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Remarks on the theory of conical refraction

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1. – The completely discontinuous behavior of light rays in a biaxial crystal that occurs, according to **Fresnel**'s theory, when a wave normal approaches an optical axis has been a cause of much concern. In fact, it would be difficult to find an analogue for it in all of physics, and it is precisely that fact that perhaps explains the sensation that the **Hamilton-Lloyd** discovery of conical refraction represented in their time.

In a previous paper ([1]), I proved that the observed phenomenon that commonly leads back to the singular behavior of a wave that propagates along an optical axis is not, in truth, based upon that. In what follows, I would like to communicate an argument that makes it likely to me that *in reality, that singular behavior does not even exist.* Since that singular behavior was not necessary for the derivation of the result that is presented in the present article, I have not introduced it here, but rather I have added it as a supplement.

In the previous paper, I showed that the modern (physical) interpretation of light rays in terms of the energy flux is not necessarily consistent with the older (geometric) one in terms of the radius vector of the wave surface. In fact, in general, it is plausible that only an *approximate* agreement between the two statements exists, and even then, only with additional hypothetical adjustments – e.g., the assumption that when several waves advance in the same space, the two energy fluxes do not mutually interact in an observable way. Such an assumption proves to be understandable in the cases where that component of the energy flux that is produced by *both* waves simultaneously and is nowhere absent will imply vanishing (temporal or spatial) *mean values* over all appreciable *times* or *spatial regions*. However, it is precisely that assumption that will not be fulfilled when one is dealing with the simultaneous propagation of an ordinary and an extraordinary wave in directions that lie close to an optical axis. As a rule, those waves are *coherent*, and as a result, the *temporal* mean value of those terms in the energy current will not be equal to zero. The difference between their velocities will *vanish* when one approaches the optical axes and as a result their phase difference will slowly alternate and ultimately become constant; thus, appealing to a *spatial* mean value would not be permissible in this case. It seems to me that it would then follow that one cannot decompose the energy flux that corresponds to the two waves in the immediate neighborhood of an optical axis, *but one must always consider it to be a unified entity.* However, one would then arrive at results that would deviate noticeably from the older theory.

([1]) **W. Voigt**, Ann. Phys. (Leipzig) **18** (1905), pp. 645.

2. – As before, I would like to employ a coordinate system whose Z-axis is the wave normal, whose X-axis is the (magnetic) direction of polarization of the ordinary one, and whose Y-axis falls along the direction of polarization of the extraordinary wave. If A, B are the components of the oscillating magnetic field strength and U, V, W are the components of the energy current then when one employs the formulas (16), (17), (18) of the previous article (when they are specialized to inactive crystals), that will yield the equations:

$$(1)\qquad \begin{cases} U = \dfrac{v}{8\pi}(\Theta_1-\Theta_3)(B^2 n_e \cos i + AB n_0 \sin i)\sin\Phi, \\ V = -\dfrac{v}{8\pi}(\Theta_1-\Theta_3)(A^2 n_o \sin i + AB n_e \cos i)\sin\Phi, \\ W = \dfrac{v}{8\pi}\left(\dfrac{A^2}{n_0}+\dfrac{B^2}{n_0}\right), \end{cases}$$

which are valid in the immediate neighborhood of an optical axis. In them, Θ_1 and Θ_2 denote the extreme values of the squares of the reciprocal index of refraction, n_o and n_e denote the indices of refraction for the ordinary and extraordinary waves, which are parallel to the XY-plane, and i is the angle between the ZX-plane and the plane of the optical axes, which subtend the angle Φ in their own right.

n_o and n_e differ only slightly in the immediate neighborhood of an optical axis, in such a way that one can replace them with their mean value n in (1). One now introduces another coordinate cross in place of the original one XY whose X_0-axis is parallel to the old one and whose Y_0-axis is normal to the plane of the optical axes, so one sets:

$$(2)\qquad \begin{cases} U_0 = U\cos i - V\sin i, \quad V_0 = U\sin i + V\cos i, \\ A_0 = A\cos i - B\sin i, \quad B_0 = A\sin i + B\cos i, \end{cases}$$

and (1) will immediately give:

$$(3)\qquad \begin{cases} U_0 = \dfrac{vn}{8\pi}(\Theta_1-\Theta_3)B_0^2\sin\Phi, \\ V_0 = -\dfrac{vn}{8\pi}(\Theta_1-\Theta_3)A_0B_0\sin\Phi, \\ W_0 = \dfrac{v}{4\pi n}(A_0^2+B_0^2). \end{cases}$$

In these formulas, the influence of the direction of propagation is found only in the abbreviations A_0 and B_0 , which consist of two components with different phases, from (2), since the ordinary oscillation A propagates with a different velocity than the extraordinary one B. For example, at a distance z from the point of entry into the crystal, one will have:

(4) $$A = F\cos\alpha\left(t - \frac{z}{\omega_0} - f\right), \quad B = G\cos\alpha\left(t - \frac{z}{\omega_0} - g\right).$$

Hence, when one neglects the weakening of the incident wave under reflection, one can think of representing it by:

(5) $$\bar{A} = F\cos\alpha\,(t - f), \qquad \bar{B} = G\cos\alpha\,(t - g),$$

or also by:

(6) $$\bar{A}_0 = F_0\cos\alpha\,(t - f_0), \qquad \bar{B}_0 = G_0\cos\alpha\,(t - g_0).$$

If the wave normal is shifted to the optical axis A_1 then the energy current will clearly change *continuously*.

2. – The general discussion of these results is obviously very involved, but the direct result of it is that when one abbreviates (3) with:

(7) $$U_0 = h B_0^2, \quad V_0 = -h A_0 B_0, \quad W_0 = k(A_0^2 + B_0^2),$$

that will yield the general relation:

(8) $$U_0 W_0 = \frac{k}{h}(U_0^2 + V_0^2).$$

Therefore, no matter what law applies to the incident oscillations, and no matter how the direction of the energy flux oscillates accordingly, for any location of the propagating wave, the latter will continually remain on a skew circular cone whose equation is (8), which is bisected by the plane of the optical axes with one edge on one optical axis, while the second edge lies in the plane of the optical axes and subtends an angle of χ with the first edge that is given by:

(9) $$\tan\chi = \frac{h}{k} = \tfrac{1}{2}\, n^2\,(\Theta_1 - \Theta_3)\sin\Phi\,.$$

Here and in what follows, if we draw the trace of the energy current at the location in question on a plane that is normal to the optical axis A_1 at a unit distance from that location then that will yield a circle of diameter $D = h / k$ that contacts the Y_0-axis at the coordinate origin (Fig. 1). Its center C_1 is very close to the trace of the ray axis that belongs to A_1 .

For the sake of simplicity, the amplitudes F_0 and G_0 of the incident oscillations might be further drawn with a unit of measurement that makes their resultant representable by a chord through the location A_1 (cf., Fig. 1).

The cone that goes through (8) is identical to the cone in the theory of internal conical refraction. However, its appearance in no way expresses, say, the idea that the energy current spreads out in that cone. Moreover, since A_0 and B_0 alternate periodically with t and z, the energy flux will move along a *helix* that itself rotates in such a way that each line element on it that corresponds to z and is in the same direction will remain such a cone.

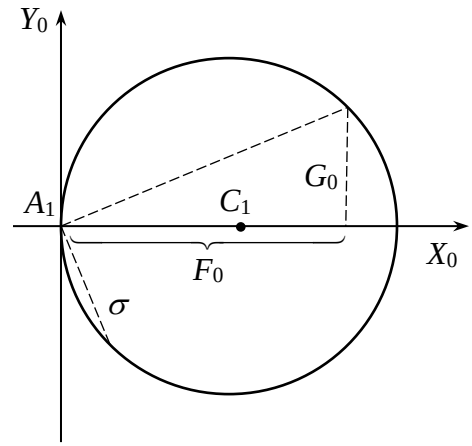

Figure 1.

4. – For the direction of the optical axis itself, for which $\omega_o = \omega_e = \omega$, we can write:

(10) $$A_0 = F_0 \cos\alpha\left(t - \frac{z}{\omega} - f_0\right), \quad B_0 = G_0 \cos\alpha\left(t - \frac{z}{\omega} - g_0\right),$$

so

(11) $$\begin{cases} U_0 = \quad hG_0^2 \cos^2\alpha\left(t - \frac{z}{\omega} - g_0\right), \\ V_0 = -hF_0 G_0 \cos\alpha\left(t - \frac{z}{\omega} - f_0\right)\cos\alpha\left(t - \frac{z}{\omega} - g_0\right), \\ W_0 = \quad k\left[F_0^2 \cos^2\alpha\left(t - \frac{z}{\omega} - f_0\right) + G_0^2 \cos^2\alpha\left(t - \frac{z}{\omega} - g_0\right)\right]. \end{cases}$$

The simplest case to which those formulas can be applied is that of *incident linearly-polarized light*, which is given by $f_0 = g_0$. Here, the direction of the energy current is constant in time and space, so one will have:

(12) $$U_0 : V_0 : W_0 = hG_0^2 : -hG_0 F_0 : k(F_0^2 + G_0^2).$$

Accordingly, the ray that corresponds to a wave normal that corresponds to the optical axis for incident linearly-polarized light will lie on the cone of the theoretical internal conical refraction in the plane that can be laid through the optical axis normal to the plane of polarization of the incident light. It is the locus of the (ordinary or extraordinary) rays that correspond to one of the wave normals with the direction of polarization in question that is close to the optical axis A_1 . Its trace in Fig. 1 is the point σ.

In the case considered, the older theory yields a ray *cone* with a maximum intensity at the location σ and a minimum at the diametrically-opposed point.

If linearly-oscillating light with a randomly-varying polarization plane in incident then the point σ will also glide randomly on the circle in the Figure, and a type of conical refraction must arise because of that.

5. – A second simple case is that of incident *circularly-polarized light*, for which one can set:

(13) $$A_0 = F_0 \sin\alpha\left(t - \frac{z}{\omega}\right), \quad B_0 = \pm F_0 \cos\alpha\left(t - \frac{z}{\omega}\right).$$

When $\alpha\,(t - z / \omega)$ is abbreviated by T, from (7), one will then have:

(14) $$U_0 = h F_0^2 \cos^2 T, \quad V_0 = \mp h F_0^2 \sin T \cos T\,, \quad W_0 = 2k F_0^2$$

here. One easily recognizes that the trace of the energy current traverses the circle around C_1 here with the same velocity, and indeed in the + or – sense according to whether the + or the – sign is true in (13), respectively. The current then traverses a helix with the C_1-direction as its axis. One circuit of the helix has a height of one wavelength. The perceptible ray in question would accordingly coincide appreciably with the direction with of the ray axis C_1 .

In the older theory, circularly-polarized light would produce the ray cone with essentially the same intensity everywhere.

6. – The more general case of incident *elliptically-polarized* light serves as a transition between the two that were previously discussed.

For it, we set:

(15) $$A_0 = F_0 \cos\,(T - \delta)\,, \quad B_0 = G_0 \cos\,(T + \delta)\,,$$

and obtain:

(16) $$\frac{V_0}{U_0} = -\frac{F_0}{G_0}\frac{\cos(T-\delta)}{\cos(T+\delta)},$$

from which, when $0 < \delta < \pi / 2$, one will have for:

$$T = \quad 0, \quad \delta, \quad \frac{\pi}{2} - \delta, \quad \frac{\pi}{2}, \quad \frac{\pi}{2} + \delta, \quad \pi = \delta,$$

$$\frac{V_0}{U_0} = -\frac{F_0}{G_0}, \quad -\frac{F_0}{G_0 \cos 2\delta}, \quad \mp\infty, \quad +\frac{F_0}{G_0}, \quad 0, \quad -\frac{F_0 \cos 2\delta}{G_0},$$

respectively, with $T = \pi$ once more set to be the origin.

These six arguments T are indicated in the following Fig. 2 for directions of the component $J = \sqrt{U_0^2 + V_0^2}$ that is parallel to the plane of the wave for $\delta = \pi / 6$. The values of the argument that belong to the six directions will advance by $\pi / 6$, so the corresponding time intervals will be *equal* for constant z. One sees that for incident elliptically-polarized light the rotation of the trace of the energy current on the base circle will take place with *non-uniform* velocity. The smallest velocity lies close to the location 2, while the largest lies close to the location 5.

The energy current will then exist on a helix with a periodically-alternating height, and the perceptible *mean* direction of the ray will deviate from the direction C_1 on the side of the point 2.

7. – The *arithmetic mean values* $\bar{U}_0$, $\bar{V}_0$, and $\bar{W}_0$ of the energy current are then given by:

$$\bar{U}_0 = \tfrac{1}{2} h G_0^2, \quad \bar{V}_0 = -\tfrac{1}{2} h F_0 G_0 \cos\alpha (f_0 - g_0), \quad \bar{W}_0 = \tfrac{1}{2} k (F_0^2 + G_0^2), \tag{17}$$

respectively. One can assume *that they correspond to the direction of the perceptible ray.* $\alpha (f_0 - g_0)$ will then replace the 2δ above. The trace of that direction can be easily constructed in the Figure.

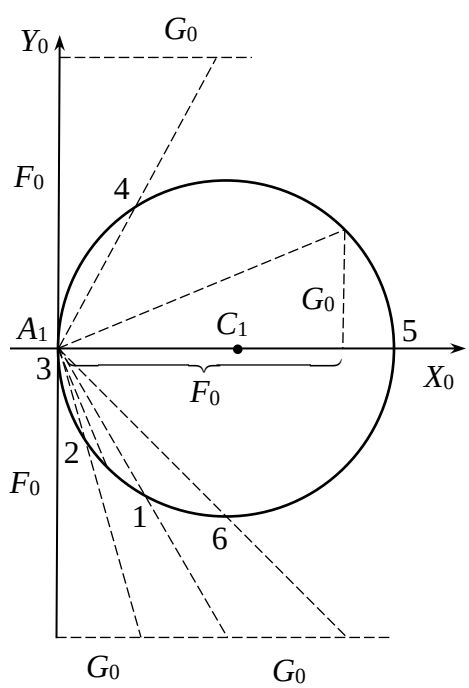

Figure 2.

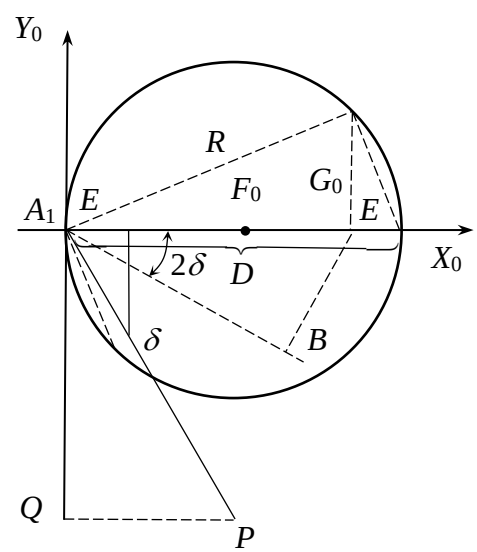

Figure 3.

Once one makes $\overline{A_1 Q} = \overline{A_1 B} = F_0 \cos 2\delta$ and $\overline{QP} = G_0$, that will imply that:

$$\frac{\bar{V}_0}{\bar{U}_0} = -\frac{F_0 \cos\ 2\delta}{G_0} = -\frac{\overline{A_1 Q}}{\overline{QP}},$$

so the trace must then lie on the line $\overline{A_1 P}$. Furthermore (cf., the Fig.):

$$\frac{\bar{U}_0}{\bar{W}_0} = \frac{h}{k}\frac{G_0^2}{F_0^2 + G_0^2} = \frac{h}{k}\frac{G_0^2}{R^2}.$$

Now, from (3), the base circle has a diameter of $D = h / k$, and furthermore, one has $D : R = R : F_0$, so one will have:

$$\bar{U}_0 / \bar{W}_0 = G_0^2 / F_0 ,$$

and since (Fig. 3) $E : G_0 = G_0 : F_0$, it will ultimately follow that:

$$\bar{U}_0 / \bar{W}_0 = E.$$

One only needs to draw E along the + X_0-axis from A_1 and erect an altitude to the X_0-axis at the endpoint. Its point of intersection with the line $\overline{A_1 P}$ will give the desired trace $\bar{\sigma}$ of the mean energy current.

That construction can greatly clarify the effect of the amplitude ratio (F_0 / G_0) and the delay (2δ) in the direction of the ray that corresponds to one of the wave normals that falls along the optical axis A_1.

If natural light is incident, which one can regard as alternating very quickly and randomly and being elliptically polarized, then the ray will change its direction randomly in such a way that its trace will assume all possible positions inside of the base circle.

Since only vanishing amounts of energy can propagate in a geometric direction, the foregoing considerations will have just as much *practical* significance as **Hamilton**'s result on the ray cone. At least, the proof that the results of the older and newer theories of rays are very different in the special cases considered does not seem unimportant, nor that the newer theory does not imply the objectionable discontinuity to which the older one led. Of course, there is still one point in the newer theory that remains unclear: viz., the type of transition that takes place from a unified energy current into two separate perceptible rays. The investigation of that question remains to be done.

For the theory of perceptible internal conical refraction that I have developed previously, it is naturally irrelevant whether the behavior of the rays that correspond to wave normals in the neighborhood of optical axes are *infinitely* fast or vary with *exceptionally high, but finite,* velocity.

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