# On the integration of sheaves of infinitesimal transformations in the case where the degree of the derived sheaf is $n+1$, if the degree of the sheaf is $n$ 

By E. VESSIOT

Translated by D. H. Delphenich

## Introduction and summary.

1.     - In 1924, I gave (Bulletin de la Société mathématiques de France, 52, pp. 336-395) ( ${ }^{1}$ ) a new general theory of integration problems that was based upon the consideration of sheaves of infinitesimal transformations.

Such a sheaf $F$ is comprised of all transformations:

$$
X=\lambda_{1} X_{1}+\lambda_{2} X_{2}+\ldots+\lambda_{n} X_{n},
$$

in which the $X_{i}$ are given transformations:

$$
X_{i}=\sum_{k=1}^{m} \xi_{i, k}\left(x_{1}, \ldots, x_{m}\right) \frac{\partial f}{\partial x_{k}} \quad(i=1,2, \ldots, n),
$$

and the $\lambda_{i}$ are arbitrary functions of the variables $x_{1}, \ldots, x_{m}$. The $X_{i}$ are supposed to be divergent (i.e., independent linear forms in the $\partial f / \partial x_{k}$ ) and constitute a basis for the sheaf. The number of them $n(n \leq m)$ is the degree of the sheaf. Two transformations of the sheaf $X f$ and $Y f$ are said to be in involution when their Jacobi bracket ( $X f, Y f$ ) belongs to the sheaf. The sheaf is called complete if all of its transformations are pair-wise in involution. One will obtain a complete system (in the Clebsch sense) when one equates the transformations of a basis for a complete sheaf to zero. A fundamental system of integrals of that complete system is then a fundamental system of invariants of the sheaf (i.e., all of the transformations of the sheaf).

Any integration problem is equivalent to the search for complete subsheaves of a certain sheaf $F$. One will get a complete integral of the problem $\left({ }^{2}\right)$ upon equating a fundamental system of

[^0]invariants of such a subsheaf to arbitrary constants, and one will get all of them when one ignores the ones that have a singular character.

A transformation of the sheaf $F$ is called distinguished when it is in involution with any transformation of the sheaf. The distinguished transformations, when they exist, define a complete subsheaf, which will be called distinguished, or rather characteristic, because the integral multiplicities of that subsheaf ${ }^{1}$ ) (which have the maximum number of dimensions) are the Cauchy characteristics of the problem.

If one confines the search for complete integrals to the search for the ones that have the maximum number of dimensions (which is, in general, the essential problem) then one will have the fundamental theorem that any complete integral is a family of multiplicities that is generated by Cauchy characteristics $\left({ }^{2}\right)$, in such a way that the complete subsheaf that provides a complete integral contains the characteristic subsheaf, and its invariants are invariants of a distinguished subsheaf, or in other words, characteristic functions.
2. - When a sheaf $F$ is not complete, the set of brackets ( $X f, Y f$ ) of its transformations, when taken pair-wise, constitutes a sheaf $F^{\prime}$ that contains $F$ : It is the derived sheaf of $F$. Similarly, $F^{\prime}$ will have a derived sheaf, and so on. The sequence of successive derived sheaves of $F$ terminates with a last derived sheaf, which is complete, and its integration will provide the invariants of $F$. Of course, they will exist only if the degree of the last derived sheaf is less than the number $m$ of variables.

If the basis for the derived sheaf $F^{\prime}$ is taken in the form:

$$
X_{1}, X_{2}, \ldots, X_{n} ; \quad Z_{1}, Z_{2}, \ldots, Z_{n}
$$

then one will have some identity-congruences for the brackets $\left(X_{i}, X_{k}\right)$ of the basis transformations of $F$ that are structure formulas and take the form:

$$
\left(X_{i}, X_{k}\right) \equiv \sum_{j=1}^{n} c_{i, j, k} Z_{j} \quad(\bmod F) \quad(i, k=1,2, \ldots, n)
$$

The congruence sign $(\bmod F)$ indicates that the difference between the two sides of the equation is a transformation of $F$. The nature of the sheaf, from the standpoint of integration, depends essentially upon its structure, and possibly on the structures of its successive derived sheaves, because it is by comparing structures that one will see whether one can pass from one sheaf to another by a change of variables.

[^1]3. - The present paper is dedicated to the simplest case $\left({ }^{1}\right)$, namely, the one in which the derived sheaf $F^{\prime}$ has degree $n+1$ when the degree of $F$ is $n$. The structure formulas then have the simple form:
\[

$$
\begin{equation*}
\left(X_{i}, X_{k}\right) \equiv c_{i, k} Z \quad(\bmod F) \quad(i, k=1,2, \ldots, n) \tag{1}
\end{equation*}
$$

\]

in which $Z$ is an arbitrary transformation of the derived sheaf (that does not belong to $F$ ).
If the number of variables is $m=n+1$ then the integration of the sheaf is equivalent to that of a Pfaff equation, i.e., to the Pfaff problem: As is known, it contains the problem of integrating system of first-order partial differential equations in one unknown function. In a note to the Comptes rendus de l'Académie des Sciences ( ${ }^{2}$ ), I summarized my method and results that relate to that case. It will be included in what follows.

In general, let $m=n+p(p \geq 1)$ be the number of variables, and set $n=2 s+r$, when $2 s$ denotes the rank of the skew-symmetric determinant whose elements are the structure functions $c_{i, k}$. The characteristic subsheaf $G$ has degree $r$, and the complete subsheaves of maximum degree that provide complete integrals will have degree $g=s+r$. Instead of looking for complete subsheaves of degree $g$, it is preferable in the present problem to seek the complete integrals directly, i.e., a system of fundamental invariants of those complete subsheaves. For each of them, those invariants, or complete integral elements, are $s+p$ in number. From the fundamental theorem on Cauchy characteristics, which was recalled above, all of them must be characteristic functions.

Furthermore, one can make a system of fundamental invariants of the given sheaf $F$ figure in each complete integral. An essential result is that for $s>1$, those invariants of $F$ will be $p-1$ in number, or in other words, that the derived sheaf $F$ is complete. If one introduces those invariants as new variables then one will come down to the case of $m=n+1$, i.e., to the Pfaff problem. However, it is pointless to make that change of variables, and all that remains is to find the other $s+1$ elements of the complete integral.

One can take one of them to be an arbitrary characteristic function, namely, $\varphi$. In order to construct the others, it will suffice to remark that they are invariants of any distinguished transformation of the subsheaf of $F$ (of maximum degree) that has $\varphi$ for its invariant. Those distinguished transformations, among which one will find those of $F$, define a sheaf of degree $r+$ 1: It will then suffice to find one of them that does not belong to the characteristic $G$ of $F$. In order to have a simple formula, one chooses $X_{v+1}, X_{v+2}, \ldots, X_{n}(n=2 s)$ to be some distinguished transformations in the basis for $F$, and the desired transformation will be, in the form of a bordered determinant:

[^2]\[

\frac{1}{\delta}\left|$$
\begin{array}{cccc}
c_{1} & \cdots & c_{1, v} & -X_{1} \varphi  \tag{2}\\
\vdots & \ddots & \vdots & \vdots \\
c_{v, 1} & \cdots & c_{v, v} & -X_{v} \varphi \\
X_{1} \varphi & \cdots & X_{v} \varphi & 0
\end{array}
$$\right|, \quad with \quad \delta=\left|$$
\begin{array}{ccc}
c_{1,1} & \cdots & c_{1, v} \\
\vdots & \ddots & \vdots \\
c_{v, 1} & \cdots & c_{v, v}
\end{array}
$$\right| .
\]

I represent that expression by $\{\varphi, f\}$ and call it the bracket of the two functions $\varphi$ and $f$ : Indeed, it is a generalization of the Poisson bracket $[\varphi, f]$. With that notation, the rule for constructing a complete integral can be stated as follows: One takes an arbitrary invariant $u_{1}$ of the characteristic complete sheaf $G$, then an arbitrary invariant $u_{2}$ of the (complete) sheaf that is the sum of $G$ and $\left\{u_{1}, f\right\}$, then an arbitrary invariant $u_{3}$ of the (complete) sheaf that is the sum of $G,\left\{u_{1}, f\right\}$, and $\left\{u_{2}\right.$, $f\}$, and so on. Finally, take an arbitrary invariant $u_{0}$ of the (complete) sheaf that is the sum of $G$, $\left\{u_{1}, f\right\}, \ldots,\left\{u_{s}, f\right\}$. The last sheaf is any one of the complete subsheaves of $F$ of (maximum) degree $g=s+r$. The elements of the corresponding complete sheaf are $u_{0}, u_{1}, \ldots, u_{s}$, and the invariants of $F$.

One recognizes the sequence of integrations in the Clebsch method, as applied to the Pfaff problem.
4. - If one introduces the elements $u_{0}, u_{1}, \ldots, u_{s}$ of a complete integral as new variables then the basis for the sheaf will reduce to the canonical form:

$$
\begin{equation*}
U_{i} f=\frac{\partial f}{\partial u_{i}}+v_{i} \frac{\partial f}{\partial u_{0}}, \quad V_{i} f=\frac{\partial f}{\partial v_{i}}, \quad \frac{\partial f}{\partial z_{1}}, \quad \ldots, \quad \frac{\partial f}{\partial z_{r}} \quad(i=1,2, \ldots, s) \tag{3}
\end{equation*}
$$

The variables $v_{1}, \ldots, v_{s}$, which are thus introduced, will be called the polar elements that are associated with the complete integral. They are defined by the identity:

$$
\begin{equation*}
\left\{u_{0}, f\right\}-\sum_{i=1}^{s} v_{i}\left\{u_{i}, f\right\} \equiv 0 \quad(\bmod G) \tag{4}
\end{equation*}
$$

The complete integral that includes polar elements is defined by the following bracket relations, in which $\rho$ is an undetermined factor:

$$
\left\{\begin{array}{ll}
\left\{u_{0}, u_{i}\right\}=0, & \left\{u_{i}, u_{k}\right\}=0,  \tag{5}\\
\left(v_{i}, v_{k}\right)=0, & \left(u_{i}, v_{k}\right)=0, \\
\left\{v_{i}, u_{i}\right\}=\rho, & \left\{v_{i}, u_{0}\right\}=\rho v_{i},
\end{array} \quad(i, k=1,2, \ldots, s ; i \neq k) .\right.
$$

The reduction of the given sheaf $F$ to the canonical form (3) reduces the integration of $F$ to that of the canonical sheaf $U_{1}, \ldots, U_{s}, V_{1}, \ldots, V_{s}$ in the $2 s+1$ variables $u_{0}, u_{1}, \ldots, u_{s}, v_{1}, \ldots, v_{s}$. That problem is equivalent to the determination of all contact transformations of $(s+1)$-dimensional
space. The preceding results will provide all of the fundamental principles of that theory of contact transformations in a very simple manner.
5. - The case of $s=1$ enters into the general theory only if the derived sheaf $F^{\prime}$ of $F$ is complete. However, it is another case in which the general solution of the problem of integration of $F$ (for $s=1$ ) is once more given by explicit formulas. That is when the degrees of the successive derived sheaves increase by one unit when one passes from each of those derived sheaves to the following one. I have then recovered the equivalent of a theorem of Cartan $\left({ }^{1}\right)$ under hypotheses that are a little more general.
6. - The result that were summarized above (no. 4) imply that the passage from one complete integral (comprised of polar elements) to another can be done by a contact transformation. The general determination of the complete integrals must then depend upon an automorphic differential system $\left(^{2}\right)$, which corresponds to the general group of transformations of $(s+1)$-dimensional space.

The automorphic systems to consider, when reduced to the canonical form that exhibits a fundamental system of differential invariants of the group, have the following first-order equations:

$$
\begin{equation*}
\rho\left[x_{i}, x_{k}\right]_{u, v}=\varphi_{i k}\left(x_{0}, x_{1}, \ldots, x_{2 s}\right) \quad(i, k=0,1,2, \ldots, 2 s), \tag{6}
\end{equation*}
$$

and a complementary equation:

$$
\begin{equation*}
\rho^{s+1} \frac{\partial\left(x_{0}, x_{1}, \ldots, x_{s}, x_{s+1}, \ldots, x_{2 s}\right)}{\partial\left(u_{0}, u_{1}, \ldots, u_{s}, v_{s+1}, \ldots, v_{2 s}\right)}=\Phi\left(x_{0}, x_{1}, \ldots, x_{2 s}\right) . \tag{7}
\end{equation*}
$$

In those equations, $\rho$ is an undetermined factor to be eliminated. The Poisson brackets $\left[x_{i}, x_{k}\right]_{u, v}$ are taken with respect to $u_{0}, u_{1}, \ldots, u_{s} ; v_{0}, v_{1}, \ldots, v_{s}$, which are considered to be independent variables.

Moreover, equation (7) is an integrability condition for equations (6), and one can calculate $\Phi$ without integration when one knows the $\varphi_{i k}$.

Conversely, such a system will correspond to a Pfaff problem. If one introduces the generalized bracket:

$$
\begin{equation*}
\{\varphi, f\}=\sum_{i=0}^{2 s} \sum_{k=0}^{2 s} \varphi_{i, k} \frac{\partial \varphi}{\partial x_{i}} \frac{\partial \varphi}{\partial x_{k}}, \tag{8}
\end{equation*}
$$

[^3]in which equations (6) express the idea that the passage from the $x$ to the $u, v$ reduces to canonical form the sheaf $F$ that is defined by the transformations $\left\{x_{i}, f\right\}$. The integrability conditions are obtained by means of the identity:
\[

$$
\begin{equation*}
(\{\theta, f\},\{\psi, f\}) \equiv\{\theta, \psi\} \cdot Z f \quad(\bmod F) \tag{9}
\end{equation*}
$$

\]

in which $\theta, \psi$ are arbitrary functions, and $Z f$ is an (undetermined) infinitesimal transformation that does not depend upon the choice of those functions. Once those conditions are fulfilled, the integration method that is summarized in no. $\mathbf{3}$ will provide the $u, v$ as functions of the $x$ : One uses the bracket (8), but one shows that it is basically no different from the bracket that was defined in no. 3.

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## I. - Search for complete integrals.

1. Structure of the sheaf. - Let $F$ be the sheaf in question, let $n$ be its degree, let $X_{1}, X_{2}, \ldots, X_{n}$ be a basis for that sheaf, and let $x_{1}, x_{2}, \ldots, x_{m}$ be variables. Set $m=n+p$ and one has $p \geq 1$. The structure formulas have the form:

$$
\begin{equation*}
\left(X_{i}, X_{k}\right) \equiv c_{i, k} Z \quad(\bmod F) \quad(i, k=1,2, \ldots, n), \tag{1}
\end{equation*}
$$

whose structure coefficients are coupled by the relations:

$$
c_{i, k}+c_{k, i}=0 \quad(i, k,=1,2, \ldots, n)
$$

The bracket of the two arbitrary transformations of $F$ :

$$
\begin{equation*}
U=\sum_{i=1}^{n} u_{i} X_{i}, \quad V=\sum_{i=1}^{n} v_{i} X_{i} \tag{3}
\end{equation*}
$$

will then be:

$$
\begin{equation*}
(U, V) \equiv \Phi(u \mid v) Z \quad(\bmod F) \tag{4}
\end{equation*}
$$

in which $\Phi$ is the alternating bilinear form:

$$
\begin{equation*}
\Phi(u \mid v)=\sum_{i} \sum_{k} c_{i, k} u_{i} v_{k}=\sum_{(i, k)} c_{i, k}\left(u_{i} v_{k}-u_{k} v_{i}\right) . \tag{5}
\end{equation*}
$$

In formulas (1) and (4), $Z$ is an arbitrary transformation of the derived sheaf $F^{\prime}$ of $F$. If one changes that transformation then $\Phi$ will be multiplied by a factor.

If one takes another basis for $F$ :

$$
\begin{equation*}
X_{h}^{\prime}=\sum_{i=1}^{n} w_{h, i} X_{i} \quad(h=1,2, \ldots, n) \tag{6}
\end{equation*}
$$

then the transformations (3) will take the forms:

$$
\begin{equation*}
U=\sum_{h=1}^{n} u_{h}^{\prime} X_{h}^{\prime}, \quad V=\sum_{h=1}^{n} v_{h}^{\prime} X_{h}^{\prime}, \tag{7}
\end{equation*}
$$

in which the $u^{\prime}$ are deduced from the $u$, and the $v^{\prime}$, from the $v$, by the same linear transformation that is contragredient to (6):

$$
\begin{equation*}
u_{i}=\sum_{h=1}^{n} w_{h, i} u_{h}^{\prime}, \quad v=\sum_{h=1}^{n} w_{h, i} v_{h}^{\prime} \quad(i=1,2, \ldots, n) \tag{8}
\end{equation*}
$$

The form $\Phi$ is then found to be transformed in such a manner that one has:

$$
\begin{equation*}
\sum_{i} \sum_{k} c_{i, k} u_{i} v_{k}=\sum_{i} \sum_{k} c_{i, k}^{\prime} u_{i}^{\prime} v_{k}^{\prime}, \tag{9}
\end{equation*}
$$

and that identity will define the new structure coefficients $c_{i, k}^{\prime}$.
One can arrange that the transformation (6), or what amounts to the same thing, the transformation (8), reduces $\Phi$, and consequently, the structure of $F$, to a canonical type. Since, from the foregoing, $\Phi$ can be multiplied by an arbitrary factor, it is the equation $\Phi(u, v)=0$ that is what is basically reduced to a canonical form. If one interprets $X_{1}, \ldots, X_{n}$ as coordinates then the determinants $\left(u_{i} v_{k}-u_{k} v_{i}\right)$ are the coordinates of the line that is the intersection of the planes $U=$ $0, V=0$, and $\Phi(u, v)$ is then interpreted as the equation of a linear complex. It is well-known that the canonical form is then:

$$
\begin{equation*}
\Phi=u_{1}^{\prime} v_{2}^{\prime}-u_{2}^{\prime} v_{1}^{\prime}+u_{3}^{\prime} v_{4}^{\prime}-u_{4}^{\prime} v_{3}^{\prime}+\cdots+u_{2 s-1}^{\prime} v_{2 s}^{\prime}-u_{2 s}^{\prime} v_{2 s-1}^{\prime} . \tag{10}
\end{equation*}
$$

One can obtain it as follows: Introduce the derivatives:

$$
\begin{equation*}
\Phi_{u i}^{\prime}=\sum_{k=1}^{n} c_{i, k} v_{k}, \quad \Phi_{v i}^{\prime}=\sum_{k=1}^{n} c_{k, i} u_{k}=-\sum_{k=1}^{n} c_{i, k} v_{k} . \tag{11}
\end{equation*}
$$

The determinant $\Delta$ of the $c_{i, k}$, which is the Hessian of $\Phi$ and the determinant of the coefficients of one and the other series of derivatives, is skew-symmetric, from (2). Let $2 s$ be its rank, which is even, as one knows. If $s$ is zero then the $c_{i, k}$ will all be zero, so $\Phi$ will be identically zero, and the sheaf $F$ will be complete. That case will be discarded.

If the $c_{i, k}$ are not all zero then one can suppose that $c_{1,2} \neq 0$, and so $\Phi_{u 1}^{\prime}$ and $\Phi_{v 1}^{\prime}$ will be two linearly-independent forms. Set:

$$
\begin{equation*}
\Psi=\Phi-\frac{1}{c_{1,2}}\left(\Phi_{u 1}^{\prime} \Phi_{v 2}^{\prime}-\Phi_{u 2}^{\prime} \Phi_{v 1}^{\prime}\right) \tag{12}
\end{equation*}
$$

From (11), that form $\Psi$ is bilinear and alternating, and its derivatives are:

$$
\begin{equation*}
\Psi_{u i}^{\prime}=\Phi_{u i}^{\prime}-\frac{1}{c_{1,2}}\left(c_{i, 2} \Phi_{u 1}^{\prime}-c_{i, 1} \Phi_{u 2}^{\prime}\right) \tag{13}
\end{equation*}
$$

One concludes from this that $\Psi_{u 1}^{\prime}, \Psi_{u 2}^{\prime}$ are identically zero and that for $i=3,4, \ldots, n$, the $\Psi_{u i}^{\prime}$ are coupled by just as many independent relations as the $\Phi_{u i}^{\prime}$ are for $i=1,2,3, \ldots, n$. As far as the
second point is concerned, it is obvious that any relation between the $\Psi_{u i}^{\prime}$ will give a relation between the $\Phi_{u i}^{\prime}$ with the same coefficients for $i=3,4, \ldots, n$. The converse is also true because a relation:

$$
\sum_{i=1}^{n} \alpha_{i} \Phi_{u i}^{\prime}=0
$$

will imply a relation:

$$
\sum_{i=3}^{n} \alpha_{i} \Psi_{u i}^{\prime}+\beta_{1} \Phi_{u 1}^{\prime}+\beta_{2} \Phi_{u 2}^{\prime}=0
$$

and the coefficients $\beta_{1}$ and $\beta_{2}$ are zero because they are, up to a factor of $\pm c_{1,2}$, the coefficients of $v_{1}$ and $v_{2}$ in that identity. Moreover, if there are several relations $\sum \alpha_{k, i} \Phi_{u i}^{\prime}=0$ then the matrix of coefficients $\alpha_{k, 3}, \ldots, \alpha_{k, n}$ will not be zero because there is no relation of that type between just $\Phi_{u 1}^{\prime}$ and $\Phi_{u 2}^{\prime}$.

Hence, $\Psi$ does not contain $u_{1}, u_{2}, v_{1}, v_{2}$, and the rank of its Hessian is fewer by two units than the rank of the Hessian of $\Phi$. Having established that, make the change of variables:

$$
\begin{equation*}
u_{1}^{\prime}=\frac{1}{\sqrt{c_{2,1}}} \Phi_{v 1}^{\prime}, \quad u_{2}^{\prime}=\frac{1}{\sqrt{c_{2,1}}} \Phi_{v 2}^{\prime}, \quad v_{1}^{\prime}=\frac{-1}{\sqrt{c_{2,1}}} \Phi_{u 1}^{\prime}, \quad v_{2}^{\prime}=\frac{-1}{\sqrt{c_{2,1}}} \Phi_{u 2}^{\prime} \tag{14}
\end{equation*}
$$

in order to replace $u_{1}, u_{2}, v_{1}, v_{2}$. It is indeed cogredient, from (11), and it will give:

$$
\begin{equation*}
\Phi=u_{1}^{\prime} v_{2}^{\prime}-u_{2}^{\prime} v_{1}^{\prime}+\Psi . \tag{15}
\end{equation*}
$$

If one operates on real quantities then one can suppose that $c_{1,2}>0$, even if it means exchanging the roles of $\Phi_{u 1}^{\prime}$ and $\Phi_{u 2}^{\prime}$, in such a way that the reduction will be real.

We can now operate on $\Psi$ as we did on $\Phi$, and so on, and if the rank of the Hessian $\Delta$ of $\Phi$ is $2 s$ then we will get the form of type (10) that we have asserted. We point out that it admits a wellknown homogeneous linear group and can consequently be obtained in an infinitude of ways.

It will be convenient to change the notation by writing $A_{1}, A_{2}, \ldots, A_{s}$ for $X_{1}^{\prime}, X_{3}^{\prime}, \ldots, X_{2 s-1}^{\prime}$, $B_{1}$, with $B_{2}, \ldots, B_{s}$ for $X_{2}^{\prime}, X_{4}^{\prime}, \ldots, X_{2 s}^{\prime}$, and $C_{1}, C_{2}, \ldots, C_{r}$ for $X_{2 s+1}, X_{2 s+2}, \ldots, X_{n}$. One then has the canonical basis:

$$
\begin{equation*}
A_{1}, A_{2}, \ldots, A_{s} ; \quad B_{2}, \ldots, B_{s} ; \quad C_{1}, C_{2}, \ldots, C_{r} \quad(n=2 s+r), \tag{16}
\end{equation*}
$$

and upon setting:

$$
\begin{equation*}
U=\sum_{i=1}^{s} a_{i} A_{i}+\sum_{i=1}^{s} b_{i} B_{i}+\sum_{j=1}^{r} c_{j} C_{j}, \quad V=\sum_{i=1}^{s} a_{i}^{\prime} A_{i}+\sum_{i=1}^{s} b_{i}^{\prime} B_{i}+\sum_{j=1}^{r} c_{j}^{\prime} C_{j}, \tag{17}
\end{equation*}
$$

one will now have:

$$
\begin{equation*}
(U, V) \equiv \Phi(a \mid b) \cdot Z \quad(\bmod F) \tag{18}
\end{equation*}
$$

with

$$
\begin{equation*}
\Phi=\sum_{i=1}^{s}\left(a_{i} b_{i}^{\prime}-b_{i} a_{i}^{\prime}\right) . \tag{19}
\end{equation*}
$$

That is equivalent to saying that the brackets of the transformations of the basis (16) are all congruent to zero, except for the brackets:

$$
\begin{equation*}
\left(A_{i}, B_{i}\right) \equiv Z \quad(\bmod F) \quad(i=1,2, \ldots, s) \tag{20}
\end{equation*}
$$

In order for $U$ to be a distinguished transformation, it is necessary and sufficient that the form (19) must be annulled for any $a^{\prime}$ and $b^{\prime}$, so $U$ will reduce to $\sum_{j=1}^{r} c_{j} C_{j}$, in which the $c_{j}$ are arbitrary. The subsheaf of distinguished transformations ( ${ }^{1}$ ) of $F$ is then the sheaf $G$ with basis $C_{1}, \ldots, C_{r}$, so it will have degree $r=n-2 s$ if the rank of the determinant of the $C_{i, k}$ is supposed to be equal to $2 s$.

The last result can be obtained immediately without using the canonical form by writing that the initial form $\Phi$ [equation (5)] is zero for any $v$.
2. Involutions and complete integrals. - It is easy to discuss the general involutions of degrees $2,3, \ldots$ of the sheaf $F$ by means of the canonical basis (16). Let an arbitrary subsheaf $F_{\alpha}$ of degree $\alpha$ have a basis that consists of the transformations:

$$
\begin{equation*}
V_{k}=\sum_{i=1}^{s} a_{k, i} A_{i}+\sum_{i=1}^{s} b_{k, i} B_{i}+\sum_{j=1}^{r} c_{k, j} C_{j} \quad(k=1,2, \ldots, \alpha) . \tag{21}
\end{equation*}
$$

The transformations $U$ [equation (17)] in involution with that sheaf are defined, from (19), by the linear equations:

$$
\begin{equation*}
\sum_{i=1}^{s}\left(a_{i} b_{k, i}-b_{i} a_{k, i}\right)=0 \quad(k=1,2, \ldots, \alpha) \tag{22}
\end{equation*}
$$

Those equations will be independent provided that no combination of the $V_{k}$ reduces to a combination of the $C_{j}$, i.e., provided that $F_{\alpha}$ contains no distinguished transformation of $F$. Under that condition, the sheaf $\Psi$ of transformations $U$ in question will have degree $n-\alpha$, and it will obviously contain the sheaf $G$ of distinguished transformations.

[^4]If one supposes, moreover, that $F_{\alpha}$ is an involution then $\Psi$ will contain $F_{\alpha}$ in such a way that one will have $n-\alpha \geq \alpha+r$, which will demand that $\alpha \leq s$. Conversely, as long as $\alpha<s$, $\Psi$ will contain a subsheaf of degree $n-2 \alpha-r$ that belongs to neither $F \alpha$ not $G$.

One concludes from this that:

1. The general involutions of degree $\alpha \leq s$ do not contain any distinguished transformation of $F$.
2. The general involutions of degree $\alpha>s$ are deduced from the general involution of degree $s$ by appending $a-s$ arbitrary distinguished transformations.
3. The general involution of maximum degree has degree $s+r$, i.e., the genus of $F$ is $s+r$.

That general involution of maximum degree $g=s+r$ can be put into the solved form:

$$
\begin{equation*}
V_{i}=A_{i}+\sum_{k=1}^{s} p_{i, k} B_{k} \quad(i=1,2, \ldots, s), \quad C_{j} \quad(j=1,2, \ldots, r), \tag{23}
\end{equation*}
$$

and the conditions for involution then reduce to:

$$
\begin{equation*}
p_{i, k}=p_{k, i} \quad(i, k=1,2, \ldots, s) . \tag{24}
\end{equation*}
$$

From the general theory $\left({ }^{1}\right)$, one knows that the partial differential equations that relate to the $p_{i, k}$ and express the idea that the sheaf (23) is complete are compatible, and that they provide all of the general complete subsheaves of maximum degree of the sheaf $F$.

The general integral multiplicities of such a complete subsheaf are represented by equations of the form:

$$
\begin{equation*}
\varphi_{h}\left(x_{1}, x_{2}, \ldots, x_{m}\right)=c_{h} \quad(h=1,2, \ldots, s+p) . \tag{25}
\end{equation*}
$$

We reserve the name of complete integrals for them. We say that the functions $\varphi_{h}$ are the elements, and we say: the complete integral $\left(\varphi_{1}, \varphi_{2}, \ldots, \varphi_{s+p}\right)$. We shall restrict the problem of integrating the sheaf $F$ to the search for its complete integrals.

In the present case, there is some advantage to looking for the complete integrals themselves, and not the complete subsheaves for which they are the fundamental invariants.

Here are some preliminary remarks in that regard:

1. The elements of any complete integral are the invariants of the distinguished subsheaf $G$ of $F$. That results from the foregoing, and it is a special case of the general theorem in the theory of Cauchy characteristics $\left({ }^{2}\right)$. We say that such invariants are characteristic functions (for $F$ ).

[^5]$\left(^{2}\right)$ M., nos. 18-19, pp. 374, et seq.
2. One can make the invariants of the sheaf $F$ (if they exist) appear among the elements of any complete integral. Those invariants will exist when the first of the successive derivatives of $F$ that is identical to the following one has degree less than $m$. It will then be complete $\left({ }^{1}\right)$, and its invariants will be those of $F$. We must therefore look for those of the elements of each complete integral that are not invariants of $F$.
3. From the preceding remark, one can first look for the invariants of $F$ by integrating a complete sheaf. Upon taking them to be new variables, one will reduce the integer $p=m-n$ by as many units as there are such (independent) invariants.

However, it will be preferable to avoid such changes of variables by leaving behind the search for other elements of the independent complete integrals in order to search for invariants of $F$. That is what we shall do. We let $q$ denote the number of independent invariants of $F$. It will be a maximum when the derived sheaf $F^{\prime}$ is complete. We will then have $q<m-n$, i.e., $q<p$.
3. Subsheaf of a characteristic function. - Our method is based upon the study of the subsheaf of $F$ that admits an arbitrary function $\varphi$ as an invariant. From the preceding remarks, we suppose that $\varphi$ is not an invariant of $F$, but a characteristic function, i.e., an invariant of $G$. We let $F_{\varphi}$ denote its subsheaf, i.e., the sheaf of transformations of $F$ that $\varphi$ admits.

Take a basis of $F$ to be, on the one hand, the distinguished transformations $C_{1}, \ldots, C_{r}$ that define $G$, and on the other, any other $v=2 s$ transformations of $F$, namely, $X_{1}, X_{2}, \ldots, X_{v}$. The latter can be considered to be a basis for a subsheaf $H$ of $F$, and say that $F$ is the resultant $\left({ }^{2}\right)$ of $G$ and H. $F_{\varphi}$ is itself the resultant then of $G$ and the subsheaf $H_{\varphi}$ of $H$ that is composed of all transformations of $H$ that $\varphi$ admits: They have the form $\sum_{i=1}^{v} u_{i} X_{i}$, with the condition that:

$$
\begin{equation*}
\sum_{i=1}^{v} u_{i} X_{i} \varphi=0 . \tag{26}
\end{equation*}
$$

That condition is not an identity because $\varphi$ is not an invariant of $H$, since it does not belong to $F$. Therefore, $F_{\varphi}$ has degree $n-1$. Let us look for the degree of its distinguished subsheaf.

All of the transformations $C$ of $G$ belong to it. Indeed, let $U$ be any transformation of $F_{\varphi}$, so the transformation $(U, C)$ will admit $\varphi$, since $U$ and $C$ admit it. Now, it belongs to $F$, since $C$ is distinguished in $F$. Thus, it belongs to $F_{\varphi}$. Q.E.D.

It results from this that if $s=1$ then $F_{\varphi}$ will be complete because it has degree $n-1=r+1$ since $r=n-2 s$, in general, and it will contain at least $r$ distinguished transformations, in such a

[^6]way that if one appends an arbitrary transformation of the sheaf to them then the all of the brackets of the transformations of the basis thus-chosen will belong to the sheaf.

Therefore, suppose that $s>1$. I say that $F_{\varphi}$ can have no more than $r+1$ divergent distinguished transformations. Indeed, suppose that there are $r+2$. We can then suppose that they are $C_{1}, \ldots, C_{r}$, $X_{1}, X_{2}$, and that $X_{3}, \ldots, X_{v-1}$ belong to $F_{\varphi} . u_{1} X_{1}+u_{2} X_{2}$ will then be distinguished for $F$, provided that one has $\left(u_{1} X_{1}+u_{2} X_{2}, X_{v}\right) \equiv 0(\bmod F)$, i.e., $c_{1, v} u_{1}+c_{2, v} u_{2}=0$, in such a way that $F$ will have more than $r$ divergent distinguished transformations, which is a contradiction. Therefore, the distinguished subsheaf of $F_{\varphi}$ has degree at least $r$ and at most $r+1$; we shall see that it has degree $r+1$.

Indeed, consider the derived sheaf $F_{\varphi}^{\prime}$ of $F_{\varphi}$. It is contained in the derived sheaf $F^{\prime}$ of $F$, and all of its transformations leave $\varphi$ invariant, since they are brackets of the transformations that $\varphi$ admits. Now, $F^{\prime}$ does not leave $\varphi$ invariant, since otherwise $\varphi$ would be an invariant of $F$. Therefore, the degree of $F_{\varphi}^{\prime}$ is less than the degree of $F^{\prime}$, i.e., the degree is equal to at most ( $n+$ 1) $-1=n$.

Moreover, that degree is equal to at most the degree $n-1$ of $F_{\varphi}$, and one cannot have equality, since $F_{\varphi}$ has at most $r-1$ distinguished transformations (which is a number less than $n-1$, since $r=n-2 s$, and $s>1$ ), so it is not complete. Therefore, the derived sheaf $F_{\varphi}^{\prime}$ of $F_{\varphi}$ has degree $n$. Hence, $F_{\varphi}$, which has degree $n-1$, has a derived sheaf of degree $n$. It then results from no. 1 that the degree of its distinguished subsheaf has the same parity as $n-1$, i.e., the same parity as $r+1$. Since that degree can be only $r$ or $r+1$, from the foregoing, we conclude that it is $r+1$.

In summary, $F_{\varphi}$ has degree $n-1$, its derived sheaf has degree $n$, and its distinguished subsheaf has degree $r+1$.

I now say that the invariants of $F_{\varphi}$ are functions of the invariants of $F$ and $\varphi$.
I will show that by proving that if $F_{\varphi}$ has exactly $q$ independent invariants then $F$ will have $q$ -1 . To that end, recall the general notations of no. 1. We can suppose that $F_{\varphi}$ is defined by $X_{1}, \ldots$, $X_{n-1}$ and that $x_{1}, \ldots, x_{q}$ are invariants of $F_{\varphi} . X_{1}, \ldots, X_{n-1}$ will then take the solved form, which is written:

$$
\begin{equation*}
X_{i}=\frac{\partial f}{\partial x_{q+i}}+\sum_{j=n+q}^{m} \zeta_{i, j} \frac{\partial f}{\partial x_{j}} \quad(i=1,2, \ldots, n-1) \tag{27}
\end{equation*}
$$

If we suppose that $x_{1}=\varphi$ then the term in $\partial f / \partial x_{1}$ cannot be missing from $X_{n}$, which we can reduce to the form:

$$
\begin{equation*}
X_{n}=\frac{\partial f}{\partial x_{1}}+\sum_{n=2}^{q} \xi_{h} \frac{\partial f}{\partial x_{h}}+\sum_{j=n+q}^{m} \zeta_{n, j} \frac{\partial f}{\partial x_{j}} . \tag{28}
\end{equation*}
$$

The derived sheaf $F_{\varphi}^{\prime}$ of $F_{\varphi}$, which has degree $n$, is obtained by combining the transformations (27) with one of their brackets, which will have the form:

$$
\begin{equation*}
Z f=\sum_{j=n+q}^{m} \zeta_{j} \frac{\partial f}{\partial x_{j}} . \tag{29}
\end{equation*}
$$

Since the derived sheaf $F^{\prime}$ of $F$ has degree $n+1$, it will be obtained by appending $X_{n}$, in addition.

Hence, the brackets $\left(X_{i}, X_{n}\right)$ for $(i=1,2, \ldots, \overline{n-1})$ must be combinations of the transformations (27), (28), (29). Since $\partial f / \partial x_{1}$ does not appear in them, they will be combinations of (27) and (29), i.e., $\frac{\partial f}{\partial x_{2}}, \ldots, \frac{\partial f}{\partial x_{q}}$ must no longer appear in them. It then results that the $\xi_{h}$ are invariants of $X_{i}$, i.e., functions of only $x_{1}, x_{2}, \ldots, x_{q}$.

However, $X_{n}$, and as a result $F$, will then admit the integrals of the equation:

$$
\begin{equation*}
\frac{\partial f}{\partial x_{1}}+\sum_{n=2}^{q} \xi_{h} \frac{\partial f}{\partial x_{h}}=0 \tag{30}
\end{equation*}
$$

as invariants, which indeed gives $q-1$ independent invariants.
4. Method of integration. - It results from the foregoing that $F_{\varphi}$ has the same character as $F$, in the sense that the degree of its derived sheaf is greater by one unit than its own degree. Moreover, when one passes from $F$ to $F \varphi$, the number $q$ of independent invariants will become $q+1$, the degree $n$ will become $n-1$, the degree $r$ of its distinguished subgroup will become $r+1$, the number $s=(n-r) / 2$ will become $s-1$, and the number $p=m-n$ will become $p+1$; hence, the genus $g \equiv s+r$ will remain constant.

One then sees that no matter what the function $\varphi$ might be (provided that it is characteristic), it can be chosen to be a complete integral element of $F$ and that the corresponding complete integrals will all be complete integrals of $F_{\varphi}$. Moreover, since one can, from the foregoing, argue with $F_{\varphi}$ as one does with $F$, one will have in that fact the basic principle for a method of constructing complete integrals of $F$ that is the following one:

One determines an arbitrary invariant $\varphi_{1}$ of the distinguished subsheaf $G$ of $F$, then an arbitrary invariant $\varphi_{2}$ of the distinguished subsheaf $G_{1}$ of the sheaf $F_{1}$ that is composed of all transformation of $F$ that admit $\varphi_{1}$, then an arbitrary invariant of the distinguished subsheaf $G_{2}$ of the sheaf $F_{2}$ that is composed of all transformations of $F_{1}$ that $\varphi_{2}$ admit, and so on. When one has chosen $\varphi_{1}, \varphi_{2}, \ldots$, $\varphi_{s-1}$, one will then be reduced to case of $s=1$, in such a way that the following operation, which is the adjunction of $\varphi_{s}$, will give a complete subsheaf $F_{s}$ of degree $g=s+r$ that defines the desired complete integral. One will know the integrals $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{s-1}$, and it will remain for one to calculate $p$ other ones. One calculates them by starting from the $q$ invariants of $F$, and it will only remain for one to find $q-p$ new invariants of $F_{s}$.

It is implicit that $\varphi_{1}$ must not be an invariant of $F$, that $\varphi_{2}$ must not be a function of $\varphi_{1}$ and some invariants of $F$, that $\varphi_{3}$ must not be a function of $\varphi_{1}, \varphi_{2}$, and some invariants of $F$, and so on.

In order to apply the method, one only has to determine the successive distinguished subsheaves $G, G_{1}, \ldots, G_{s}$, while the last one coincides with $F_{s}$, moreover. It results from what we have seen that each one will contain the preceding one and will have a degree that is higher by one unit. In order to pass from one of those sheaves to the following one, one must then append a conveniently-chosen new transformation to it.

To that end, we remark that $G_{k}$ is the resultant of its distinguished subsheaves of $F_{\varphi_{1}}, F_{\varphi_{2}}, \ldots$, $F_{\varphi_{k}}$, which we would like to denote by $G_{\varphi_{1}}, G_{\varphi_{2}}, \ldots, G_{\varphi_{k}}$, respectively. Indeed, we know that $G_{k}$ contains $G_{1}$, which is nothing but $G_{\varphi_{1}}$. As a result, it will contain $G_{\varphi_{2}}, \ldots, G_{\varphi_{k}}$, because $F_{k}$ is the largest subsheaf that is common to $F_{\varphi_{1}}, F_{\varphi_{2}}, \ldots, F_{\varphi_{k}}$, in such a way that the functions $\varphi_{1}, \varphi_{2}, \ldots$, $\varphi_{k}$ will have equivalent roles with respect to them. Therefore, $G_{k}$ contains $G_{k-1}$ and $G_{\varphi_{k}}$, and since its degree surpasses that of $G_{k-1}$ by one unit, it will be the resultant of $G_{k-1}$ and $G_{\varphi_{k}}$, as long as $G_{\varphi_{k}}$ is contained in $G_{k-1}$.

In order to see that the latter situation is impossible, it will suffice to remark that if one composes $F_{k-1}$ and $F_{\varphi_{k}}$ then one reconstitutes $F$, because $F_{k-1}$ has degree $n-k+1, F_{\varphi_{k}}$ has degree $n-1$, and their largest common subsheaf, which is $F_{k}$, has degree $n-k$. The resultant of $F_{k-1}$ and $F_{\varphi_{k}}$ will then have degree:

$$
(n-k+1)+(n-1)-(n-k)=n .
$$

Since it is contained in $F$, whose degree is equal to $n$, that resultant will coincide with the latter.
From that, one sees that if $G_{\varphi_{k}}$ is contained in $G_{k-1}$ then it will leave each of the sheaves $F_{\varphi_{k}}$ and $F_{k-1}$ invariant, and consequently their resultant $F$. Now, that is not true, because that resultant is not contained in the distinguished subsheaf $G$ of $F$.

We have then proved that $G_{k}$ is the resultant of $G_{k-1}$ and $G_{\varphi_{k}}$. One then concludes, by recurrence, that it is the resultant of $G_{\varphi_{1}}, G_{\varphi_{2}}, \ldots, G_{\varphi_{k}}$, as we first stated.

The new transformation that one must append to $G_{k-1}$ in order to get $G_{k}$ will then be a transformation of $G_{\varphi_{k}}$, and everything comes down to finding a general transformation of $G_{\varphi}$ that does not belong to any of the other analogous subsheaves $G_{\psi}$. Since, from the foregoing, they have only $G$ in common, which has degree $r$, and $G_{\varphi}$ has degree $r+1$, by definition, it will suffice to find a transformation of $G_{\varphi}$ that does not belong to $G$. That is what we shall do in the following subsection, and our conclusion will be that if $T_{\varphi}$ is one such transformation then $G_{k}$ will be the resultant of $G$ and $T_{\varphi_{1}}, T_{\varphi_{2}}, \ldots, T_{\varphi_{k}}$, i.e., that its basis will be:

$$
\begin{equation*}
C_{1}, C_{2}, \ldots, C_{k}, T_{\varphi_{1}}, T_{\varphi_{2}}, \ldots, T_{\varphi_{k}} \tag{31}
\end{equation*}
$$

Upon equating the symbols of those transformations (31) to zero, one will then get the complete system, for which $\varphi_{k+1}$ will be one integral (which is distinct from $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{k}$, and the invariants of $F$ ).
5. The bracket of two functions. - Recall the notations of no. $\mathbf{1}$ and suppose, as in no. $\mathbf{3}$, that if $X_{1}, X_{2}, \ldots, X_{v}$ are $v=2 s$ arbitrary transformations of $F$ then one chooses $X_{v+1}, \ldots, X_{n}$ to be $r=n-$ $n$ distinguished transformations. All of the functions $c_{i, k}$ for which $i$ and $k$ exceed $v$ will then be zero, and the determinant of the remaining $c_{i, k}(i, k=1,2, \ldots, v)$ will be non-zero.

The sheaf $F_{\varphi}$ is composed of the transformations:

$$
U=\sum_{i=1}^{n} u_{i} X_{i}
$$

for which one has $U \varphi=0$, i.e., since $X_{\nu+1} \varphi, \ldots, X_{n} \varphi$ are zero, by hypothesis, one will have:

$$
\begin{equation*}
\sum_{h=1}^{v} u_{h} X_{h} \varphi=0 . \tag{32}
\end{equation*}
$$

A transformation:

$$
V=\sum_{i=1}^{n} v_{i} X_{i}
$$

will be a distinguished transformation of $F_{\varphi}$ only if it is in involution with all of the transformations $U$ that were just defined. That will give the necessary condition for the equation:

$$
\begin{equation*}
\sum_{h=1}^{v} \sum_{k=1}^{v} c_{h, k} u_{h} v_{k}=0 \tag{33}
\end{equation*}
$$

to be a consequence of (32), or that one should have:

$$
\begin{equation*}
\sum_{k=1}^{v} c_{h, k} u_{h}=\rho \cdot X_{h} \varphi \quad(h=1,2, \ldots, v), \tag{34}
\end{equation*}
$$

in which $\rho$ is an auxiliary unknown.
That will determine $v_{1}, \ldots, v_{v}$, up to a factor $\rho$, and leave $v_{v+1}, \ldots, v_{n}$ arbitrary. One then obtains a sheaf whose basis includes not only $X_{v+1}, \ldots, X_{n}$, but also the transformation that one deduces from (34) by supposing that $v_{v+1}=0, \ldots, v_{n}=0$. Since that is defined up to a factor, one can take its expression to the be following bordered determinant, which corresponds to $\rho=1$, and which we call the bracket $\{\varphi, f\}$ :

$$
\{\varphi, f\}=\frac{1}{\delta}\left|\begin{array}{cccc}
c_{1,1} & \cdots & c_{1, v} & -X_{1} \varphi  \tag{35}\\
\vdots & \ddots & \vdots & \vdots \\
c_{v, 1} & \cdots & c_{v, v} & -X_{v} \varphi \\
X_{1} \varphi & \cdots & X_{v} \varphi & 0
\end{array}\right|, \quad \text { with } \quad \delta=\left|\begin{array}{ccc}
c_{1,1} & \cdots & c_{1, v} \\
\vdots & \ddots & \vdots \\
c_{v, 1} & \cdots & c_{v, v}
\end{array}\right| .
$$

That is the desired transformation $T_{\varphi}$, because the sheaf that we just obtained has degree $r+1$, like the subsheaf $G_{\varphi}$, and since the argument that we applied to it implies that it contains $G_{\varphi}$, it will coincide with the latter. Moreover, formula (35) makes it obvious that $\{\varphi, f\}$ leaves $\varphi$ invariant, because $\{\varphi, \varphi\}$ is zero, since it is the skew-symmetric determinant of odd degree ( $v+1=2 s+1$ ).

We also point out the skew-symmetry property of the bracket:

$$
\begin{equation*}
\{\varphi, f\}+\{f, \varphi\}=0 . \tag{36}
\end{equation*}
$$

One establishes that by switching the rows and columns in the determinant (35), and then replacing each $c_{i, k}$ with $c_{k, i}$ at the same time that one changes $\varphi$ into $-\varphi$ and $f$ into $-f$. The latter operations will change the signs of all elements in the determinant, which was, by definition, multiplied by $(-1)^{v+1}$, i.e., by $(-1)$, since $v=2 s$. Since the right-hand side of (35) will then become $\{f, \varphi\}$, the property is established.

The bracket $\{\varphi, f\}$ takes a very simple form when one starts from a canonical basis for the sheaf. Let (16) be that basis, and recall the preceding calculation, with:

$$
U=\sum_{i=1}^{s}\left(a_{i} A_{i}+b_{i} B_{i}\right) \quad \text { and } \quad V=\sum_{i=1}^{s}\left(a_{i}^{\prime} A_{i}+b_{i}^{\prime} B_{i}\right) .
$$

Equations (34), with $\rho=1$, will then be replaced with the identity in $a_{1}, \ldots, a_{s}, b_{1}, \ldots, b_{s}$ :

$$
\sum_{i=1}^{s}\left(a_{i} b_{i}^{\prime}-b_{i} a_{i}^{\prime}\right)=\sum\left(a_{i} A_{i} \varphi+b_{i} B_{i} \varphi\right),
$$

so one concludes that $a_{i}^{\prime}=-B_{i} \varphi, b_{i}^{\prime}=A_{i} \varphi$, and consequently:

$$
\begin{equation*}
\{\varphi, f\}=\sum_{i=1}^{s}\left(A_{i} \varphi \cdot B_{i} f-B_{i} \varphi \cdot A_{i} f\right), \tag{37}
\end{equation*}
$$

because one sees quite easily that $\delta$ is then equal to 1 .
In that simple form, upon taking into account formulas (20), one effortlessly verifies that one has the formula:

$$
\begin{equation*}
(\{\varphi, f\},\{\psi, f\}) \equiv\{\varphi, \psi\} Z \quad(\bmod F) \tag{38}
\end{equation*}
$$

for the Jacobi bracket of two transformations of the type $\{\varphi, f\},\{\psi, f\}$.
We shall see that this extends to the general case of the bracket (35) by virtue of the property of the invariance of the bracket under changes of basis for the sheaf. Here is what that invariance consists of:

Recall the notations at the beginning of this subsection. Equations (34), with $\rho=1$, which gave us $\{\varphi, f\}$, are equivalent to the single condition:

$$
\begin{equation*}
(U, V) \equiv U \varphi \cdot Z \quad(\bmod F) \tag{39}
\end{equation*}
$$

From our calculations, that congruence must be true for any transformation $U$ of the sheaf $X_{1}, \ldots$, $X_{v}$ (or the sheaf $H$ of no. 3). However, it will again be true if we adds any distinguished transformation $C$ to $U$, provided that we impose the condition on $\varphi$ that it must be a characteristic function, as we shall do. The condition:

$$
(U+C, V) \equiv(U \varphi+C \varphi) \cdot Z \quad(\bmod F)
$$

will then indeed reduce to (39), due to the fact that $(C, V) \equiv 0$ and $C \varphi=0$.
Therefore, the congruence (39) is true for any transformation $U$ of $F$, and from what we have seen, since it is equivalent to the equations of condition (34) (with $\rho=1$ ), it will define $V$ entirely when one wishes that $V$ should belong to $H$. If one suppresses that restriction then one must append an arbitrary distinguished transformation to $V$.

One then sees that the bracket $\{\varphi, f\}$ is thus defined by a property that is independent of the choice of basis. However, that must imply that one can then add an arbitrary distinguished transformation of the sheaf $F$ to it. In other words, if one changes the basis for $F$ then one will modify $\{\varphi, f\}$ by at most the addition of a distinguished transformation of $F$. That is the stated invariance. It is obvious that it implies that the formula (38) will be preserved at the moment when one supposes that $\varphi$ and $\psi$ are characteristic functions.

The defining identity (39) further shows that if one changes the transformation $Z$ into another one $\rho Z+X$ of the derived sheaf $F^{\prime}$ ( $X$ being an arbitrary transformation of $F$ ) then $V$ will be found to be multiplied by $\rho$, in such a way that the identity (38) will not be modified.

Finally, as the formula (35) shows, $V$ is a covariant of the sheaf $F$ relative to any change of variables.
6. Bracket relations between the elements of an arbitrary complete integral. - By definition, the method that was given in no. $\mathbf{4}$ for constructing an arbitrary complete integral takes the following precise form:

If the number of variables is $m=n+p(p \geq 1)$ and the rank of the determinant of the $c_{i, k}$ is $2 s$ then one sets $n=2 s+r$, and $r$ is the degree of a characteristic subsheaf $G$ of $F$. The elements $\varphi_{i}$ of a complete integral are $s+p$ in number. They include the possible invariants of $F$; let the number of them be $q$ : It is less than $p$. Starting from those invariants, one calculates by integrating the last of the successive derived sheaves of $F$ : The last derived sheaf is complete.

On the one hand, one determines a basis $C_{1}, \ldots, C_{r}$ for the characteristic subsheaf $G$, and one forms the expression for the bracket $\{\varphi, f\}$. Having done that, the operations to be performed are the following ones: One calculates an invariant $\varphi_{1}$ of the complete sheaf $G$, then an invariant $\varphi_{2}$ of the complete sheaf that is composed of $G$ and $\left\{\varphi_{1}, f\right\}$, then an invariant $\varphi_{3}$ of the complete sheaf that if composed of $G,\left\{\varphi_{1}, f\right\},\left\{\varphi_{2}, f\right\}$, and so on. At each operation, it is implied that the new element $\varphi$ that one calculates must not be a function of the elements that were calculated before.

Once one has thus arrived at the complete sheaf that is defined by $G,\left\{\varphi_{1}, f\right\},\left\{\varphi_{2}, f\right\}, \ldots,\left\{\varphi_{s}\right.$, $f\}$, one will know the $q$ invariants of $F$ and the functions $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{s}$ as its invariants. One thus calculates $p-q$ mutually-distinct invariants, and one will then have all of the elements of the complete integral.

We remark that the principle of the method basically consists of the fact that the transformation $\{\varphi, f\}$ belongs to any complete subsheaf that provides a complete integral that $\varphi$ belongs to. Since such a complete subsheaf has degree $s+r$, it will then result that once one has calculated $\varphi_{1}, \varphi_{2}$, $\ldots, \varphi_{s}$, the operations will be obstructed by the fact that $\left\{\varphi_{s+1}, f\right\},\left\{\varphi_{s+2}, f\right\}, \ldots$ can be only combinations of $\left\{\varphi_{1}, f\right\}, \ldots,\left\{\varphi_{s}, f\right\}$, and some distinguished transformations $C_{1}, \ldots, C_{r}$.

We finally conclude that from the foregoing, a complete integral is a set of $s+p$ independent characteristic functions $\varphi_{i}$ that satisfy the bracket relations:

$$
\begin{equation*}
\left\{\varphi_{i}, \varphi_{k}\right\}=0 \quad(i, k=1,2, \ldots, s+p) . \tag{40}
\end{equation*}
$$

I recall that we intend that a "characteristic function" should mean an invariant of the characteristic subsheaf $G$, and add that, from the foregoing, those of the relations (40) for which $i$ has the values $1,2, \ldots, s$, and $k$ has all indicated values will imply the other ones as consequences.

## II. - Passing from one complete integral to another.

7. Reducing a sheaf to a canonical form. - Knowing a complete integral will permit one to reduce the given sheaf $F$ to a canonical form by a change of variables. One can then look for other complete integrals in that canonical form. That is the topic of the present subsection.

Let $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{s+p}$ be the elements of any particular complete integral. Take them to be the new variables while keeping some of the other variables - for example, $x_{1}, x_{2}, \ldots, x_{s+p}-$ which will make $2 s+r+p=n+p=m$ variables, in all.

The complete subsheaf whose fundamental invariants are $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{s+p}$ is then $\frac{\partial f}{\partial x_{1}}, \frac{\partial f}{\partial x_{2}}, \ldots$, $\frac{\partial f}{\partial x_{s+r}}$. It contains the distinguished transformations, which consequently have the form:

$$
\begin{equation*}
C_{j}=\sum_{\alpha=1}^{s+r} \chi_{j, \alpha} \frac{\partial f}{\partial x_{\alpha}} \quad(j=1,2, \ldots, r), \tag{41}
\end{equation*}
$$

and if the notations are chosen conveniently then one can, on the other hand, keep the transformations:

$$
\begin{equation*}
\frac{\partial f}{\partial x_{1}}, \frac{\partial f}{\partial x_{2}}, \ldots, \frac{\partial f}{\partial x_{s+r}} \tag{42}
\end{equation*}
$$

to serve as the basis for the subsheaf in question. In order to complete the basis for $F$, one can take the transformations when they have been solved for $s$ of the derivatives $\partial f / \partial \varphi_{i}$, namely:

$$
\begin{equation*}
\Phi_{i}=\frac{\partial f}{\partial \varphi_{i}}+\sum_{\beta=1}^{p} \theta_{i, \beta} \frac{\partial f}{\partial \varphi_{s+\beta}} \quad(i=1,2, \ldots, s) \tag{43}
\end{equation*}
$$

One can take a transformation of the form:

$$
\sum_{\beta=1}^{p} \theta_{i, \beta} \frac{\partial f}{\partial \varphi_{s+\beta}}
$$

to be the transformation $Z$ of the derived sheaf that appears in the structure formulas (no. $\mathbf{1}$ ) and solve it for one of the derivatives that enter into it. It might then take the form:

$$
\begin{equation*}
Z=\frac{\partial f}{\partial \varphi_{s+1}}+\sum_{\gamma=2}^{p} \theta_{\gamma} \frac{\partial f}{\partial \varphi_{s+\gamma}} \tag{44}
\end{equation*}
$$

for example. In all of that, one supposes that $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{s}, \varphi_{s+1}$ are not invariants of $F$, which is legitimate.

Having done that, consider, as in no. 5, the determinant $\delta$ of the structure functions that correspond to the $2 s$ basis transformations (42) and (43). One will have the relations $\left({ }^{1}\right)$ :

$$
\left(\frac{\partial f}{\partial x_{i}}, \Phi_{k}\right)=\frac{\partial \theta_{k, i}}{\partial x_{i}} Z \quad(i, k=1,2, \ldots, s)
$$

in such a way that the determinant $\delta$ will reduce to the square of a functional determinant $\frac{\partial\left(\theta_{1,1}, \theta_{2,1}, \ldots, \theta_{s, 1}\right)}{\partial\left(x_{1}, x_{2}, \ldots, x_{s}\right)}$. It will then result that this functional determinant is not zero, and that one can take functions $\psi_{i}=\theta_{i, 1}(i=1,2, \ldots, s)$ to be the new variables in place of the $x_{1}, x_{2}, \ldots, x_{s}$. The transformations of the basis (42) can then be replaced with:

$$
\begin{equation*}
\Psi_{i}=\frac{\partial f}{\partial \psi_{i}} \quad(i=1,2, \ldots, s) \tag{45}
\end{equation*}
$$

and the transformations (43) can be written:

$$
\begin{equation*}
\Phi_{i}=\frac{\partial f}{\partial \varphi_{i}}+\psi_{i} \frac{\partial f}{\partial \varphi_{s+1}}+R_{i} \quad(i=1,2, \ldots, s) \tag{46}
\end{equation*}
$$

[^7]in which the $R_{i}$ depend upon only the derivatives $\frac{\partial f}{\partial \varphi_{s+2}}, \ldots, \frac{\partial f}{\partial \varphi_{s+p}}$.
We similarly write:
\[

$$
\begin{equation*}
Z=\frac{\partial f}{\partial \varphi_{s+1}}+R \tag{47}
\end{equation*}
$$

\]

Furthermore, we remark that we must have the structure relations $\left(C_{j}, \Phi_{i}\right) \equiv 0$, which implies that $C_{j} \psi_{i}=0$. Hence, from the introduction of the $\psi_{i}$, the derivatives $\partial f / \partial \psi_{i}$ will not figure in the $C_{j}$, and we can replace them with the transformations:

$$
\begin{equation*}
\frac{\partial f}{\partial x_{s+1}}, \frac{\partial f}{\partial x_{s+2}}, \ldots, \frac{\partial f}{\partial x_{s+r}} \tag{48}
\end{equation*}
$$

which will then be the distinguished transformations that serve as the basis for the characteristic subsheaf $G$. We conclude that the variables $x_{s+1}, x_{s+2}, \ldots, x_{s+r}$ have disappeared from the coefficients of the transformations $R_{i}$ because we must have $\left(\frac{\partial f}{\partial x_{s+j}}, \Phi_{i}\right)=0$. Those variables will likewise disappear from $Z$, since we then have $\left(\frac{\partial f}{\partial \psi_{i}}, \Phi_{i}\right)=Z$. Hence, the variables $x_{s+1}, \ldots, x_{s+r}$ will now disappear from the calculations.

Now consider the structure relations that relate to the brackets $\left(\Psi_{i}, \Phi_{k}\right)$. They lead to the identities:

$$
\frac{\partial R_{i}}{\partial \psi_{i}}=R, \quad \frac{\partial R_{i}}{\partial \psi_{k}}=0 \quad(i \neq k, i, k=1,2, \ldots, s) .
$$

Until further notice, we shall exclude the case of $s=1$. We conclude that the variables $\psi$ do not enter into the coefficients of $R$ and that one has:

$$
\begin{equation*}
R_{i}=R \psi_{i}+S_{i} \tag{49}
\end{equation*}
$$

in which the $S_{i}$ no longer depend upon the variables $\varphi$. We can then write:

$$
\begin{equation*}
\Phi_{i}=\frac{\partial f}{\partial \varphi_{i}}+\psi_{i} Z+S_{i} \quad(i=1,2, \ldots, s) \tag{50}
\end{equation*}
$$

Having said that, we pass on to the brackets $\left(\Phi_{i}, \Phi_{k}\right)$. Since, from the form of the $\Phi_{i}$, only the derivatives $\frac{\partial f}{\partial \varphi_{s+2}}, \ldots, \frac{\partial f}{\partial \varphi_{s+p}}$ can enter into them, they will be identically zero. Now, one has:

$$
\left(\Phi_{i}, \Phi_{k}\right)=\left(\frac{\partial f}{\partial \varphi_{i}}+S_{i}, \frac{\partial f}{\partial \varphi_{k}}+S_{k}\right)+\psi_{k}\left(\frac{\partial f}{\partial \varphi_{i}}+S_{i}, Z\right)-\psi_{i}\left(\frac{\partial f}{\partial \varphi_{k}}+S_{k}, Z\right)
$$

and since $\psi_{i}$ and $\psi_{k}$ do not enter into any coefficient of $Z$ or $S_{i}$, one can conclude that:

$$
\left(\frac{\partial f}{\partial \varphi_{i}}+S_{i}, \frac{\partial f}{\partial \varphi_{k}}+S_{k}\right)=0, \quad\left(\frac{\partial f}{\partial \varphi_{i}}+S_{i}, Z\right)=0
$$

for all values of $i$ and $k$.
It then results that the sheaf $F^{\prime}$ is complete, i.e., that the number of independent invariants of $F$, which we have denoted by $q$, is equal to $m-(n+1)=p-1$.

Now, from the calculations that were indicated in no. 6, we can suppose that those invariants are found among the $\varphi$, and in other words, that $\varphi_{s+2}, \ldots, \varphi_{s+p}$ are all invariants of $F$ and $F^{\prime}$. In other words, $R$ and the $R_{i}$ are identically zero.

Therefore, the changes of variables that we made in this subsection will lead directly to the following canonical form:

$$
\begin{equation*}
\Psi_{i}=\frac{\partial f}{\partial \psi_{i}}, \quad \Phi_{i}=\frac{\partial f}{\partial \varphi_{i}}+\psi_{i} \frac{\partial f}{\partial \varphi_{0}}, \quad C_{i}=\frac{\partial f}{\partial x_{s+j}} \quad(i=1,2, \ldots, s ; j=1,2, \ldots, r), \tag{51}
\end{equation*}
$$

and the chosen transformation $Z$ reduces to:

$$
\begin{equation*}
Z=\frac{\partial f}{\partial \varphi_{0}} . \tag{52}
\end{equation*}
$$

We have just denoted $\varphi_{s+1}$ by only $\varphi_{0}$ to simplify the writing.
8. Polar functions of a complete integral. - If we summarize the results that we just obtained then we will first see that for $s>1$, everything basically takes place as if the number of variables were $v+1$, the sheaf $F$ had even degree $v=2 s$, and it had no distinguished transformations, since we were no longer concerned with the variables $\varphi$ and $\psi$.

Moreover, the variables $\varphi$ are arbitrary elements, which are $s+1$ in number, of an arbitrary complete integral under the single condition that none of them should be an invariant of $F$ and that no function of those invariants should be such an invariant.

We say that the functions $\psi_{i}$ that are associated with the $\varphi_{i}$ in the canonical form (51) are the polar functions of the $\varphi_{i}$. They are defined by the analysis of the preceding subsection, which gave the means to calculate them when one knew the $\varphi_{i}$ as soon as one had chosen the complete integral element that one took to be $\varphi_{0}$. We shall show that, on the other hand, they satisfy some bracket relations that characterize them.

The basis (51) for the sheaf $F$ is a canonical basis (no. 1), because one has the identities:

$$
\begin{equation*}
\left(\Psi_{i}, \Phi_{i}\right)=Z \quad(i=1,2, \ldots, s) \tag{53}
\end{equation*}
$$

while the other brackets of the $\Psi_{i}, \Phi_{i}$, and $C_{i}$ are zero. Furthermore, one sees that the congruences of no. $\mathbf{1}$ are equalities here.

Upon taking that basis (51) to be the starting point, the general bracket $\{f, g\}$ will become the Poisson bracket:

$$
\begin{equation*}
[f, g]=\sum_{i=1}^{s}\left(\frac{\partial f}{\partial \psi_{i}} \Phi_{i} g-\frac{\partial g}{\partial \psi_{i}} \Phi_{i} f\right) \tag{54}
\end{equation*}
$$

because the $\Phi_{i}$ can be interpreted as total derivatives with respect to the $\varphi_{i}(i=1,2, \ldots, s)$, while the $\psi_{i}$ are considered to be the partial derivatives $\partial \varphi_{0} / \partial \varphi_{i}$. Regardless of that interpretation, with the notation (54), one will have the immediate formulas:

$$
\left\{\begin{array}{lll}
{\left[\varphi_{0}, \varphi_{i}\right]=0,} & {\left[\varphi_{i}, \varphi_{k}\right]=0,} & {\left[\psi_{i}, \psi_{k}\right]=0,}  \tag{55}\\
{\left[\varphi_{i}, \psi_{k}\right]=0,} & {\left[\varphi_{i}, \psi_{i}\right]=-1,} & {\left[\varphi_{0}, \psi_{i}\right]=-\psi_{i},}
\end{array} \quad(i, k=1,2, \ldots, s ; i \neq k) .\right.
$$

One can also write, for $i=1,2, \ldots, s$ :

$$
\begin{equation*}
\left[\psi_{i}, f\right]=\Phi_{i}, \quad\left[\varphi_{i}, f\right]=-\Psi_{i}, \quad\left[\varphi_{0}, f\right]=-\sum_{k=1}^{s} \psi_{k} \Psi_{k}, \tag{56}
\end{equation*}
$$

so

$$
\begin{equation*}
\left[\varphi_{0}, f\right]=\sum_{i=1}^{s} \psi_{i}\left[\varphi_{i}, f\right] . \tag{57}
\end{equation*}
$$

If we return to the general bracket $\{f, g\}$ that is constructed from an arbitrary basis of $F$ and an arbitrary transformation $Z$ of the derived sheaf $Z^{\prime}$ then from the property of invariance of the bracket that was established in no. $\mathbf{5}$, we will have the identity:

$$
\begin{equation*}
\{f, g\} \equiv \rho[f, g] \quad(\bmod G) \tag{58}
\end{equation*}
$$

in which the factor $\rho$ is defined by:

$$
\begin{equation*}
Z \equiv \rho \frac{\partial f}{\partial \varphi_{0}} \quad(\bmod F) \tag{59}
\end{equation*}
$$

Formulas (55), (56), (57) then give the following ones, since the $\varphi$ and the $\psi$ are invariants of the characteristic subsheaf $G$ :

$$
\left\{\begin{array}{ll}
\left\{\varphi_{0}, \varphi_{i}\right\}=0, & \left\{\varphi_{i}, \varphi_{k}\right\}=0,  \tag{60}\\
\left\{\varphi_{i}, \psi_{k}\right\}=0, & \left\{\psi_{i}, \psi_{k}\right\}=0, \\
\left.\varphi_{i}\right\}=-\rho, & \left\{\varphi_{0}, \psi_{i}\right\}=-\rho \psi_{i},
\end{array} \quad(i, k=1,2, \ldots, s ; i \neq k) .\right.
$$

$$
\begin{align*}
\left\{\psi_{i}, f\right\} \equiv \rho \Phi_{i}, \quad\left\{\varphi_{i}, f\right\} \equiv-\rho \Psi_{i}, \quad\left\{\varphi_{0}, f\right\} \equiv-\rho \sum_{k=1}^{s} \psi_{k} \Psi_{k} \quad(\bmod G)  \tag{61}\\
\left\{\varphi_{0}, f\right\} \equiv \sum_{k=1}^{s} \psi_{k}\left\{\varphi_{k}, f\right\} \quad(\bmod G) \tag{62}
\end{align*}
$$

The formulas (60), when combined with the hypothesis that the $\varphi$ and the $\psi$ are invariants of $G$, characterize the system of $\varphi$ and $\psi$ as being composed of the elements $\varphi_{0}, \varphi_{1}, \ldots, \varphi_{\mathrm{s}}$ of a complete integral and their polar $\psi_{1}, \ldots, \psi_{s}$, in which it is intended that one must add some independent invariants (of maximum number) from $F$ to the $\varphi$.

Indeed, if one takes the variables to be the $\varphi$, the $\psi$, the invariants in questions $\chi_{1}, \chi_{2}, \ldots, \chi_{p-1}$, and $r$ other arbitrary variables $z_{1}, \ldots, z_{r}$ then the transformations of $G$ will have the form:

$$
C=\sum_{j=1}^{r} \zeta_{j} \frac{\partial f}{\partial x_{j}}
$$

since the $C \varphi$, the $C \psi$, and the $C \chi$ are zero. On the other hand, any transformation of the type $\{g$, $f\}$ has the expression:

$$
\{g, f\}=\sum_{\alpha=0}^{s}\left\{g, \varphi_{\alpha}\right\} \frac{\partial f}{\partial \varphi_{\alpha}}+\sum_{\beta=1}^{s}\left\{g, \psi_{\beta}\right\} \frac{\partial f}{\partial \psi_{\beta}}+\sum_{\gamma=1}^{r}\left\{g, z_{\gamma}\right\} \frac{\partial f}{\partial z_{\gamma}},
$$

because the $\left\{g, \chi_{h}\right\}$ are zero, due to the fact that $\{g, f\}$ is a transformation of $F$. One can further write that as:

$$
\begin{equation*}
\{g, f\} \equiv \sum_{\alpha=0}^{s}\left\{g, \varphi_{\alpha}\right\} \frac{\partial f}{\partial \varphi_{\alpha}}+\sum_{\beta=1}^{s}\left\{g, \psi_{\beta}\right\} \frac{\partial f}{\partial \psi_{\beta}} \quad(\bmod G) . \tag{63}
\end{equation*}
$$

One can profit from the relations (60) to conclude that:

$$
\begin{equation*}
\left\{\psi_{i}, f\right\} \equiv \rho \Phi_{i}, \quad\left\{\varphi_{i}, f\right\} \equiv-\rho \Psi_{i} \quad(\bmod G), \quad\left\{\varphi_{0}, f\right\} \equiv-\rho \sum_{k=1}^{s} \psi_{k} \Psi_{k} \quad(\bmod G) \tag{64}
\end{equation*}
$$

Those equations indeed prove that the $\psi_{i}$ are associated with the $\varphi_{i}$ in such a way as to give the sheaf $F$ its canonical form (51). Q.E.D.

Moreover, equations (64) will imply the formula (62), whose importance is just as fundamental, because it suffices to define the $\psi_{i}$. That results from the fact that the complete subsheaf that has the $\varphi_{i}$ for its invariants has a basis (no. 6) that consists of the transformations $\left\{\varphi_{1}, f\right\},\left\{\varphi_{2}, f\right\}, \ldots,\left\{\varphi_{s}, f\right\}, C_{1}, C_{2}, \ldots, C_{r}$. They are then divergent, which excludes the possibility of there being two distinct identity-congruences of the form (58).
9. Explicit integration of the canonical sheaf. - It is clear from the foregoing that once one knows a particular complete integral, the integration of $F$ will come down to the integration of the canonical sheaf, which is composed of the $\Psi_{i}$ and the $\Phi_{i}$. With a change of notation, that will be the sheaf $K$ that has basis:

$$
\begin{equation*}
P_{i}=\frac{\partial f}{\partial p_{i}}, \quad X_{i}=\frac{\partial f}{\partial x_{i}}+p_{i} \frac{\partial f}{\partial x_{0}} \quad(i=1,2, \ldots, s) \tag{65}
\end{equation*}
$$

That sheaf $K$ is the dual $\left({ }^{1}\right)$ of the Pfaff equation:

$$
\begin{equation*}
d x_{0}-p_{1} d x_{1}-p_{2} d x_{2}-\ldots-p_{s} d x_{s}=0 . \tag{65}
\end{equation*}
$$

We can then use some well-known results here. However, it would seem interesting to us to deduce everything from our theory of sheaves.

One will first get an intuitive solution upon remarking that the sheaf (65) results from the prolongation $\left({ }^{2}\right)$ of the sheaf:

$$
\frac{\partial f}{\partial x_{0}}, \quad \frac{\partial f}{\partial x_{1}}, \quad \ldots, \quad \frac{\partial f}{\partial x_{s}}
$$

when one considers $x_{0}$ to be a function of $x_{1}, \ldots, x_{s}$, and the $p$ to be the partial derivatives $\partial x_{0} / \partial x_{i}$. Therefore, in order to get an $s$-dimensional integral of (65) that depends upon $s+1$ arbitrary constants, it would suffice to prolong an $s$-dimensional multiplicity in the space $x_{0}, x_{1}, \ldots, x_{s}$, when that multiplicity depends upon $s+1$ arbitrary constants $a_{0}, a_{1}, \ldots, a_{s}$. Hence, one will get the solution:

$$
\begin{equation*}
x_{0}=W\left(x_{1}, \ldots, x_{s} ; a_{0}, a_{1}, \ldots, a_{s}\right), \quad \rho_{i}=\frac{\partial W}{\partial x_{i}} \quad(i=1,2, \ldots, s), \tag{66}
\end{equation*}
$$

in which $W$ is an arbitrary function of its arguments.
However, there is good reason to see whether there is no other solution and to argue directly with the sheaf $K$ without invoking the notion of prolongation.

Therefore, imagine a complete subsheaf $K_{1}$ of $K$ of degree $s$. It is defined by $s$ independent combinations of the $X_{i}$ and the $P_{i}$.

1. First suppose that one can solve it for the $X_{i}$ in such a way that the basis for the subsheaf $K_{1}$ has the form:

$$
\begin{equation*}
X_{i}=\sum_{j=1}^{s} \lambda_{i, j} P_{j} \quad(i=1,2, \ldots, s) \tag{67}
\end{equation*}
$$

From the known properties of complete systems, one can then solve the complete integral for the $x_{0}, p_{1}, \ldots, p_{s}$, and it will have the form:

[^8]\[

$$
\begin{equation*}
x_{0}=W\left(x_{1}, \ldots, x_{s} ; a_{0}, a_{1}, \ldots, a_{s}\right), \quad p_{j}=\Theta_{j}\left(x_{1}, \ldots, x_{s} ; a_{0}, a_{1}, \ldots, a_{s}\right), \quad(j=1,2, \ldots, s) \tag{68}
\end{equation*}
$$

\]

We write down that it admits the transformations (67). It will become:

$$
p_{i}=\frac{\partial W}{\partial x_{i}}, \quad \lambda_{i, j}=\frac{\partial \Theta_{j}}{\partial x_{i}} \quad(i, j=1,2, \ldots, s),
$$

which must be a consequence of equations (68). One will then find that the necessary and sufficient conditions for the possibility of that are the formulas:

$$
\Theta_{j}=\frac{\partial W}{\partial x_{j}} \quad(j=1,2, \ldots, s)
$$

with the condition that the equations (68) thus-constructed can be solved for the constants $a_{0}, a_{1}$, $\ldots, a_{s}$ that will then be the solution (66) that was introduced before.
2. In the second place, suppose that the basis for $K_{1}$ can be solved for $\sigma$ of the $X_{i}$ and no more $\left.{ }^{1}\right)(0<\sigma<s)$, and suppose, to simplify notations, that they are $X_{1}, \ldots, X_{\sigma}$. One will then have $s$ $\sigma$ independent combinations of $P_{1}, \ldots, P_{s}$ in $K_{1}$. I say that they must be soluble for the $P_{\sigma+1}, \ldots$, $P_{s}$.

Indeed, suppose that the contrary is true: There will be at least one transformation of the form $\mu_{1} P_{1}+\ldots+\mu_{\sigma} P_{\sigma}$ in $K_{1}$. However, that is impossible, because the brackets with the transformations that are solved for $X_{1}, \ldots, X_{\sigma}$ will be $\mu_{1} Z, \ldots, \mu_{\sigma} Z$, which are transformations that will not all be identically zero and will not belong to $K_{1}$.

The basis for $K_{1}$ can then be taken in the following form here:

$$
\begin{cases}X_{h}+\sum_{\alpha=1}^{s-\sigma} \lambda_{h, \alpha} X_{\sigma+\alpha}+\sum_{\beta=1}^{\sigma} \mu_{h, \beta} P_{\beta} & (h=1,2, \ldots, \sigma),  \tag{69}\\ P_{\sigma+l}+\sum_{\beta=1}^{\sigma} v_{l, \beta} P_{\beta} & (l=1,2, \ldots, s-\sigma) .\end{cases}
$$

One then concludes that the complete integral can be solved for $x_{0}, x_{\sigma+1}, \ldots, x_{s}, p_{1}, \ldots, p_{s}$. One then writes:

$$
\begin{equation*}
x_{0}=W_{0}, \quad x_{\sigma+l}=W_{l}, \quad p_{h}=\Theta_{h} \quad(l=1,2, \ldots, s-\sigma ; h=1,2, \ldots, s) \tag{70}
\end{equation*}
$$

The right-hand sides of those equations are functions of $x_{1}, \ldots, x_{\sigma}, p_{\sigma+1}, \ldots, p_{s}$, and some arbitrary constants $a_{0}, a_{1}, \ldots, a_{s}$.

We now express the idea that they admit the transformations (69). That will give the conditions:
( ${ }^{1}$ ) The hypothesis that $\sigma=0$ gives the complete integral $x_{0}=a_{0}, x_{1}=a_{1}, \ldots, x_{s}=a_{s}$.

$$
p_{h}+\sum_{\alpha=1}^{s-\sigma} \lambda_{h, \alpha} p_{\sigma+\alpha}=\frac{\partial W}{\partial x_{h}}, \quad \lambda_{h, l}=\frac{\partial W_{l}}{\partial x_{h}}, \quad \frac{\partial W_{0}}{\partial p_{\sigma+l}}=0, \quad \frac{\partial W_{h}}{\partial p_{\sigma+l}}=0
$$

(for $h=1,2, \ldots, \sigma$ and $l=1,2, \ldots, s-\sigma$ ),
and in addition:

$$
\mu_{h, \beta}=\frac{\partial \Theta_{\beta}}{\partial x_{h}}, \quad v_{h, \beta}=\frac{\partial \Theta_{\beta}}{\partial p_{\sigma+l}} \quad(h, \beta=1,2, \ldots, \sigma ; l=1,2, \ldots, s-\sigma) .
$$

One concludes that one has the general solution:

$$
\begin{equation*}
x_{0}=W_{0}, \quad x_{\sigma+l}=W_{l}, \quad p_{h}=\frac{\partial W}{\partial x_{h}}-\sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha} \frac{\partial W_{\alpha}}{\partial x_{h}} \quad(h=1,2, \ldots, s ; l=1,2, \ldots, s-\sigma), \tag{71}
\end{equation*}
$$

in which $W_{0}$ and $W_{l}$ are functions of $x_{1}, \ldots, x_{n}, a_{0}, a_{1}, \ldots, a_{s}$, which are arbitrary, with the only reservation that those equations (71) can be solved for $a_{0}, a_{1}, \ldots, a_{s}$.
3. If one sets, with Sophus Lie:

$$
\begin{equation*}
W=W_{0}-\sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha} W_{\alpha} \tag{72}
\end{equation*}
$$

then one can replace formulas (71) with:

$$
\begin{gather*}
x_{0}=W-\sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha} \frac{\partial W}{\partial p_{\sigma+\alpha}}, \quad x_{\sigma+l}=-\frac{\partial W}{\partial p_{\sigma+\alpha}}, \quad p_{h}=\frac{\partial W}{\partial x_{h}},  \tag{73}\\
W=\text { functions of } p_{\sigma+l}, \ldots, p_{s}, x_{1}, \ldots, x_{\sigma}, a_{0}, a_{1}, \ldots, a_{s}, \tag{74}
\end{gather*}
$$

in which one has $h=1,2, \ldots, \sigma ; l=1,2, \ldots, s-\sigma$, as always, and one confirms that one then has a complete integral no matter what one chooses $W$ to be.

Indeed, express the idea that the system admits the transformation:

$$
\begin{equation*}
U=\sum_{i=1}^{s} \lambda_{i} X_{i}+\sum_{i=1}^{s} \mu_{i} P_{i} \tag{75}
\end{equation*}
$$

and we find the conditions:

$$
\begin{align*}
& \sum_{i=1}^{s} \lambda_{i} p_{i}=\sum_{\beta=1}^{\sigma} \lambda_{\beta}\left(\frac{\partial W}{\partial x_{\beta}}-\sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha} \frac{\partial^{2} W}{\partial p_{\sigma+\alpha} \partial x_{\beta}}\right)-\sum_{\gamma=1}^{s-\sigma} \mu_{s+\gamma} \sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha} \frac{\partial^{2} W}{\partial p_{\sigma+\alpha} \partial p_{\sigma+\gamma}},  \tag{76}\\
& \lambda_{\sigma+l}=-\sum_{\beta=1}^{\sigma} \lambda_{\beta} \frac{\partial^{2} W}{\partial p_{\sigma+\alpha} \partial x_{\beta}}-\sum_{\alpha=1}^{s-\sigma} \mu_{\sigma+\alpha} \frac{\partial^{2} W}{\partial p_{\sigma+l} \partial p_{\sigma+\alpha}} \quad(l=1,2, \ldots, s-\sigma), \tag{77}
\end{align*}
$$

$$
\begin{equation*}
\mu_{h}=\sum_{\beta=1}^{\sigma} \lambda_{\beta} \frac{\partial^{2} W}{\partial p_{h} \partial x_{\beta}}-\sum_{\alpha=1}^{s-\sigma} \mu_{\sigma+\alpha} \frac{\partial^{2} W}{\partial x_{h} \partial p_{\sigma+\alpha}} \quad(h=1,2, \ldots, \sigma) . \tag{78}
\end{equation*}
$$

Now, equation (76) will become an identity when one takes into account equations (77) and (78), along with equations (73). All that will remain then are equations (77) and (78), which leave $\lambda_{1}, \ldots, \lambda_{\sigma}$ and $\mu_{\sigma+1}, \ldots, \mu_{s}$ arbitrary. One will then indeed have a subsheaf of degree $s$ that leaves the multiplicity (73) invariant.
10. Continuation. Calculating the polar functions. - We have thus defined the most complete general integral by means of formulas (75), which we can think of as including the normal formulas (66) for $\sigma=s$. Since those equations define the functions $a_{0}, a_{1}, \ldots, a_{s}$ of the variables $x_{0}, x_{1}, \ldots$, $x_{s} ; p_{1}, \ldots, p_{s}$, it remains for us to find the polar functions $b_{1}, \ldots, b_{s}$ that they are associated with. In order to do that, we need only apply the rule that was given at the end of no. 8, and to that effect, we take the brackets with $f$ of the two sides of the identity:

$$
\begin{equation*}
x_{0}=W\left(x_{1}, \ldots, x_{s} ; p_{1}, \ldots, p_{s}\right) \tag{79}
\end{equation*}
$$

for the normal case, and for the general case, the brackets of the two sides of the identity:

$$
\begin{equation*}
x_{0}=W\left(x_{1}, \ldots, x_{\sigma} ; p_{\sigma+1}, \ldots, p_{s} ; a_{0}, a_{1}, \ldots, a_{s}\right)+\sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha} x_{\sigma+\alpha} \tag{80}
\end{equation*}
$$

which is an immediate consequence of formulas (73).
Since the operation $\{\varphi, f\}$ is linear and homogeneous with respect to the derivatives of $\varphi$, one can apply the rule for the derivation of composed functions to it. We remark that the bracket is the Poisson bracket here, and that the formulas (56) will give:

$$
\begin{equation*}
\left\{x_{i}, f\right\}=-P_{i}, \quad\left[p_{i}, f\right]=X_{i}, \quad\left[x_{0}, f\right]=-\sum_{i=1}^{s} p_{i} x_{i} \tag{81}
\end{equation*}
$$

with the present notations.
With the identity (79), we will then get:

$$
-\sum_{i=1}^{s} p_{i} P_{i}=\sum_{i=1}^{s} \frac{\partial W}{\partial x_{i}}\left(-P_{i}\right)+\sum_{j=0}^{s} \frac{\partial W}{\partial a_{j}}\left[a_{i}, f\right],
$$

and when one takes formulas (66) into account, that will reduce to:

$$
\begin{equation*}
\sum_{j=0}^{s} \frac{\partial W}{\partial a_{j}}\left[a_{i}, f\right]=0 \tag{82}
\end{equation*}
$$

With the identity (80), we will get:

$$
-\sum_{i=1}^{s} p_{i} P_{i}=\sum_{h=1}^{\sigma} \frac{\partial W}{\partial x_{h}}\left(-P_{h}\right)+\sum_{l=1}^{s-\sigma} \frac{\partial W}{\partial p_{\sigma+l}} X_{\sigma+l}+\sum_{j=0}^{s} \frac{\partial W}{\partial a_{j}}\left[a_{i}, f\right]+\sum_{\alpha=1}^{s-\sigma} p_{\sigma+\alpha}\left(-P_{\sigma+\alpha}\right)+\sum_{\alpha=1}^{s-\sigma} x_{\sigma+\alpha} X_{\sigma+\alpha} .
$$

On the other hand, since the $b_{i}$ are defined by the identity:

$$
\begin{equation*}
\left[a_{i}, f\right]=\sum_{i=1}^{s} b_{i}\left[a_{i}, f\right], \tag{83}
\end{equation*}
$$

from the final result in no. $\mathbf{8}$, so it will suffice to compare that with (82) to conclude that the $b_{i}$ are defined here by the system of equations:

$$
\begin{equation*}
\frac{\partial W}{\partial a_{i}}+b_{i} \frac{\partial W}{\partial a_{0}}=0 \quad(i=1,2, \ldots, s) \tag{84}
\end{equation*}
$$

which must, of course, be associated with the ones that define $a_{0}, a_{1}, \ldots, a_{s}$, i.e., formulas (66) or (73), according to the case.
11. Contact transformations. - Let us combine the formulas that were obtained. Upon putting $y_{i}$ in place of $a_{i}, q_{i}$ in place of $b_{i}$, and changing $\sigma$ into $s-\sigma$, we will get two systems: For the normal case:

$$
\left\{\begin{array}{c}
x_{0}=W\left(x_{1}, \ldots, x_{s} ; y_{0}, y_{1}, \ldots, y_{s}\right),  \tag{83}\\
p_{i}=\frac{\partial W}{\partial x_{i}}, \quad \frac{\partial W}{\partial y_{1}}+q_{i} \frac{\partial W}{\partial y_{0}}=0 \quad(i=1,2, \ldots, s),
\end{array}\right.
$$

and for the most general case:

$$
\begin{gather*}
x_{0}-\sum_{\alpha=1}^{\sigma} p_{\alpha} x_{\alpha}=W\left(x_{\sigma+1}, \ldots, x_{1} ; p_{1}, \ldots, p_{\sigma} ; y_{0}, y_{1}, \ldots, y_{s}\right), \\
x_{h}=-\frac{\partial W}{\partial p_{h}}, \quad p_{\sigma+l}=\frac{\partial W}{\partial x_{\sigma+l}}, \quad \frac{\partial W}{\partial y_{1}}+q_{i} \frac{\partial W}{\partial y_{0}}=0  \tag{86}\\
(h=1,2, \ldots, \sigma ; l=1,2, \ldots, s-\sigma ; i=1,2, \ldots, s) .
\end{gather*}
$$

One and the other of those systems define a transformation of $x_{0}, x_{1}, \ldots, x_{s}, p_{1}, \ldots, p_{s}$ into $y_{0}$, $y_{1}, \ldots, y_{s}, q_{1}, \ldots, q_{s}$, and it will result from the manner by which we arrived at the sheaf $K$ that this transformation will change the sheaf into:

$$
\begin{equation*}
Q_{i}=\frac{\partial f}{\partial q_{i}}, \quad Y_{i}=\frac{\partial f}{\partial y_{i}}+q_{i} \frac{\partial f}{\partial y_{s}} \quad(i=1,2, \ldots, s), \tag{87}
\end{equation*}
$$

because $y_{0}, y_{1}, \ldots, y_{s}$ is a complete integral and $q_{1}, \ldots, q_{s}$ are polar functions. In other words, the transformations in question leave the sheaf $K$ invariant.

Conversely, if one transformation of the $x_{0}, x_{1}, \ldots, x_{s}, p_{1}, \ldots, p_{s}$ into $y_{0}, y_{1}, \ldots, y_{s}, q_{1}, \ldots, q_{s}$ leaves $K$ invariant, i.e., it permits one to put its basis into the form (87), then it is clear that $y_{0}, y_{1}$, $\ldots, y_{s}$ will be independent invariants of the complete subsheaf $Q_{1}, \ldots, Q_{s}$, in such a way that they are the elements of a complete integral, and the form of the $Y_{i}$ indicates that $q_{1}, \ldots, q_{s}$ are the polar elements that are associated with that complete integral.

We have thus obtained the general form of the transformations that leave the sheaf $K$ invariant, and what amounts to the same thing, the dual Pfaff equation:

$$
d x_{0}-\sum_{i=1}^{s} p_{i} d x_{i}=0
$$

i.e., the contact transformations of the space $x_{0}, x_{1}, \ldots, x_{s}$.

From the analysis of no. 9 , the function $W$ of formulas (86) can be supposed to be linear in $p_{1}$, $\ldots, p_{s}$ :
(86, cont.)

$$
W=W_{0}-\sum_{h=1}^{\sigma} p_{h} W_{h},
$$

in such a way that the formulas (86) include $\sigma+1$ point-like relations (into $x_{0}, x_{1}, \ldots, x_{s}$ and $y_{0}, y_{1}$, $\ldots, y_{s}$ ), namely:

$$
x_{0}=W_{0}, \quad x_{h}=W_{h} \quad(h=1,2, \ldots, \sigma),
$$

whereas formulas (85) include only one.
Of course, all of that conforms to the classical theory of Sophus Lie. However, we did not find formulas (86) in his book.

From what we found in no. $\mathbf{8}$ in regard to the bracket relations that characterize the complete integrals [equations (60)], one will immediately conclude, on the one hand, the following theorem, which is fundamental to the theory of contact transformations. In order for the equations:

$$
\left\{\begin{array}{c}
y_{h}=y_{h}\left(x_{0}, x_{1}, \ldots, x_{s} ; p_{1}, \ldots, p_{s}\right), \quad q_{i}=q_{i}\left(x_{0}, x_{1}, \ldots, x_{s} ; y_{1}, \ldots, y_{s}\right)  \tag{88}\\
(h=0,1,2, \ldots, s ; i=1,2, \ldots, s)
\end{array}\right.
$$

to define a contact transformation, it is necessary and sufficient that the functions $y_{h}$ and $q_{i}$ should satisfy the bracket relations:

$$
\left\{\begin{array}{c}
{\left[y_{i}, y_{k}\right]=0, \quad\left[y_{0}, y_{i}\right]=0, \quad\left[q_{i}, q_{k}\right]=0, \quad\left[y_{i}, q_{k}\right]=0, \quad\left[q_{i}, y_{i}\right]=\rho, \quad\left[q_{i}, y_{0}\right]=\rho q_{i},}  \tag{89}\\
(i, k,=1,2, \ldots, s ; i \neq k) .
\end{array}\right.
$$

It is convenient to recall that those relations, in which $\rho$ cannot be identically zero, assure the independence of the functions $y_{h}$ and $q_{i}$. Indeed, if one has an identity relation:

$$
\begin{equation*}
F\left(y_{0}, y_{1}, \ldots, y_{s} ; q_{1}, \ldots, q_{s}\right)=0 \tag{90}
\end{equation*}
$$

then it cannot contain the $q_{i}$ because when it is solved for $q_{1}$, for example, it will take the form:

$$
q_{1}=G\left(y_{0}, y_{1}, \ldots, y_{s} ; q_{2}, \ldots, q_{s}\right)
$$

and upon forming the bracket of the two numbers with $y_{1}$, one will conclude that $\rho=0$. Hence, the relation (90) cannot contain $q_{1}, \ldots, q_{s}$, and will contain $y_{1}$, for example, since it cannot reduce to $y_{0}=$ const., due to the last formulas in (89). Now, if one supposes that one has:

$$
y_{1}=H\left(y_{0}, y_{1}, \ldots, y_{s}\right)
$$

then upon forming the bracket of the two numbers with $q_{1}$, one will conclude that:

$$
\rho\left[1-q_{1} \frac{\partial H}{\partial y_{0}}\right]=0
$$

which is impossible, because it will be a relation that contains $q_{1}$, or rather, it will reduce to $\rho=0$.
As far as the factor $\rho$ is concerned, from (59), it is defined by the condition:

$$
\begin{equation*}
\frac{\partial f}{\partial x_{0}}-\rho \frac{\partial f}{\partial y_{0}} \equiv 0 \quad(\bmod K) \tag{91}
\end{equation*}
$$

in which the first derivative corresponds to the set of independent variables $x_{h}, p_{i}$, and the second one corresponds to the set of new variables $y_{h}, q_{i}$.

Moreover, one can make that more precise by using the obvious identity:

$$
\frac{\partial f}{\partial x_{0}}=\frac{\partial f}{\partial y_{0}}\left(\frac{\partial y_{0}}{\partial x_{0}}-\sum_{i=1}^{s} q_{i} \frac{\partial y_{i}}{\partial x_{0}}\right)+\sum_{i=1}^{s} Y_{i} \frac{\partial y_{i}}{\partial x_{0}}+\sum_{i=1}^{s} Q_{i} \frac{\partial y_{i}}{\partial x_{0}}
$$

which will give:

$$
\begin{equation*}
\rho=\frac{\partial x_{0}}{\partial y_{0}}-\sum_{i=1}^{s} q_{i} \frac{\partial y_{i}}{\partial x_{0}} \tag{92}
\end{equation*}
$$

In order to apply that to the transformations (85) and (86), it will suffice to differentiate the first of the formulas with respect to $x_{0}$ in both cases, which will give:

$$
1=\frac{\partial W}{\partial y_{0}} \frac{\partial y_{0}}{\partial x_{0}}+\sum_{i=1}^{s} \frac{\partial W}{\partial y_{0}} \frac{\partial y_{i}}{\partial x_{0}},
$$

so, upon taking into account formulas (85) and (86), one will have:

$$
1=\frac{\partial W}{\partial y_{0}}\left(\frac{\partial y_{0}}{\partial x_{0}}-\sum_{i=1}^{s} q_{i} \frac{\partial y_{i}}{\partial x_{0}}\right) .
$$

One will then have simply:

$$
\begin{equation*}
\rho=\frac{1}{\left(\frac{\partial W}{\partial y_{0}}\right)} \tag{93}
\end{equation*}
$$

We finally remark that the theorem regarding composite functions gives the identities:

$$
\begin{aligned}
& X_{i}=\frac{\partial f}{\partial y_{0}}\left(X_{i} y_{0}-\sum_{k=1}^{s} q_{k} X_{i} y_{k}\right)+\sum_{k=1}^{s} Y_{k} \cdot X_{i} y_{k}+\sum_{k=1}^{s} Q_{k} \cdot X_{i} q_{k}, \\
& P_{i}=\frac{\partial f}{\partial y_{0}}\left(P_{i} y_{0}-\sum_{k=1}^{s} q_{k} P_{i} y_{k}\right)+\sum_{k=1}^{s} Y_{k} \cdot P_{i} y_{k}+\sum_{k=1}^{s} Q_{k} \cdot P_{i} q_{k} .
\end{aligned}
$$

Now, in order for the transformation to be a contact transformation, it must leave the sheaf $K$ invariant, which is equivalent to saying that the term in $\partial f / \partial y_{0}$ must disappear from those formulas. One will then have the conditions:

$$
\begin{equation*}
X_{i} y_{0}-\sum_{k=1}^{s} q_{k} X_{i} y_{k}=0, \quad P_{i} y_{0}-\sum_{k=1}^{s} q_{k} P_{i} y_{k}=0 \quad(i=1,2, \ldots, s) \tag{94}
\end{equation*}
$$

which also characterize the contact transformations.
One sees, moreover, that the following law by which the contact transformation exchanges the transformations of the sheaf is given by the following formulas, in which $i=1,2, \ldots, s$ :

$$
\begin{equation*}
X_{i}=\sum_{k=1}^{s}\left(Y_{k} \cdot X_{i} y_{k}+Q_{k} \cdot X_{i} q_{k}\right), \quad \quad P_{i}=\sum_{k=1}^{s}\left(Y_{k} \cdot P_{i} y_{k}+Q_{k} \cdot P_{i} q_{k}\right) \tag{95}
\end{equation*}
$$

12. Passing from one complete integral to another. - One can interpret the preceding results from the viewpoint of the integration of the sheaf $F$ in the following manner:

First, suppose the particular case of $r=0, p=1$, and consequently $n=2 s$. Hence, knowing one complete integral $u_{0}, u_{1}, \ldots, u_{s}$, from which one can deduce the polar functions $v_{0}, v_{1}, \ldots, v_{s}$, will reduce $F$ to the form $K$. In order to do that, it will suffice to make a change of variables by taking the variables $u_{0}, u_{1}, \ldots, u_{s} ; v_{0}, v_{1}, \ldots, v_{s}$ in place of the initial $2 s+1$ variables. It will then remain for one to integrate $K$, and an arbitrary complete integral is given when one starts from $u_{0}, u_{1}, \ldots$, $u_{s} ; v_{0}, v_{1}, \ldots, v_{s}$, by the formulas for an arbitrary contact transformation in the space of $u_{0}, u_{1}, \ldots$, $u_{s}$.

Consequently, if one agrees to extend the meaning of the term complete integral by including not only the elements of the complete integral, properly speaking (such as $u_{0}, u_{1}, \ldots, u_{s}$ ), but also the associated polar elements (such as $v_{1}, \ldots, v_{s}$ ), one can say that passing from one complete integral to another is effected by a contact transformation.

From the applications of no. 8, that will still be true when $r$ is not zero and $p$ is greater than 1 . It is only necessary that the contact transformation in question should depend upon $p-1$ invariants of $F$ in an arbitrary manner, which will involve just as many arbitrary constant parameters. Moreover, those invariants must be appended to the complete integral elements that are provided by the contact transformation, even if they are found to be appended to the complete integral elements $u_{0}, u_{1}, \ldots, u_{s}$, which are supposed to be calculated first.

The result will break down for $s=1$ and $p>1$ only when $F$ has less than $p-1$ distinct invariants, i.e., when its derived sheaf $F^{\prime}$ is not complete. It will then remain for us to study that case, which will be the subject of the following section. Recall that when $s$ is greater than $1, F^{\prime}$ will always be complete, as we saw in no. 7.

## III. - Study of the exceptional case.

13. Case in which the second derived sheaf has degree $n+2$. - From the analysis of no. 7, we obtained the form for $F$ in the case of $s=1$ that we called semi-canonical:

$$
\begin{equation*}
X=\frac{\partial f}{\partial x}+x_{1} \frac{\partial f}{\partial x_{0}}+Y, \quad X_{1}=\frac{\partial f}{\partial x_{1}}, \quad Z_{i}=\frac{\partial f}{\partial x_{j}}, \quad(j=1,2, \ldots, r) \tag{96}
\end{equation*}
$$

Since the $Z_{j}$ are distinguished transformations, $Y$ has the form:

$$
\begin{equation*}
Y=\sum_{\alpha=1}^{l} \eta_{\alpha} \frac{\partial f}{\partial y_{\alpha}}, \tag{97}
\end{equation*}
$$

in which the $\eta_{\alpha}$ do not depend upon the $z_{j}$ but can depend upon all of the other variables $t_{1}, t_{2}, \ldots$, $t_{q}$ that the possible invariants of $F$ might be comprised of.

The degree of $F$ is $n=2 s+r=2+r$, and the total number of variables is $m=n+1+l+q$, in such a way that $l=(p-1)-q$. As we saw in no. 12, the case in which $l=0(q=p-1)$ will occur in the general case, because $Y$ will disappear then. We shall see that in the case of $l>0$, we can also have a canonical form under certain hypotheses that relate to the successive derivatives of $F$.

The derived sheaf $F^{\prime}$ is deduced from (96) by appending the transformation $\left(X_{1}, X\right)$, i.e.:

$$
\begin{equation*}
Z=\frac{\partial f}{\partial x_{0}}+\frac{\partial Y}{\partial x_{1}} \quad\left(\frac{\partial Y}{\partial x_{1}}=\sum_{\alpha=1}^{l} \frac{\partial \eta_{\alpha}}{\partial x_{1}} \frac{\partial f}{\partial y_{\alpha}}\right) . \tag{98}
\end{equation*}
$$

Hence, the transformations $Z_{j}$ are once more distinguished for $F^{\prime}$.
In order to pass on to following derived sheaf $F^{\prime \prime}$, we must append $(X, Z)$ and $\left(X_{1}, Z\right)$. Hence, $F^{\prime \prime}$ has degree $n+2$ or $n+3$. We shall examine the case in which it has degree $n+2$.

The derivative $F^{\prime}$, like $F$, then has a sheaf of degree greater by one unit for its derived sheaf. We can then apply the results of our study. Set $n^{\prime}=n+1$ and let $r^{\prime}$ be the degree of the characteristic sheaf of $F^{\prime}$. From the remark that was just made, we will have $r^{\prime} \geq r$. Hence, $n^{\prime}-r^{\prime}$ $\leq n-r+2$, since $F^{\prime}$ is not complete. Hence, $n^{\prime}-r^{\prime}=2$, i.e., the value of $s$ again 1 for $F^{\prime}$. Moreover, $F^{\prime}$ can have no other invariants than $t_{1}, t_{2}, \ldots, t_{q}$, in such a way that the value of $l$ will become $l^{\prime}=l-1$ for $F^{\prime}$. By definition, the semi-canonical form of $F^{\prime}$ is:

$$
\begin{equation*}
X^{\prime}=\frac{\partial f}{\partial x^{\prime}}+x_{1}^{\prime} \frac{\partial Y}{\partial x_{0}^{\prime}}+Y^{\prime}, \quad X_{1}^{\prime}=\frac{\partial f}{\partial x_{1}^{\prime}}, \quad \frac{\partial f}{\partial z}, \quad Z_{j} \quad(j=1,2, \ldots, r), \tag{99}
\end{equation*}
$$

with

$$
\begin{equation*}
Y^{\prime}=\sum_{\beta=1}^{l-1} \eta_{\beta}^{\prime} \frac{\partial f}{\partial y_{\beta}^{\prime}} \tag{100}
\end{equation*}
$$

The transformation $\partial f / \partial z$ is the new distinguished transformation of $F^{\prime}$ (other than $Z_{j}$ ). Hence, the $\eta_{\beta}^{\prime}$ will not depend upon either the $z_{j}$ or $z$.

Having said that, consider $F$ to be a subsheaf of $F^{\prime}$. It will be defined by the $Z_{j}$, which it contains by hypothesis, and two other basis transformations of the form $\lambda X^{\prime}+\lambda_{1}^{\prime} X_{1}^{\prime}+\mu \frac{\partial f}{\partial z}$. Those two transformations are independent relative to $X^{\prime}$ and $X_{1}^{\prime}$, or rather $\partial f / \partial z$ belongs to $F$.

Under the first hypothesis, the two basis transformations in question can be taken to have the form:

$$
X^{\prime}+\lambda \frac{\partial f}{\partial z}, \quad X_{1}^{\prime}+\lambda_{1} \frac{\partial f}{\partial z}
$$

However, if the bracket of two such transformations that contains $\frac{\partial f}{\partial x_{0}^{\prime}}$, but not $\frac{\partial f}{\partial x^{\prime}}$, then it cannot belong to $F^{\prime}$. The hypothesis must then be rejected since the bracket of two arbitrary transformations of $F$ must belong to its derived sheaf $F^{\prime}$.

Therefore, $F$ will contain $\partial f / \partial z$, and its basis can be taken in the form:

$$
\begin{equation*}
X^{\prime \prime}=X^{\prime}+\zeta X_{1}^{\prime}, \quad \frac{\partial f}{\partial z}, \quad Z_{1}, \ldots, Z_{r} \tag{101}
\end{equation*}
$$

It is, moreover, impossible for $\zeta$ to be independent of $z$ without the sheaf (101) being complete. Hence, one can take $\zeta$ to be the new variables in place of $z$, and upon changing the notations, one will arrive at a semi-canonical form that is more precise then (96):

$$
\begin{equation*}
X=\frac{\partial f}{\partial x}+x_{1} \frac{\partial f}{\partial x_{0}}+x_{2} \frac{\partial f}{\partial x_{1}}+Y, \quad X_{2}=\frac{\partial f}{\partial x_{2}}, \quad Z_{j} \quad(j=1,2, \ldots, r), \tag{102}
\end{equation*}
$$

in which $Y$ involve only the $l-1$ variables $y_{\alpha}$, instead of $l$ :

$$
\begin{equation*}
Y=\sum_{\alpha=1}^{l-1} \eta_{\alpha} \frac{\partial f}{\partial y_{\alpha}} \tag{103}
\end{equation*}
$$

Moreover, it will result from the way that we arrived at that result that the $\eta_{\alpha}$ will not depend upon either the $z_{j}$ or $x_{2}$.
14. - Generalization. - That result can be generalized step-by-step. We shall suppose that the degrees of $F^{\prime}, F^{\prime \prime}, \ldots, F^{(k)}$ are $n+1, n+2, \ldots, n+k$, respectively, and we shall see that in those cases one can take the semi-canonical form of $F$ to be:

$$
\left\{\begin{array}{c}
X=\frac{\partial f}{\partial x}+x_{1} \frac{\partial f}{\partial x_{0}}+x_{2} \frac{\partial f}{\partial x_{1}}+\cdots+x_{k} \frac{\partial f}{\partial x_{k-1}}+Y ; \quad \frac{\partial f}{\partial x_{k}} ; \quad Z_{j}  \tag{104}\\
(j=1,2, \ldots, r),
\end{array}\right.
$$

with

$$
\begin{equation*}
Y=\sum_{\alpha=1}^{l-k+1} \eta_{\alpha} \frac{\partial f}{\partial y_{\alpha}} \tag{105}
\end{equation*}
$$

in which the $\eta_{\alpha}$ do not depend upon either the $z_{j}$ or the $x_{2}, x_{3}, \ldots, x_{k}$.
If that theorem is supposed to be true for $k=1,2, \ldots, h$ then it will suffice to prove that it is true for $k=h+1$. To that effect, we then apply it to $F^{\prime}$, which will then be, with a distinguished transformation (namely, $\partial f / \partial z$ ), moreover:

$$
\begin{equation*}
X^{\prime}=\frac{\partial f}{\partial x^{\prime}}+x_{1}^{\prime} \frac{\partial f}{\partial x_{0}^{\prime}}+x_{2}^{\prime} \frac{\partial f}{\partial x_{1}^{\prime}}+\cdots+x_{k}^{\prime} \frac{\partial f}{\partial x_{k-1}^{\prime}}+Y^{\prime}, \quad \frac{\partial f}{\partial x_{k}^{\prime}}, \quad \frac{\partial f}{\partial z}, \quad Z_{j} \tag{106}
\end{equation*}
$$

with

$$
\begin{equation*}
Y^{\prime}=\sum_{\alpha=1}^{l-k} \eta_{\alpha}^{\prime} \frac{\partial f}{\partial y_{\alpha}^{\prime}} \tag{107}
\end{equation*}
$$

In order to deduce $F$, it will suffice to associate $Z_{j}$ with two transformations of the form:

$$
\lambda X^{\prime}+\mu \frac{\partial f}{\partial x_{k}^{\prime}}+v \frac{\partial f}{\partial z},
$$

and one will see, as above, that one can deduce a combination that contains only $\partial f / \partial z$. Since those two transformations are then:

$$
X^{\prime}+\zeta \frac{\partial f}{\partial x_{k}^{\prime}}, \quad \frac{\partial f}{\partial z}
$$

one will confirm that $\zeta$ can be taken to be the variable in place of $z$ by reasoning as in no. 13, and all that will remain is to make a change of notations in order to obtain the stated result.

The converse of that theorem is true, because since $F$ has the form (104), one will deduce $F^{\prime}$ by appending the bracket $\left(\frac{\partial f}{\partial x_{k}}, X\right)$, i.e., $\frac{\partial f}{\partial x_{k-1}}$. One will then get the following form for $F^{\prime}$ :

$$
\frac{\partial f}{\partial x}+x_{1} \frac{\partial f}{\partial x_{0}}+\cdots+x_{k-1} \frac{\partial f}{\partial x_{k-2}}+Y ; \quad \frac{\partial f}{\partial x_{k-1}}, \frac{\partial f}{\partial x_{k}} ; \quad \frac{\partial f}{\partial z_{1}}, \ldots, \frac{\partial f}{\partial z_{2}}
$$

Since $Y$ does not depend upon $x_{k-1}$ and $x_{k}$, one similarly passes on to $F^{\prime \prime}$ by appending $\frac{\partial f}{\partial x_{k-2}}$, and so on, up to $F^{(k-1)}$, which is:

$$
\begin{equation*}
\frac{\partial f}{\partial x}+x_{1} \frac{\partial f}{\partial x_{0}}+Y, \quad \frac{\partial f}{\partial x_{1}}, \frac{\partial f}{\partial x_{2}}, \ldots, \frac{\partial f}{\partial x_{k}} ; \quad \frac{\partial f}{\partial z_{1}}, \ldots, \frac{\partial f}{\partial z_{2}} \tag{108}
\end{equation*}
$$

Finally, $F^{(k)}$ is deduced from (108) by appending $\frac{\partial f}{\partial x_{0}}+\frac{\partial Y}{\partial x_{1}}$.
On the other hand, we remark that all of the successive derived sheaves have $q$ invariants, like $F$, and that as a result, their degree cannot exceed the limit:

$$
m-q=n+p-q=n+l+1
$$

and must attain it. It will then be the case that $k=l+1$, and since $F^{(k)}$ has degree $n+k$, it will be complete, and furthermore, from (105), $Y$ will disappear from formula (104), and if it so happens that $k=l$ then since $F^{(k)}$ has degree $n+k$ (i.e., $n+l$ ), $F^{(k+1)}$ must have degree $n+l+1$, i.e., $n+k$ +1 , and it will be complete.

Therefore, either the reduction can be continued up to $k=l+1=p-q$ and given the canonical form:

$$
\begin{equation*}
\frac{\partial f}{\partial x}+x_{1} \frac{\partial f}{\partial x_{0}}+x_{2} \frac{\partial f}{\partial x_{1}}+\cdots+x_{l+1} \frac{\partial f}{\partial x_{l}} ; \quad \frac{\partial f}{\partial x_{l+1}} \quad ; \quad \frac{\partial f}{\partial z_{1}}, \ldots, \frac{\partial f}{\partial z_{r}} \tag{109}
\end{equation*}
$$

or the reduction will stop before $k$ has attained the value l, and one will have the semi-canonical form (104), in which the variables $y_{\alpha}$ are at least two in number.

In the first case, the degrees of the successive derived sheaves will increase by one unit until the last one $F^{(l+1)}$. In the second case, the degrees of the successive derived sheaves will increase by one unit up to $F^{(k)}$ inclusive $(k<l)$, and $F^{(k+1)}$ will have degree $n+k+2$, i.e., its degree will be greater by two units than that of $F^{(k)}$.

As for the manner by which that reduction is accomplished, one first reduces $F^{(k-1)}$ to the form (108), which one can do by integrating a complete system (see no. 7). One then passes from $F^{(k-1)}$ to $F^{(k-2)}$, and so on, by the method that was presented in the proof of the present (direct) theorem. That will require no more than successive changes of variables without any new integration.
15. On the integration of the sheaf in question. -As far as the integration of the given sheaf $F$ is concerned, it comes down to that of the sheaves that have types of the form (104) or (109). Now they present themselves as the results of successive prolongations $\left({ }^{1}\right)$ of the sheaf:

$$
\begin{equation*}
X=\frac{\partial f}{\partial x}+\eta_{1} \frac{\partial f}{\partial y_{1}}+\cdots+\eta_{h} \frac{\partial f}{\partial y_{h}}+x_{1} \frac{\partial f}{\partial x_{0}}, \quad \frac{\partial f}{\partial x_{1}}, \tag{110}
\end{equation*}
$$

in the first case and of the sheaf:

$$
\begin{equation*}
\frac{\partial f}{\partial x_{0}}, \quad \frac{\partial f}{\partial x} \tag{111}
\end{equation*}
$$

in the second case. In the two cases, the prolongation can be made by considering $x_{0}$ to be a function of $x$ whose successive derivatives are $x_{1}, x_{2}, \ldots$

In the second case [form (109)], one will then have the explicit solution immediately, which has $l+2$ arbitrary constants:

$$
\begin{equation*}
x_{0}=W\left(x ; a_{0}, a_{1}, \ldots, a_{l+1}\right), \quad x_{1}=\frac{\partial W}{\partial x}, \quad x_{2}=\frac{\partial^{2} W}{\partial x^{2}}, \quad \ldots, \quad x_{l+1}=\frac{\partial^{l+1} W}{\partial x^{l+1}} . \tag{112}
\end{equation*}
$$

It is implicit that the invariants $t_{1}, \ldots, t_{q}$, which must be equal to constants, can enter into $W$ in an arbitrary manner.

Independently of the theory of prolongation, one sees, moreover, that the complete subsheaves in question are obtained by appending a transformation of the form $X+\omega \frac{\partial f}{\partial x_{l+1}}$ ( $\omega$ being arbitrary)
${ }^{(1)}$ ) M., no. 15, pp. 367.
to $\frac{\partial f}{\partial z_{j}}$, and that everything comes down to finding their invariants. Now, that is equivalent to integrating the system:

$$
d x=\frac{d x_{0}}{x_{1}}=\frac{d x_{1}}{x_{2}}=\ldots=\frac{d x_{l}}{x_{l+1}}=\frac{d x_{l+1}}{\omega\left(x, x_{0}, x_{1}, \ldots, x_{l+1}\right)},
$$

i.e., to integrating an arbitrary differential equation:

$$
\frac{d^{l+2} x_{0}}{d x^{l+2}}=\omega\left(x, x_{0}, \ldots, \frac{d x_{0}}{d x}, \ldots, \frac{d^{l+1} x_{0}}{d x^{l+1}}\right),
$$

so the formulas (112).
In the first case [form (104)], one will be likewise led to integrate a system of the form ( $h=l$ $-k+1)$ :

$$
\begin{align*}
d x & =\frac{d x_{0}}{x_{1}}=\frac{d y_{1}}{\eta_{1}\left(x, x_{0}, x_{1}, y_{1}, \ldots, y_{h}\right)}=\ldots=\frac{d y_{h}}{\eta_{h}\left(x, x_{0}, x_{1}, y_{1}, \ldots, y_{h}\right)}  \tag{113}\\
& =\frac{d x_{1}}{x_{2}}=\frac{d x_{2}}{x_{3}}=\ldots=\frac{d x_{k-1}}{x_{k}}=\frac{d x_{k}}{\omega\left(x, x_{0}, \ldots, x_{k}, y_{1}, \ldots, y_{h}\right)},
\end{align*}
$$

in which the function $\omega$ is arbitrary.
First consider the partial system that results from this:

$$
\begin{equation*}
\frac{d y_{l}}{d x}=\eta_{l}\left(x, x_{0}, \frac{d x_{0}}{d x}, y_{1}, \ldots, y_{h}\right) \quad(i=1,2, \ldots, h) \tag{114}
\end{equation*}
$$

It defines the $y_{i}$ as functionals of $x$ and $x_{0}$, and as a result, $\omega$ as a functional of $x$ and $x_{0}$, in such a way that will have a functional equation for $x_{0}$ :

$$
\frac{d^{k+1} x_{0}}{d x^{k+1}}=\bar{\omega}\left[\left(x, x_{0}, \frac{d x_{0}}{d x}, \ldots, \frac{d^{k} x_{0}}{d x^{k}}\right)\right] .
$$

Leaving aside the determination of all the complete integrals, one then proposes only to look for all of their $(r+1)$-dimensional integral multiplicities, and it would be legitimate to confine oneself to the system (114) while considering $x_{0}$ to be an arbitrary function of $x$.

That is the reduction of the integration to which one arrives in this case. It is equivalent to considering that one is reduced to the integration of the sheaf (110) with $h+3$ variables, whose second derived sheaf has degree 5 , by hypothesis.

We shall return to the case of the canonical form (109) and adopt the viewpoint of $(r+1)$ dimensional integral multiplicities, as we just did for the general case. Let $\xi_{1}, \xi_{2}, \ldots, \xi_{m}$ ( $m=n+$ $p, n=r+2, p=q+l+1)$ be the initial variables. The change of variables that led to the canonical
form will give the expressions for those $\xi_{i}$ as functions of $x, x_{0}, x_{1}, \ldots, x_{l+1} ; t_{1}, \ldots, t_{q} ; z_{1}, \ldots, z_{r}$, and the desired general multiplicity is obtained by replacing $x_{0}$ with an arbitrary function of $x$ in those formulas, along with replacing $x_{1}, \ldots, x_{l+1}$ with the successive derivatives of that functions, and finally replacing $t_{1}, \ldots, t_{q}$ with arbitrary constants. We can express that fact by saying that the sheaf $F$ has an explicit general integral.

Conversely, any sheaf that has an explicit general integral of the preceding form (in the sense that was just explained) will have successive derivatives of degrees $n+1, n+2, n+3, \ldots$ if it has degree $n$.

Indeed, let $F$ be that sheaf, while $\xi_{1}, \xi_{2}, \ldots, \xi_{m}$ are the variables that appear in it, and:

$$
\begin{equation*}
\xi_{i}=\mathcal{F}_{i}\left(x, x_{0}, x_{1}, \ldots, x_{l+1} ; t_{1}, \ldots, t_{q} ; z_{1}, \ldots, z_{r}\right) \quad(i=1,2, \ldots, m) \tag{115}
\end{equation*}
$$

are the formulas that define the general integral when one sets:

$$
\begin{equation*}
x_{0}=\psi(x), \quad x_{1}=\psi^{\prime}(x), \quad \ldots, \quad x_{l+1}=\psi^{(l+1)}(x), \quad t_{1}=a_{1}, \quad \ldots, \quad t_{q}=a_{q} \tag{116}
\end{equation*}
$$

in which $\psi(x)$ is an arbitrary function. The functions $\mathcal{F}_{i}$ are independent functions relative to the arguments that appear in them since there can exist no relation between just the $\xi_{i}$.

That being the case, if $m=l+3+q+r$ then one can consider equations (115) to define a change of variables that will make the sheaf $F$ pass to a sheaf $\Phi$ for which formulas (116) will give the general solution. From this, the sheaf $\Phi$ will be the dual to the Pfaff system:

$$
d x_{0}-x_{1} d x=0, \quad d x_{1}-x_{2} d x=0, \quad \ldots, \quad d x_{l}-x_{l+1} d x=0, \quad d t_{1}=0, \quad \ldots, \quad d t_{q}=0
$$

and will be nothing but the sheaf (109). Hence, $F$ will have (109) for its canonical form and will consequently possess the stated property.

Let us now examine the case in which $m$ is less than the number of arguments in the functions (115). We combine them with some arbitrary formulas of the same form:

$$
\begin{equation*}
\eta_{j}=\mathcal{G}_{j}\left(x, x_{0}, x_{1}, \ldots, x_{l+1} ; t_{1}, \ldots, t_{q} ; z_{1}, \ldots, z_{r}\right) \quad(j=1,2, \ldots, \mu) . \tag{117}
\end{equation*}
$$

in such a manner that $m+\mu=l+3+q+r$, and we will then have a change of variables again. If we apply it to the sheaf (109) whose general integral is (116), then we will get a sheaf $U$ whose basis transformations have the form:

$$
\mathcal{U}_{h}=\mathcal{X}_{h}+\mathcal{Y}_{h},
$$

in which $\mathcal{X}_{h}$ is a transformation of the given sheaf $F$ and $\mathcal{Y}_{h}$ has the form:

$$
\mathcal{Y}_{h}=\sum_{j=1}^{\mu} H_{h, j}\left(\xi_{1}, \ldots, \xi_{m} ; \eta_{1}, \ldots, \eta_{\mu}\right) \frac{\partial f}{\partial \eta_{j}} .
$$

From the argument that was just made, that sheaf $U$ will possess the stated properties. Now one has:

$$
\left(\mathcal{U}_{h}, \mathcal{U}_{k}\right)=\left(\mathcal{X}_{h}, \mathcal{X}_{k}\right)+\sum_{j=1}^{\mu} H_{h, k \mid j}(\xi \mid \eta) \frac{\partial f}{\partial \eta_{j}},
$$

because the $\mathcal{X}$ do not depend upon the $\eta$. One then concludes that the basis for the derived sheaf $U^{\prime}$ of $U$ will be defined, like that of $U$, by transformations of the type $\mathcal{X}+\mathcal{Y}$, where $\mathcal{X}$ does not depend upon the $\eta$. Since the structure formulas for $U$ have the form:

$$
\left(\mathcal{U}_{h}, \mathcal{U}_{k}\right) \equiv c_{h k} \mathcal{V} \quad(\bmod U)
$$

with $\mathcal{V}=\mathcal{X}+\mathcal{Y}$, one will then have the structure formulas for $F$ :

$$
\left(\mathcal{X}_{h}, \mathcal{X}_{k}\right) \equiv c_{h k} \mathcal{X} \quad(\bmod F) .
$$

That argument, in which it is not supposed that the transformations $\mathcal{X}_{h}$ are necessarily divergent, will suffice to prove that the degree of $F^{\prime}$ is greater than the degree of $F$ by at most one. Since one can then argue with $F^{\prime}$ and $U^{\prime}$ as one did with $F$ and $U$, the stated proposition is then found to be established entirely ( ${ }^{1}$ ).

## IV. - Automorphic systems relative to the general group of contact transformations.

16. Defining equations of the infinitesimal transformations of the general group of contact transformations. - In no. 12, we saw that as soon as we take the unknowns in the integration of a sheaf $F$ of the type that is considered in this treatise to be the system of functions $u_{0}, u_{1}, \ldots, u_{s} ; v_{1}$, $\ldots, v_{s}$ that are composed of the variable elements of the complete integrals $u_{0}, u_{1}, \ldots, u_{s}$ (the others being the invariants of $F$ ) and the polar functions $v_{1}, \ldots, v_{s}$ that are associated with $u_{1}, \ldots, u_{s}$, respectively, the most general solution is deduced from an arbitrary particular solution upon performing the general contact transformation of the space $u_{0}, u_{1}, \ldots, u_{s}$ on the functions $u$ and $v$.

Those $2 s+1$ unknowns $u$ and $v$ are then, in fact, found to satisfy an automorphic system $\left({ }^{2}\right)$ whose associated group is the general group of contact transformations in (s+1)-dimensional space.

[^9]It would be interesting to make that fact obvious by comparing the canonical form to which such an automorphic system is susceptible with the equations that were obtained before for the determination of those functions. In order to do that, it is necessary to first find the canonical form for the defining equations for the group $G$. Of course, if one deals with the defining equations for the finite transformations of the group then it would be good to first examine what happens with the defining equations of the infinitesimal transformations.

I shall then recall the sheaf (65), namely:

$$
\begin{equation*}
P_{i} f=\frac{\partial f}{\partial p_{i}}, \quad X_{i} f=\frac{\partial f}{\partial x_{i}}+p_{i} \frac{\partial f}{\partial x_{0}}=\frac{\partial f}{\partial x_{i}} \quad(i=1,2, \ldots, s), \tag{118}
\end{equation*}
$$

and look for the infinitesimal transformations that leave it invariant. Let:

$$
\begin{equation*}
Z f=\sum_{i=1}^{m} \varpi_{i} \frac{\partial f}{\partial p_{i}}+\sum_{i=1}^{m} \xi_{i} \frac{\partial f}{\partial x_{i}}+\lambda \frac{\partial f}{\partial x_{0}} \tag{119}
\end{equation*}
$$

be such a transformation. The bracket relations:

$$
\left(Z f, P_{i} f\right) \equiv 0, \quad\left(Z f, X_{i} f\right) \equiv 0 \quad\left(\bmod P_{i}, X_{i}\right)
$$

immediately give:

$$
\begin{equation*}
\xi_{i}+\frac{\partial \lambda}{\partial p_{i}}=0, \quad \varpi_{i}-\frac{\partial \lambda}{\partial x_{i}}=0 \quad(i=1,2, \ldots, s) . \tag{120}
\end{equation*}
$$

Those are the well-known Lie formulas: On the other hand, the coefficient of $\partial f / \partial x_{0}$ in $Z f$ is:

$$
\begin{equation*}
\xi_{0}=\lambda+\sum_{i=1}^{m} p_{i} \xi_{i} \tag{121}
\end{equation*}
$$

The unknowns are: $\lambda$, the $\xi_{i}$, and the $\varpi_{i}$, and equations (120) are $2 s$ first-order equations. One deduces $s(2 s-1)$ new first-order equations from them by differentiation:

$$
\left\{\begin{array}{cc}
\frac{\partial \xi_{i}}{\partial p_{k}}-\frac{\partial \xi_{k}}{\partial p_{i}}=0, & \frac{\partial \varpi_{i}}{\partial x_{k}}-\frac{d \varpi_{k}}{d x_{i}}=0,  \tag{122}\\
\frac{d \xi_{i}}{d x_{k}}+\frac{\partial \xi_{k}}{\partial x_{i}}=0 & (i \neq k=1,2, \ldots . k) \\
\frac{d \xi_{i}}{\partial p_{i}}=\frac{\partial \varpi_{i}}{\partial x_{0}} & (i=1,2, \ldots, s)
\end{array}\right.
$$

The system (120), (122) is composed of $s(2 s+1)$ first-order equations. We shall verify that it is completely integrable $\left({ }^{1}\right)$.

[^10]I first solve them in the following manner:
(a)

$$
\frac{\partial \xi_{i}}{\partial p_{k}}=\frac{\partial \xi_{k}}{\partial p_{i}} \quad(i=2,3, \ldots, s ; k<i)
$$

(b)

$$
\frac{\partial \varpi_{i}}{\partial p_{i}}=\frac{\partial \lambda}{\partial x_{0}}-\frac{d \xi_{i}}{d x_{i}} \quad(i=1,2, \ldots, s)
$$

$$
\frac{\partial \varpi_{i}}{\partial p_{k}}=-\frac{d \xi_{i}}{d x_{i}} \quad(i=1,2, \ldots, s)
$$

$$
\frac{d \varpi_{i}}{d x_{k}}=\frac{d \varpi_{i}}{d x_{i}}
$$

$$
(i=2,3, \ldots, s ; k<i)
$$

(c)

$$
\begin{array}{ll}
\frac{\partial \lambda}{\partial p_{k}}=-\xi_{k} & (k=1,2, \ldots, s), \\
\frac{d \lambda}{d x_{k}}=\varpi_{k} & (k=1,2, \ldots, s)
\end{array}
$$

It is intended that equations $\left(b^{\prime \prime}\right)$ and $\left(c^{\prime}\right)$ are regarded as being solved for $\frac{\partial \varpi_{i}}{\partial x_{k}}$ and $\frac{\partial \lambda}{\partial x_{k}}$.
One effortlessly verifies that one will get all of the derivatives of the left-hand sides by successively differentiating those left-hand sides, and only once for each of them, upon differentiating in all (different) ways:

1. The (a) with respect to $x_{0}, x_{1}, \ldots, x_{m}$ and the $p_{j}$ for which one has $1 \leq j \leq k$ or $i \leq j$.
2. The $(b)$ with respect to all of the variables.
3. The $\left(b^{\prime}\right)$ with respect to $x_{0}, x_{1}, \ldots, x_{m}$ and the $p_{j}$ for which one has $1 \leq j \leq k$ or $j \neq i$.
4. The $\left(b^{\prime \prime}\right)$ with respect to the $x_{j}$ for which one will have $0 \leq j \leq k$ or $i \leq j$.
5. The (c) with respect to $x_{0}, x_{1}, \ldots, x_{m}$ and the $p_{j}$ for which one has $1 \leq j \leq k$.
6. The ( $c^{\prime}$ ) with respect to the $x_{j}$ for which one has $0 \leq j \leq k$.

The form that was given to the system is then complete, and it remains to be proved that if one differentiates it only once in all possible ways then one will not get equations that are consequences of differentiating equations $(E)$ in all of the ways that were specified above.

The verification is pointless for equations $(c)$ and $\left(c^{\prime}\right)$, whose integrability conditions are expressed by equations ( $b$ ) and ( $b^{\prime}$ ).

We have equations $(E)$ for equations $(a)$ :

$$
\frac{\partial^{2} \xi_{i}}{\partial p_{k} \partial p_{j}}=\frac{\partial^{2} \xi_{k}}{\partial p_{i} \partial p_{j}} \quad(j \leq k<i \text { or } k<i \leq j)
$$

and upon differentiating in the opposite order if $j \neq k$, one will also have:

$$
\frac{\partial^{2} \xi_{i}}{\partial p_{j} \partial p_{k}}=\frac{\partial^{2} \xi_{j}}{\partial p_{i} \partial p_{k}} \quad(j<i)
$$

One will then have two formulas if $j<k<i$, and upon equating them, one will get:

$$
\frac{\partial}{\partial p_{i}}\left(\frac{\partial \xi_{k}}{\partial p_{j}}-\frac{\partial \xi_{j}}{\partial p_{k}}\right)=0
$$

which is an equation $(E)$, since one has $j<k$ and $k<i$.
The same verification will be true for the $\left(b^{\prime \prime}\right)$ in regard to the differentiations that are made with respect to the $x_{j}$.

Like equations $(E)$, the (b) give:

$$
\frac{d}{d x_{k}} \frac{\partial \varpi_{i}}{\partial p_{i}}=\frac{d}{d x_{k}} \frac{\partial \lambda}{\partial x_{0}}-\frac{d}{d x_{k}} \frac{d \xi_{i}}{d x_{i}}=\frac{\partial}{\partial x_{0}} \frac{d \lambda}{d x_{k}}-\frac{d}{d x_{i}} \frac{d \xi_{i}}{d x_{k}}
$$

or

$$
\begin{equation*}
\frac{d}{d x_{k}} \frac{\partial \varpi_{i}}{\partial p_{i}}=\frac{\partial \varpi_{k}}{\partial x_{0}}-\frac{d}{d x_{i}} \frac{d \xi_{i}}{d p_{k}} . \tag{123}
\end{equation*}
$$

On the other hand, if $1 \leq k<i$ then one will have:

$$
\frac{\partial}{\partial p_{i}} \frac{d \varpi_{i}}{d x_{k}}=\frac{\partial}{\partial p_{i}} \frac{d \varpi_{k}}{d x_{i}}=\frac{\partial \varpi_{k}}{\partial x_{0}}+\frac{d}{d x_{i}} \frac{d \varpi_{k}}{d p_{i}}
$$

or

$$
\begin{equation*}
\frac{\partial}{\partial p_{i}} \frac{d \varpi_{i}}{d x_{k}}=\frac{\partial \varpi_{k}}{\partial x_{0}}-\frac{d}{d x_{i}} \frac{d \xi_{i}}{d x_{k}} \tag{124}
\end{equation*}
$$

because since $k<i$, one can use $\left(b^{\prime}\right)$ and apply the differentiation $d / d x_{i}$. One will then see that the expressions (123) and (124) are identical.

For $j \neq i,(b)$, like equations $(E)$, will again give:

$$
\frac{\partial^{2} \varpi_{i}}{\partial p_{i} \partial p_{j}}=\frac{\partial^{2} \lambda}{\partial x_{0} \partial p_{i}}-\frac{\partial}{\partial p_{j}} \frac{d \xi_{i}}{d x_{i}},
$$

so

$$
\begin{equation*}
\frac{\partial^{2} \varpi_{i}}{\partial p_{i} \partial p_{j}}=-\frac{\partial \xi_{j}}{\partial x_{0}}-\frac{d}{d x_{i}} \frac{\partial \xi_{i}}{\partial p_{j}} . \tag{125}
\end{equation*}
$$

On the other hand, from $\left(b^{\prime}\right)$, one will have:

$$
\begin{equation*}
\frac{\partial^{2} \varpi_{i}}{\partial p_{j} \partial p_{i}}=-\frac{\partial}{\partial p_{i}} \frac{d \xi_{j}}{d x_{i}}=-\frac{\partial \xi_{j}}{\partial x_{0}}-\frac{d}{d x_{i}} \frac{\partial \xi_{j}}{\partial p_{i}} . \tag{126}
\end{equation*}
$$

The identities in the right-hand sides of (125) and (126) result from (a).
Like equations $(E),\left(b^{\prime}\right)$ will first give:

$$
\begin{equation*}
\frac{d}{d x_{j}} \frac{\partial \varpi_{i}}{\partial p_{k}}=-\frac{d}{d x_{j}} \frac{\partial \xi_{k}}{\partial x_{i}}=-\frac{d}{d x_{i}} \frac{\partial \xi_{k}}{\partial x_{j}} \tag{127}
\end{equation*}
$$

and on the other hand, one will have from $\left(b^{\prime \prime}\right)$ that for $j<i$ (since $\left.i \neq k\right)$ :

$$
\begin{equation*}
\frac{\partial}{\partial p_{k}} \frac{d \varpi_{i}}{d x_{j}}=\frac{\partial}{\partial p_{k}} \frac{d \varpi_{j}}{d x_{i}}=\frac{d}{d x_{i}} \frac{\partial \varpi_{j}}{\partial p_{k}} . \tag{128}
\end{equation*}
$$

In the case of $j=k$, the right-hand side of (128) can be calculated by means of $(b)$, and due to $\left(c^{\prime}\right)$, it will become:

$$
\frac{d}{d x_{i}} \frac{\partial \lambda}{\partial x_{0}}-\frac{d}{d x_{i}} \frac{\partial \xi_{k}}{\partial x_{k}}=\frac{\partial \varpi_{i}}{\partial x_{0}}-\frac{d}{d x_{i}} \frac{\partial \xi_{k}}{\partial x_{k}} .
$$

That is equal to the right-hand side of $(127)$ (since $j=k$ ), plus $\partial \varpi_{i} / \partial x_{0}$, but the same relation will be true between the left-hand sides of (127) and (128). The verification then once more achieved.

For $1 \leq j \leq k$ and $j \neq i$, the $\left(b^{\prime}\right)$ then will then $(E)$ :

$$
\begin{equation*}
\frac{\partial^{2} \varpi_{i}}{\partial p_{k} \partial p_{j}}=-\frac{\partial}{\partial p_{j}} \frac{d \xi_{k}}{d x_{i}}=-\frac{d}{d x_{i}} \frac{\partial \xi_{k}}{\partial p_{j}}, \tag{129}
\end{equation*}
$$

and they will give, in addition:

$$
\begin{equation*}
\frac{\partial^{2} \varpi_{i}}{\partial p_{j} \partial p_{k}}=-\frac{\partial}{\partial p_{k}} \frac{d \xi_{j}}{d x_{i}}=-\frac{d}{d x_{i}} \frac{\partial \xi_{j}}{\partial p_{k}} . \tag{130}
\end{equation*}
$$

The identity of the right-hand sides of (129) and (130) will result from equations $(E)$ when deduced from (a).

The verification of the integrability conditions is thus concluded.
17. Defining equations for the general group of contact transformations (finite transformations). - The (finite) contact transformations are characterized by the bracket relations (89) in no. 11. However, those relations do not constitute a complete-integrable system.

Indeed, if they constitute a completely-integrable system then when one expresses the idea that the infinitesimal transformation:

$$
\begin{equation*}
Z f=\sum_{i=1}^{m} \varpi_{i} \frac{\partial f}{\partial p_{i}}+\sum_{j=0}^{m} \xi_{j} \frac{\partial f}{\partial x_{j}} \tag{131}
\end{equation*}
$$

leaves the system (89) invariant (when one applies it to the independent variables), one will have found all of the first-order defining equations of the infinitesimal contact transformations.

However, that is not true. In order to see that, it will suffice to point out that those equations were deduced from the invariance property of the bracket:

$$
\begin{equation*}
[f, g]_{x, p}=\rho[f, g]_{y, p} \tag{132}
\end{equation*}
$$

when applied to the particular functions $y_{i}$ and $q_{j}$, and that, conversely, they imply the identity (132) by virtue of the formula that gives the bracket of two functions $f, g$ of $2 s+1$ arbitrary variables $z_{0}, z_{1}, \ldots, z_{2 s}$, namely:

$$
\begin{equation*}
[f, g]_{x, p}=\sum_{(\alpha, \beta)} \frac{\partial(f, g)}{\partial\left(z_{\alpha}, z_{\beta}\right)}\left[z_{\alpha}, z_{\beta}\right]_{x, p} . \tag{133}
\end{equation*}
$$

One then concludes that instead of expressing the invariance of the system (89) under $Z f$, it would amount to the same thing to express the invariance of the general identity (132). One does that by writing out that:

$$
Z\left([f, g]_{x, p}\right)=Z \rho \cdot[f, g]_{x, q}
$$

is a consequence of (132), i.e., that one has:

$$
\begin{equation*}
Z\left([f, g]_{x, p}\right)=\frac{Z \rho}{\rho} \cdot[f, g]_{x, p} \tag{134}
\end{equation*}
$$

We set:

$$
\begin{equation*}
Z \rho=\mu \cdot \rho \tag{135}
\end{equation*}
$$

to abbreviate, and the identity (134) will become:

$$
\begin{equation*}
Z\left([f, g]_{x, p}\right)=\mu \cdot[f, g]_{x, p} . \tag{136}
\end{equation*}
$$

In order to make that explicit, one must prolong $Z f$, while considering $f$ and $g$ to be untransformed functions. Consequently, one will have:

$$
0=Z d f=Z\left(\sum_{i=1}^{s} \frac{\partial f}{\partial p_{i}} d p_{i}+\sum_{j=0}^{s} \frac{\partial f}{\partial x_{j}} d x_{j}\right),
$$

so

$$
\begin{array}{rlr}
Z\left(\frac{\partial f}{\partial p_{h}}\right)=-\sum_{i=1}^{s} \frac{\partial f}{\partial p_{i}} \frac{\partial \varpi_{i}}{\partial p_{h}}-\sum_{j=0}^{s} \frac{\partial f}{\partial x_{j}} \frac{\partial \xi_{j}}{\partial p_{h}} & (h=1,2, \ldots, s), \\
Z\left(\frac{\partial f}{\partial x_{k}}\right)=-\sum_{i=1}^{s} \frac{\partial f}{\partial p_{i}} \frac{\partial \varpi_{i}}{\partial x_{k}}-\sum_{j=0}^{s} \frac{\partial f}{\partial x_{j}} \frac{\partial \xi_{j}}{\partial x_{k}} & (k=0,1,2, \ldots, s) .
\end{array}
$$

If one introduces the total derivatives $d / d x_{k}$ and sets:

$$
\begin{equation*}
\lambda=\xi_{0}-\sum_{i=1}^{s} p_{i} \xi_{i} \tag{137}
\end{equation*}
$$

as in no. $\mathbf{1 6}$ [eq. (121)], then one can replace those formulas with the following ones:

$$
\begin{align*}
& Z\left(\frac{\partial f}{\partial p_{h}}\right)=-\sum_{i=1}^{s} \frac{\partial f}{\partial p_{i}} \frac{\partial \varpi_{i}}{\partial p_{h}}-\sum_{i=1}^{s} \frac{d f}{d x_{i}} \frac{\partial \xi_{i}}{\partial p_{h}}-\frac{\partial f}{\partial x_{0}}\left(\frac{\partial \lambda}{\partial p_{h}}+\xi_{h}\right),  \tag{138}\\
& Z\left(\frac{\partial f}{\partial x_{h}}\right)=-\sum_{i=1}^{s} \frac{\partial f}{\partial p_{i}} \frac{\partial \varpi_{i}}{\partial x_{h}}-\sum_{i=1}^{s} \frac{d f}{d x_{i}} \frac{d \xi_{i}}{d x_{h}}-\frac{\partial f}{\partial x_{0}}\left(\frac{d \lambda}{d x_{h}}-\varpi_{h}\right),  \tag{139}\\
& Z\left(\frac{\partial f}{\partial x_{0}}\right)=-\sum_{i=1}^{s} \frac{\partial f}{\partial p_{i}} \frac{\partial \varpi_{i}}{\partial x_{0}}-\sum_{i=1}^{s} \frac{d f}{d x_{i}} \frac{\partial \xi_{i}}{\partial x_{h}}-\frac{\partial f}{\partial x_{0}} \frac{d \lambda}{d x_{0}} . \tag{140}
\end{align*}
$$

On the other hand, the identity (136) is written:

$$
\sum_{h=1}^{s}\left\{\left|\begin{array}{cc}
Z \frac{\partial f}{\partial p_{h}} & \frac{d f}{d x_{h}}  \tag{141}\\
Z \frac{\partial g}{\partial p_{h}} & \frac{d g}{d x_{h}}
\end{array}\right|+\left|\begin{array}{cc}
\frac{\partial f}{\partial p_{h}} & Z \frac{d f}{d x_{h}} \\
\frac{\partial g}{\partial p_{h}} & Z \frac{d g}{d x_{h}}
\end{array}\right|\right\}=\mu[f, g]_{x, p}
$$

I first apply it by taking $f$ and $g$ to be all pairs of independent variables, and I find in succession that:

1. With $x_{j}, x_{k}(j, k>0)$, one has the conditions:

$$
\frac{\partial \xi_{k}}{\partial p_{j}}-\frac{\partial \xi_{j}}{\partial p_{k}}=0 .
$$

2. With $p_{j}, p_{k}$, one has the conditions:

$$
\frac{d \varpi_{k}}{d x_{j}}-\frac{d \varpi_{j}}{d x_{k}}=0 .
$$

3. With $x_{j}, p_{k}(j>0, j \neq 0, j \neq k)$, one has the conditions:

$$
\frac{\partial \varpi_{k}}{\partial p_{j}}+\frac{d \xi_{j}}{d x_{k}}=0
$$

4. With $x_{j}, p_{j}$, one has the conditions:

$$
\frac{\partial \varpi_{j}}{\partial p_{j}}+\frac{d \xi_{j}}{d x_{j}}=-\mu .
$$

5. With $x_{0}, x_{j}$, one has the conditions:

$$
\frac{\partial \lambda}{\partial p_{j}}+\xi_{j}=0
$$

6. With $x_{0}, p_{j}$ (upon taking into account the results $\mathbf{3}$ and $\mathbf{4}$ ) one has the conditions:

$$
\frac{\partial \lambda}{\partial x_{j}}-\varpi_{j}=0
$$

One then verifies that those conditions will suffice for the general relation (141) to be true.
We then see that if we are to have all of equations (120), (121) then we are still lacking the complementary relation:

$$
\begin{equation*}
\frac{\partial \lambda}{\partial x_{0}}+\mu=0 \tag{142}
\end{equation*}
$$

18. The complementary equation. - One has to point out, moreover, that whereas we have $s$ $(2 s+1)$ first-order defining equations for the infinitesimal contact transformations, the bracket relations (89) will give us only $s(2 s+1)-1$ equations for the finite transformations (after eliminating $\rho$ ).

There is then one complementary equation that is yet to be found. It will be provided upon considering the functional determinant:

$$
\begin{equation*}
\Delta=\frac{\partial\left(y_{0}, y_{1}, \ldots, y_{s}, q_{1}, \ldots, q_{s}\right)}{\partial\left(x_{0}, x_{1}, \ldots, x_{s}, p_{1}, \ldots, p_{s}\right)} . \tag{143}
\end{equation*}
$$

More generally, let:

$$
\begin{equation*}
D=\frac{\partial\left(f_{0}, f_{1}, \ldots, f_{2 s}\right)}{\partial\left(x_{0}, x_{1}, \ldots, x_{s}, p_{1}, \ldots, p_{s}\right)}, \tag{144}
\end{equation*}
$$

in which $f_{0}, f_{1}, \ldots, f_{2 s}$ are the untransformed functions. The formulas (138), (139), (140) give:

$$
Z D=-D\left(\frac{\partial \lambda}{\partial x_{0}}+\sum_{i=1}^{s} \frac{d \xi_{i}}{d x_{i}}+\sum_{i=1}^{s} \frac{\partial \varpi_{i}}{\partial p_{i}}\right)
$$

because one can replace the general row in $D$ with:

$$
\frac{\partial f_{h}}{\partial x_{0}}, \quad \frac{\partial f_{h}}{\partial x_{1}}, \quad \ldots, \quad \frac{\partial f_{h}}{\partial x_{s}}, \quad \frac{\partial f_{h}}{\partial p_{1}}, \quad \ldots, \quad \frac{\partial f_{h}}{\partial p_{s}}
$$

If one then takes into account only the relations (4.) of the preceding subsection:

$$
Z D=-D\left(\frac{\partial \lambda}{\partial x_{0}}-s \mu\right)
$$

in such a way that formula (142) is equivalent to:

$$
Z D=(s+1) \mu \cdot D .
$$

If one compares that to formula (136) then one can conclude that:

$$
\frac{Z D}{D}=(s+1) \frac{Z[f, g]}{[f, g]}
$$

which is equivalent to the invariance of the quotient:

$$
\begin{equation*}
D:([f, g])^{s+1} \tag{145}
\end{equation*}
$$

under any contact transformation:
If we apply that to the determinant $\Delta$ and to the transformation (88) then we will have $\Delta_{x, p}=\Delta$, $\Delta_{x, p}=1$, and the relation (132) will give us:

$$
\begin{equation*}
\Delta=\rho^{s+1} \tag{146}
\end{equation*}
$$

That is the desired complementary equation.
19. Automorphic systems. -The fundamental invariants to consider are then the ratios of the Poisson brackets that are formed from $2 s+1$ and the ratio of the fundamental determinant of those functions to the $(s+1)^{\text {th }}$ power of one of the brackets.

To simplify, I shall confine myself to the case in which there are as many unknown functions as independent variables (automorphic systems of the first type). I shall denote the unknown functions by $x_{0}, x_{1}, \ldots, x_{2 s}$, and the independent variables by $u_{0}, u_{1}, \ldots, u_{s}, v_{1}, \ldots, v_{s}$. The bracket will be:

$$
[f, g]_{u, v}=\sum_{i=1}^{s}\left|\begin{array}{ll}
\frac{\partial f}{\partial v_{i}} & \frac{\partial f}{\partial u_{i}}+v_{i} \frac{\partial f}{\partial u_{0}}  \tag{147}\\
\frac{\partial g}{\partial v_{i}} & \frac{\partial g}{\partial u_{i}}+v_{i} \frac{\partial g}{\partial u_{0}}
\end{array}\right|
$$

If $\sigma$ is an undetermined factor then the automorphic system will then be written:

$$
\begin{gather*}
{\left[x_{i}, x_{k}\right]_{u, v}=\sigma \cdot \varphi_{i k}(x) \quad(i, k=0,1,2, \ldots, 2 s),}  \tag{148}\\
\Delta_{u, v}\left(x_{0}, x_{1}, \ldots, x_{2 m}\right)=\sigma^{s+1} \varphi(x) \tag{149}
\end{gather*}
$$

In the right-hand sides, the symbol $x$ is written to denote the set of variables $x_{0}, x_{1}, \ldots, x_{2 s}$, and the left-hand side of (149) denotes the functional determinant of the functions in parentheses with respect to the variables $u_{0}, u_{1}, \ldots, u_{s}, v_{1}, \ldots, v_{s}$.
20. Case of the Pfaff problem. - In order to reduce to that invariant form the system that defines the reduction of the given sheaf $F$ to the canonical form (65) in the case of $r=0, n=2 s, p=1$, which is:

$$
\begin{equation*}
V_{i} f=\frac{\partial f}{\partial v_{i}}, \quad U_{i} f=\frac{\partial f}{\partial u_{i}}+v_{i} \frac{\partial f}{\partial u_{0}} \quad(i=1,2, \ldots, s), \tag{150}
\end{equation*}
$$

with the present notations, we shall first take that system in the form (60), i.e., here it would be:

$$
\left\{\begin{array}{lll}
\left\{v_{i}, u_{k}\right\}=0, & \left\{v_{i}, u_{i}\right\}=\rho, & \left\{v_{i}, u_{0}\right\}=\rho v_{i},  \tag{151}\\
\left\{v_{i}, v_{k}\right\}=0, & \left\{u_{i}, u_{k}\right\}=0, & \left\{u_{0}, u_{i}\right\}=0
\end{array} \quad(i, k=1,2, \ldots, s ; i \neq k\} .\right.
$$

Upon arguing as we did in no. 17 for the identity (132), we see that one can use those equations to express the invariance of the general bracket that was established in no. 8, which will give:

$$
\begin{equation*}
\{f, g\}=\rho[f, g]_{u, v} \tag{152}
\end{equation*}
$$

here, since there are no distinguished transformations in $F$. Now, in order to express the idea that this identity is true, it will suffice to write down that it will be true when one replaces $f$ and $g$ with any two of the variables $x_{0}, x_{1}, \ldots, x_{2 s}$. Indeed, the formula (133) applies to the bracket $\{f, g\}$, because it results from the formula:

$$
\{f, g\}=\sum_{h=0}^{2 s}\left\{f, x_{h}\right\} \frac{\partial g}{\partial x_{h}}
$$

which is itself a consequence of the theorem regarding composite functions.
Hence, if one then sets [cf., eq. (35)]:

$$
\left\{x_{i}, x_{k}\right\}=\frac{1}{\delta}\left|\begin{array}{cccc}
c_{1,1} & \cdots & c_{1, n} & -\xi_{1, i}  \tag{153}\\
\vdots & \cdots & \vdots & \vdots \\
c_{n, 1} & \cdots & c_{n, n} & -\xi_{n, i} \\
\xi_{1, k} & \cdots & \xi_{n, k} & 0
\end{array}\right|=\varphi_{i, k}\left(x_{0}, x_{1}, \ldots, x_{n}\right)=\varphi_{i, k}(x)
$$

(with $n=2 s ; i, k=0,1,2, \ldots, n$ ) then one will have the system:

$$
\begin{equation*}
\rho\left[x_{i}, x_{k}\right]=\varphi_{i, k}(x) \quad(i, k=0,1,2, \ldots, 2 s), \tag{154}
\end{equation*}
$$

which has the form (148), with $\rho \sigma=1$.
The functions $\varphi_{i, k}$ thus-introduced, enjoy some properties that we shall now indicate.
One will first have $\varphi_{i, k}=-\varphi_{k, i}$ (no. 5).
On the other hand, the transformations:

$$
\left\{x_{i}, f\right\}=\frac{1}{\delta}\left|\begin{array}{cccc}
c_{1,1} & \cdots & c_{1, n} & -\xi_{1, i}  \tag{155}\\
\vdots & \cdots & \vdots & \vdots \\
c_{n, 1} & \cdots & c_{n, n} & -\xi_{n, i} \\
X_{1} f & \cdots & X_{n} f & 0
\end{array}\right|=\sum_{k=0}^{n} \varphi_{i, k}(x) \frac{\partial f}{\partial x_{k}} \quad(i=0,1,2, \ldots, n)
$$

are transformations of the sheaf. There are $n$ divergent ones among them. Indeed, in the matrix of $\xi_{k, i}$, there is at least one determinant of degree $n$ that is non-zero. Suppose, to fix ideas, that it is the one that corresponds to $i=1,2, \ldots, n$ : The $\left\{x_{i}, f\right\}$ will then be divergent for $i=1,2, \ldots, n$.

In order to prove that, suppose that the contrary is true, i.e., that one has an identity:

$$
\begin{equation*}
\sum_{i=1}^{n} \lambda_{i}\left\{x_{i}, f\right\}=0 \tag{156}
\end{equation*}
$$

and set:

$$
\begin{equation*}
\sum_{i=1}^{n} \lambda_{i} \xi_{k i}=\xi_{k} \quad(k=1,2, \ldots, n) \tag{157}
\end{equation*}
$$

The $\xi_{k}$ are not all zero, and the identity (156) will be written:

$$
\left|\begin{array}{cccc}
c_{1,1} & \cdots & c_{1, n} & \xi_{1}  \tag{158}\\
\vdots & \cdots & \vdots & \vdots \\
c_{n, 1} & \cdots & c_{n, n} & \xi_{n} \\
X_{1} f & \cdots & X_{n} f & 0
\end{array}\right|=0
$$

Now, the determinant of the $c_{i, k}$ is not zero, so one can satisfy the equations:

$$
\sum_{i=1}^{n} c_{k, i} \rho_{i}+\xi_{h}=0 \quad(k=1,2, \ldots, n)
$$

and the $\rho_{i}$ are not all zero. However, it will then suffice to multiply the first $n$ columns of (158) by $\rho_{1}, \ldots, \rho_{n}$ and add them to the last one in order to reduce equation (158) to the form:

$$
\sum_{i=1}^{n} \rho_{i} X_{i} f=0
$$

which is an identity that is impossible. There is a contradiction then.
Hence, the $\left\{x_{i}, f\right\}$ define the given sheaf $F$ entirely.
Furthermore, observe the formula:

$$
\begin{equation*}
\{\varphi, f\}=\sum_{i=0}^{n} \sum_{k=0}^{n} \varphi_{i k}(x) \frac{\partial \varphi}{\partial x_{i}} \frac{\partial f}{\partial x_{k}}, \tag{159}
\end{equation*}
$$

which results from equations (155) and can also be written:

$$
\{\varphi, f\}=\sum_{(i, k)} \varphi_{i, k}(x)\left|\begin{array}{ll}
\frac{\partial \varphi}{\partial x_{i}} & \frac{\partial \varphi}{\partial x_{k}}  \tag{160}\\
\frac{\partial f}{\partial x_{i}} & \frac{\partial f}{\partial x_{k}}
\end{array}\right| .
$$

Finally, I say that if $Z f$ is a conveniently-chosen infinitesimal transformation then one will have:

$$
\begin{equation*}
\left(\left\{x_{i}, f\right\},\left\{x_{j}, f\right\}\right) \equiv\left\{x_{i}, x_{j}\right\} Z f \quad(\bmod F), \tag{161}
\end{equation*}
$$

in such a way that the $\varphi_{i, j}$ are structure functions for the sheaf when put into the form (155), at the same time as those functions are coefficients of the basis transformation (155).

In order to see that, it will suffice to refer to the Poisson brackets themselves. With the notations (150), one has:

$$
\begin{gathered}
{\left[x_{i}, f\right]=\sum_{k=1}^{s}\left(\frac{\partial x_{i}}{\partial v_{k}} U_{k} f-\frac{d x_{i}}{d u_{k}} V_{k} f\right),} \\
\left(V_{k} f, U_{k} f\right)=\frac{\partial f}{\partial u_{k}} .
\end{gathered}
$$

Hence:

$$
\begin{equation*}
\left(\left[x_{i}, f\right],\left[x_{j}, f\right]\right) \equiv \sum_{k=1}^{s}\left(\frac{\partial x_{i}}{\partial v_{k}} \frac{d x_{j}}{d u_{k}}-\frac{\partial x_{j}}{\partial v_{k}} \frac{d x_{i}}{d u_{k}}\right) \frac{\partial f}{\partial u_{0}} \quad(\bmod F), \tag{162}
\end{equation*}
$$

i.e.:

$$
\begin{equation*}
\left(\left[x_{i}, f\right],\left[x_{j}, f\right]\right) \equiv\left[x_{i}, x_{j}\right] \frac{\partial f}{\partial u_{0}} \quad(\bmod F) \tag{163}
\end{equation*}
$$

Upon taking (152) into account, one will conclude that:

$$
\begin{equation*}
\left(\left[x_{i}, f\right],\left[x_{j}, f\right]\right) \equiv \rho\left\{x_{i}, x_{j}\right\} \frac{\partial f}{\partial u_{0}}, \tag{164}
\end{equation*}
$$

i.e., the formula (161), with [cf., no. 8, eq. (59)]:

$$
\begin{equation*}
Z f=\rho \frac{\partial f}{\partial u_{0}} \tag{165}
\end{equation*}
$$

21. Calculating the complementary equation. - It remains for us to calculate equation (149). To that end, I shall first consider the determinant:

$$
\begin{equation*}
\Delta_{0}=\frac{\partial\left(x_{1}, \ldots, x_{m} ; x_{m+1}, \ldots, x_{n}\right)}{\partial\left(u_{1}, \ldots, u_{m} ; v_{1}, \ldots, v_{m}\right)} . \tag{166}
\end{equation*}
$$

Its rows can be replaced with:

$$
\frac{d x_{k}}{d u_{1}}, \ldots, \quad \frac{d x_{k}}{d u_{m}}, \frac{d x_{k}}{d v_{1}}, \ldots, \quad \frac{d x_{k}}{d v_{m}} \quad(k=1,2, \ldots, n)
$$

or by

$$
-\frac{d x_{k}}{d v_{1}}, \ldots, \quad-\frac{d x_{k}}{d v_{m}}, \frac{d x_{k}}{d u_{1}}, \quad \ldots, \quad \frac{d x_{k}}{d u_{m}} \quad(k=1,2, \ldots, n)
$$

If one multiplies the two determinants that were just written down then one will get the determinant whose general element is the bracket $\left[x_{h}, x_{k}\right]$. The skew-symmetric determinant of even degree $n$ $=2 s$ is then the square of $\Delta_{0}$. However, one knows how to form the expression for the square root of a skew-symmetric determinant of even degree $n=2 s$ as a function of its $s(2 s-1)$ independent elements. One can then consider the expression for $\Delta_{0}$ as an entire rational function that is homogeneous of degree $s$ in terms of the brackets $\left[x_{h}, x_{k}\right]$ to be known. Upon taking into account the identity (152), one will then have:

$$
\begin{equation*}
\rho^{s} \Delta_{0}=G_{0}\left(\left\{x_{i}, x_{l}\right\}\right) \tag{167}
\end{equation*}
$$

in which the right-hand side is the square root of the determinant that is defined by the $\left\{x_{i}, x_{l}\right\}=$ $\varphi_{i, j}, i, j=1,2, \ldots, n$.

The same result applies mutatis mutandis to the other coefficients $\Delta_{1}, \ldots, \Delta_{n}$ of the determinant $\Delta$, which is supposed to be developed in the form:

$$
\begin{equation*}
\Delta=\Delta_{0} \frac{\partial x_{0}}{\partial u_{0}}+\Delta_{1} \frac{\partial x_{1}}{\partial u_{0}}+\cdots+\Delta_{n} \frac{\partial x_{n}}{\partial u_{0}} . \tag{168}
\end{equation*}
$$

As for the factors $\partial x_{k} / \partial u_{0}$, one can replace them in the $\Delta$ with quantities of the form:

$$
\frac{\partial x_{k}}{\partial u_{0}}+\sum_{h=1}^{m} \lambda_{h} \frac{d x_{1}}{d u_{h}}+\sum_{h=1}^{m} \mu_{h} \frac{\partial x_{k}}{\partial v_{h}} \quad(k=1,2, \ldots, n) .
$$

One can then substitute the values that are inferred from (163) or (164) upon choosing the same system of integer pairs $i, j$ for all of those values. The bracket of the two known infinitesimal transformations:

$$
\left\{x_{i}, f\right\}=\sum_{\alpha=0}^{n} \varphi_{i, \alpha} \frac{\partial f}{\partial x_{\alpha}}, \quad\left\{x_{j}, f\right\}=\sum_{\alpha=0}^{n} \varphi_{j, \alpha} \frac{\partial f}{\partial x_{\alpha}}
$$

then appears in the left-hand side of (164). One will then have:

$$
\begin{equation*}
\rho \frac{\partial f}{\partial u_{0}} \equiv \sum_{\alpha=0}^{n} \psi_{\alpha} \frac{\partial f}{\partial x_{\alpha}} \quad(\bmod F) \tag{169}
\end{equation*}
$$

in which the $\psi_{\alpha}$ are known functions of the $\varphi_{h, k}$ and their first-order partial derivatives, and one will have the desired expression:

$$
\begin{equation*}
\rho^{s+1} \Delta=\sum_{\alpha=0}^{n} \psi_{\alpha} \Delta_{\alpha}=\varphi(x) \tag{170}
\end{equation*}
$$

which is necessarily found to be independent of the particular choice of the indices $i, j$ that was found in the course of the calculations that were just indicated. That will result from the identity (161) that was established above.

One will then recover the essential character of equation (170) that it represents the result of the integrability conditions on the system (154).
22. Converse. - Conversely, take some equations of the type (154). Not all of the minors of degree $n$ in the determinant of the $\varphi_{i, k}$ (which must be skew-symmetric due to its left-hand side) can be zero, for the same reason.

Those equations are equivalent to the identities:

$$
\begin{equation*}
\rho\left[x_{i}, f\right]=\sum_{k=0}^{n} \varphi_{i, k}(x) \frac{\partial f}{\partial x_{k}} \quad(i=0,1,2, \ldots, n), \tag{171}
\end{equation*}
$$

and one can denote the right-hand sides by the notation $\left\{x_{i}, f\right\}$, and consider the sheaf $F$ that is defined by $n$ of those mutually-divergent transformations.

One will get the integrability conditions upon defining the brackets of corresponding sides of the equations with all of the pairs of the two equations (171). If we set:

$$
\begin{equation*}
\{\varphi, f\}=\sum_{i=0}^{n} \sum_{k=0}^{n} \varphi_{i, k}(x) \frac{\partial \varphi}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} \tag{172}
\end{equation*}
$$

then the right-hand sides can be written $\left\{x_{i}, f\right\}$, and as in no. 20, one will find that:

$$
\begin{equation*}
\rho \frac{\partial f}{\partial u_{0}}\left\{x_{i}, x_{j}\right\} \equiv\left(\left\{x_{i}, f\right\},\left\{x_{j}, f\right\}\right) \quad(\bmod F) \tag{173}
\end{equation*}
$$

for all the pairs of indices $i, j$. One has, moreover:

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=\varphi_{i j} \quad(i, j=0,1,2, \ldots, n) \tag{174}
\end{equation*}
$$

The integrability conditions in question are then written:

$$
\begin{equation*}
\left(\left\{x_{i}, f\right\},\left\{x_{j}, f\right\}\right) \equiv \varphi_{i j} Z f \quad(\bmod F) \tag{175}
\end{equation*}
$$

upon denoting a certain infinitesimal transformation by $Z f$.
Upon introducing two arbitrary functions $\theta, \psi$ of the variables $x_{0}, x_{1}, \ldots, x_{n}$, one can replace then with the single formula:

$$
\begin{equation*}
(\{\theta, f\},\{\psi, f\}) \equiv\{\theta, \psi\} Z f \quad(\bmod F) \tag{176}
\end{equation*}
$$

Upon taking into account the formulas thus-acquired, the calculation of the complementary equation is achieved as in no. 21. Moreover, it does not contribute to the integration and will preserve its role as the integrability conditions for equations (171), as we have seen before.

It remains to be proved that the integrability conditions (175) are sufficient and to indicate how one can carry out the integration.

To that end, I point out that the infinitesimal transformation $\{\varphi, f\}$ is in involution with all transformations of the sheaf $F$ that leave the function $\varphi$ invariant, because from (176), the bracket of $\left\{\varphi_{1}, f\right\}$ with an arbitrary transformation of $F$ :

$$
\begin{equation*}
\sum_{\alpha=0}^{n} \lambda_{\alpha}\left\{x_{\alpha}, f\right\} \tag{177}
\end{equation*}
$$

is congruent $(\bmod F)$ to:

$$
-\sum_{\alpha=0}^{n} \lambda_{\alpha}\left\{x_{\alpha}, \varphi\right\} Z f,
$$

in such a way that it is zero for:

$$
\sum_{\alpha=0}^{n} \lambda_{\alpha}\left\{x_{\alpha}, \varphi\right\}=0
$$

i.e., when $\varphi$ is an invariant of the transformation (177).

Moreover, the bracket $\{\varphi, f\}$ has the same significance relative to the sheaf $F$ as the one that was introduced in no. $\mathbf{5}$, and one can repeat all of the analysis of no. $\mathbf{6}$.

One can then give $u_{0}$ arbitrarily, take $u_{1}$ to be an integral of $\left\{u_{0}, f\right\}=0$, and take $u_{2}$ to be an integral of the (complete) system $\left\{u_{0}, f\right\}=0,\left\{u_{1}, f\right\}=0$, and so on, and finally take $u_{s}$ to be an integral of the (complete) system:

$$
\left\{u_{0}, f\right\}=0, \quad\left\{u_{1}, f\right\}=0, \quad \ldots, \quad\left\{u_{s-1}, f\right\}=0
$$

As for the calculation of the $v_{k}$, one will quite easily get a method that is patterned on the one in no. $\mathbf{8}$ upon pointing out that one has:

$$
\left[u_{0}, f\right]=-\sum_{k=1}^{s} v_{k} \frac{\partial f}{\partial u_{k}}, \quad\left[u_{h}, f\right]=-\frac{\partial f}{\partial v_{k}} \quad(h=1,2, \ldots, s),
$$

and thus, the identity:

$$
\begin{equation*}
\left[u_{0}, f\right]=\sum_{k=1}^{s} v_{k}\left[u_{k}, f\right] . \tag{178}
\end{equation*}
$$

On the other hand, since equations (171) are equivalent to the identity:

$$
\begin{equation*}
\{\varphi, f\}=\rho[\varphi, f], \tag{179}
\end{equation*}
$$

one can replace (178) with:

$$
\begin{equation*}
\left\{u_{0}, f\right\}=\sum_{k=1}^{s} v_{k}\left\{u_{k}, f\right\}, \tag{180}
\end{equation*}
$$

and that identity is what will provide the $v_{k}$ without a new integration.


[^0]:    $\left({ }^{1}\right)$ In the footnotes that refer to that paper, we shall denote it by the letter $M$, to abbreviate.
    $\left({ }^{2}\right)$ That is, one that is composed of a family of integral multiplicities (see the following footnote), such that one multiplicity passes through each point $\left(x_{1}, \ldots, x_{m}\right)$.

[^1]:    ( ${ }^{1}$ ) An $r$-dimensional multiplicity is an integral multiplicity of a sheaf when it admits $r$ (divergent) transformations of that sheaf.
    $\left({ }^{2}\right)$ Of course, that implies that those characteristics must exist to begin with.

[^2]:    $\left({ }^{1}\right)$ The case that corresponds to it in the theory of Pfaff systems was studied by Cartan [Bulletin de la Société mathématique de France 29 (1901), pp. 233], and previously by E. von Weber [Münchener Sitzungsberichte 25 (1895), pp. 423].
    $\left({ }^{2}\right)$ C. R. Acad. Sci. Paris 184 (1927), pp. 143.

[^3]:    $\left({ }^{1}\right)$ Bulletin de la Société mathématique de France 42 (1914), pp. 14-15.
    $\left(^{2}\right)$ See my article in Acta mathematica 28 (1904), pp. 311.

[^4]:    $\left({ }^{1}\right)$ It is what we call the distinguished subsheaf or the characteristic subsheaf of $F$.

[^5]:    ${ }^{1}{ }^{1}$ M., nos. 13-14, pp. 362, et seq.

[^6]:    ${ }^{1}$ ) M., no. 2, pp. 345.
    $\left({ }^{2}\right)$ More generally, the resultant of two sheaves is defined by all linear, homogeneous combinations of the transformations of one and the other sheaf. One will obtain it by the juxtaposing the bases for the two sheaves when they do not have a common subsheaf.

[^7]:    $\left({ }^{1}\right)$ Because the bracket of two arbitrary transformations of $F$ is a combination of transformations of the basis for $F$ and the transformation $Z$. One must always keep that fact in mind in all of this subsection.

[^8]:    ( ${ }^{1}$ ) M., no. 4, pp. 346-347.
    $\left(^{2}\right)$ M., no. 15, pp. 367.

[^9]:    $\left({ }^{1}\right)$ I recall that this theorem was given in the case where one does not introduce the variables $t_{1}, \ldots, t_{q} ; z_{1}, \ldots, z_{r}$ (but with a different statement, since one deals with a Pfaff system) by Cartan [Bulletin de la Société mathématique de France 42 (1914), pp. 14-15.]
    $\left(^{2}\right)$ See my treatise in Acta mathematica 28 (1904), pp. 311. I remark that one can attach the property in question to the fact that the sheaf $F$ admits a group of transformations that is isomorphic to the general group of contact transformations, but we shall depart from that viewpoint here. I also remark that for $s=t$, the property in question is exact in the stated form only in the case where the derived sheaf $F^{\prime}$ of $F$ is complete. (See the end of no. 12 and section III of the present treatise.)

[^10]:    $\left(^{1}\right)$ For the method that is followed, see JANET, Journal de Mathématiques pures et appliquées (8) 3 (1920), pp. 111.

