational auxiliary equation reduces the group G then it will reduce to an invariant subgroup ([173]).

There can then be no method that is more advantageous than the foregoing one. In particular, one concludes from this that $P(y) = 0$ is *integrable by quadratures* if and only if its rationality group is an integrable group (II A 6, no. **17**) ([174]). $P(y) = 0$ is *algebraically integrable* if and only if its rationality group includes only a *finite* number of transformations ([174]).

The integration method that was just discussed is closely connected with the following problem: Integrate the equation $P(y) = 0$ (or an n^{th}-order linear system) when one knows that the solutions of a fundamental system fulfill one or more relations of the form $F_k(y_1, y_2, \ldots, y_n) = 0$ ($k = 1, 2, \ldots, m$). Most often, one examines the case in which those relations are algebraic and homogeneous. *G. Darboux* ([175]) showed the role that was played by the covariants of the systems of algebraic forms F_k in that way. *Laguerre* and, in fact, *Halphen* ([176]) have investigated the relationship between the problem and theory of invariants of linear differential equations, especially quadratic relations. *L. Fuchs* ([177]) treated the problem from a function-theoretic viewpoint. His results along that direction were completed by *Wallenberg* ([178]). *E. Picard* ([179]) showed how the problem was implied very naturally by the theory that was developed just now.

([171]) **Vessiot**, thesis, pp. 39.

([172]) **Vessiot**, Toul. Ann. **8** (1894), H, pp. 29.

([173]) For the case in which the auxiliary equation possesses fundamental systems of solutions, one has the theorem of **Vessiot**, Thesis, pp. 44; more generally, **Picard**, *Traité* **3**, pp. 562.

([174]) **Vessiot**, Thesis, pps. 43, 46, 68.

([175]) C. R. Acad. Sci. Paris **90** (1880), pps. 524, 596.

([176]) Cf., footnotes 143 and 145.

([177]) Acta math. **1** (1882), pp. 321; Berliner Ber. (1882), pp. 703 and (1890), pp. 469. Cf., **L. Schlesinger**, Diss. Berl. 1887.

([178]) J. f. Math. **113** (1894), pp. 1; *ibid.*, **114** (1895), pp. 181.

([179]) *Traité* **3**, pp. 550.

Finally, *S. Lie* ([180]) showed that one could treat the question quite completely when one reduced it to the study of a differential system that admits a known transformation group.

38. Extension of the theory to Lie systems. Theory of J. Drach for arbitrary first-order equations. – **E. Vessiot** ([181]) showed that the essential points of the foregoing theory could be adapted to any class of *Lie* system (no. **29**). One can also restrict oneself to the case in which the associated group G is *simply transitive*. The simply-transitive group that is *reciprocal* to it (II A 6, no. **10**) then plays the same role as the general linear group in the foregoing theory.

E. Picard ([182]) made some statements about the means for extending the method to equations of arbitrary order, which allowed him to arrive at the general theorems in no. **37**.

Finally, **J. Drach** ([183]) sketched out a rational theory of integration for the general system (1), or rather, the associated partial differential equation (3). The group that arises is that of *all point-transformations:*

$$\overline{z}_i = F_i(z_1, z_2, \ldots, z_n) \tag{151}$$

that give the relationship between any fundamental systems of solutions of $L f = 0$ and the most general such systems. One must consider rational differential functions V that are constructed from $z_1, z_2, \ldots, z_n$, and their partial derivatives with respect to $x_1, x_2, \ldots, x_n$. The properties of each of them are linked with the nature of the subgroup of (151) that leaves it invariant. In particular, one has theorems that are analogous to the theorem of symmetric functions and *Lagrange*'s theorem (I B 3 c, no. **16**).

A system (1) is *special* when (3) is compatible with any system of relations that one obtains when one sets any function V equal to a rational function of $x, x_1, \ldots, x_n$. The character of each special system is specified by the fact that a subgroup G of (151) exists whose differential invariants will all include rational values in $x, x_1, \ldots, x_n$ as soon as one replaces the indeterminates $z_1, z_2, \ldots, z_n$ in them with the solutions of certain fundamental systems of (3). That subgroup is the *rationality group* of (1).

That yields yet another integration method that consists of the progressive reduction of that rationality group to distinguished subgroups that is consistent with a *normal decomposition* of that group. The auxiliary systems that appear are general systems of partial differential equations with fundamental solutions (no. **31**). The groups that are associated with it are simple, and their structure is defined by the normal decomposition of G.

([180]) Leipziger Ber. (1891), pp. 253; *ibid.* (1896), pp. 396.

([181]) Toul. Ann. **8** (1894), H, pp. 21.

([182]) C. R. Acad. Sci. Paris (1895).

([183]) C. R. Acad. Sci. Paris **8/5** (1893), **14/1** (1895), **26/10** (1897); Paris thesis 1898. **Drach** gave some consideration to the logical essence of the integration problem and the definition of transcendents. Moreover, he generalized the concept of reducibility for arbitrary differential systems. Cf., also **Vessiot**, C. R. Acad. Sci. Paris **128** (1899), pp. 544.