FOUNDATIONS

OF A

THEORY OF CURVATURE

OF

FAMILIES OF CURVES

BY

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FOREWORD

If one imagines that the coordinates of the points of a curve depend upon one or two (undetermined) parameters, in addition to the one variable, then one will get a simply-infinite or doubly-infinite *family of curves*, or, as the French and Italian mathematicians say, a *congruence of curves*.

The curvature properties of such a family depend upon the curvature properties of the individual curves of the family, which are studied using the methods of the theory of curves, and upon the way that the curves are arranged. The investigation of that arrangement makes it necessary to consider the orthogonal trajectories of the family, and therefore to introduce composed differentiations that can be extended to operations that can be called *derivatives with respect to arc-length* when one adds the requirement of invariability. The first part of the following treatise, which deals with simply-infinite families of curves, and indeed ones that lie in the planes, as well as curved surfaces, is dedicated to presenting and applying that concept.

A doubly-infinite family of curves can be given by finite equations, as well as by differential equations. The first of those cases will be treated in the second part of this paper, where, at the same time, the more important questions about families of surfaces will be discussed; the third part of the paper is dedicated to the second case.

All of this can be regarded as a generalization of the theory of surfaces, as well as a theory of the most general curvilinear, but rectangular, coordinates. Namely, the Cartesian conception of coordinates is capable of two types of generalization. Firstly, one can replace the three mutually-perpendicular families of parallel planes with three mutually-perpendicular families of curves, and thus obtain Lamé's theory, whose methods coincide with the ones that are employed in the theory of surfaces.

Secondly, however, one can replace the three mutually-perpendicular families of parallel straight lines with three mutually-perpendicular families of curved lines, or what amounts to the same thing, a family of curves, along with the two mutually-perpendicular families of its orthogonal trajectories. Should the last two be determined by the first one alone then the two families of lines of curvature of the first kind of the first family of curves could naturally stand in place of them.

In regard to the presentation of ideas, the author has endeavored to assume as little prior knowledge as possible on the part of the reader.

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R. v. Lilienthal

TABLE OF CONTENTS

PART ONE

Simply-infinite family of curves

PART TWO

Doubly-infinite families of curves that are established by finite equations

---

PART THREE

Doubly-infinite families of surface defined by differential equations

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PART ONE

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Simply-infinite families of curves

§ 1. – Curvature of a planar, simply-infinite family of curves.

A planar, simply-infinite family of curves can be represented in two different ways: In one case, it is represented by *finite* equations in which the Cartesian coordinates x, y of a point on the curve are given as functions of two variables p and t, say in the form:

$$x = f_1 (p, t), \qquad y = f_2 (p, t),$$

in which t remains fixed along each individual curve of the family and its value varies only from one curve to another, and in the other case, by a first-order *differential equation*, say:

$$dx : dy = \varphi_1 (x, y) : \varphi_2 (x, y) .$$

A family of curves is most closely linked with the family of its orthogonal trajectories. Both families collectively define a rectilinear system of curvilinear coordinate curves.

The tangents to the curves t = const. subtend an angle with the X-axis (Y-axis, resp.) whose cosine will be called κ (λ, resp.). Likewise, ξ (η, resp.) shall denote the cosine of the angle that the tangents to the orthogonal trajectories of the curves t = const. subtend with the X-axis (Y-axis, resp.). If we use the first way of representing a family of curves as our basis and set:

$$a_{11} = \left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2$$

then that will imply that:

$$\kappa = \frac{1}{\sqrt{a_{11}}}\frac{\partial x}{\partial p}, \qquad \lambda = \frac{1}{\sqrt{a_{11}}}\frac{\partial y}{\partial p}.$$

We further take:

$$\xi = -\lambda, \qquad \eta = \kappa.$$

The increases dx, dy in the Cartesian coordinates of a point along a curve that goes through the point can be represented in the form:

$$dx = \kappa T_1 - \lambda T_0 , \qquad dy = \lambda T_1 + \kappa T_0 ,$$

when:

$$T_1 = \frac{a_{11}\, dp + a_{12}\, dt}{\sqrt{a_{11}}}, \quad T_1 = \frac{\dfrac{\partial x}{\partial p}\dfrac{\partial y}{\partial t} - \dfrac{\partial y}{\partial p}\dfrac{\partial x}{\partial t}}{\sqrt{a_{11}}}\, dt = \frac{\Delta}{\sqrt{a_{11}}}\, dt,$$

and

$$a_{12} = \frac{\partial x}{\partial p}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial t}.$$

One can regard the linear differential forms T_1 and T_0 as arc-length elements of the coordinate lines, since the arc-length of an arbitrary curve that goes through the point (x, y) will contain the expression $\sqrt{T_1^2 + T_0^2}$. That way of looking at things can be regarded as an extension of the usual one, in which T_1 represents the arc-length of the curves of the family for $T_0 = 0$, and T_0 represents the arc-length of their orthogonal trajectories for $T_1 = 0$.

Since:

$$dt = \frac{\sqrt{a_{11}}}{\Delta} T_0, \qquad dp = \frac{\Delta T_1 - a_{12} T_0}{\Delta\sqrt{a_{11}}},$$

the differential of an arbitrary function $\mathfrak{F}$ of p and t will be a linear form in T_0 and T_1.

We set:

$$d\,\mathfrak{F} = (d\mathfrak{F})_{T_1} T_1 + (d\mathfrak{F})_{T_0} T_0,$$

in which:

$$(d\mathfrak{F})_{T_1} = \frac{1}{\sqrt{a_{11}}}\frac{\partial\mathfrak{F}}{\partial p}, \quad (d\mathfrak{F})_{T_0} = \frac{1}{\Delta\sqrt{a_{11}}}\left(-a_{12}\frac{\partial\mathfrak{F}}{\partial p} + a_{11}\frac{\partial\mathfrak{F}}{\partial t}\right).$$

The operations $(d\mathfrak{F})_{T_1}$ and $(d\mathfrak{F})_{T_0}$ shall be called *derivatives of* $\mathfrak{F}$ *with respect to the arc-lengths of the coordinate lines.*

The notations $\frac{\partial\mathfrak{F}}{\partial s}$, $\frac{\partial\mathfrak{F}}{\partial n}$ have been applied many times in place of the notations $(d\mathfrak{F})_{T_1}$ and $(d\mathfrak{F})_{T_0}$, in which s (n, resp.) are understood to mean the arc-lengths of the curves t = const. (their orthogonal trajectories, resp.). This notation might easily lead one to believe that one can take s and n to be independent variables. However, the latter is the case only when T_0 and T_1 are complete differentials, which emerges from the equations:

$$\frac{\partial x}{\partial p}\frac{\partial^2 y}{\partial p^2} - \frac{\partial y}{\partial p}\frac{\partial^2 x}{\partial p^2} = 0,$$

$$\frac{\partial x}{\partial p}\frac{\partial^2 y}{\partial p\,\partial t} - \frac{\partial y}{\partial p}\frac{\partial^2 x}{\partial p\,\partial t} = 0,$$

or

$$\frac{\partial\kappa}{\partial p} = \frac{\partial\kappa}{\partial t} = 0,$$

as is easy to see. The fulfillment of those equations expresses the idea that the given family of curves consists of a system of parallel straight lines, and the notation in question is justified only in that case.

Operations such as $(d\mathfrak{F})_{T_1}$, $(d\mathfrak{F})_{T_0}$, which are linear, homogeneous combinations of differentiations, have also been recently called *differential parameters.* However, I will employ that terminology exclusively for the things that **Lamé**, who was the author of that not-exactly-fortunate word construction, understood it to mean.

We shall employ the notations:

$$d\,(d\mathfrak{F})_{T_1} = (d\mathfrak{F})_{T_1^2} T_1 + (d\mathfrak{F})_{T_1 T_0} T_0, \qquad d\,(d\mathfrak{F})_{T_0} = (d\mathfrak{F})_{T_0 T_1} T_1 + (d\mathfrak{F})_{T_0^2} T_0$$

for the repeated application of the operations $(d\mathfrak{F})_{T_1}$, $(d\mathfrak{F})_{T_0}$.

Now, a key question relates to the effect of transposing those two operations. It is answered by a general theorem.

We take:

$$\nu_1\, T_1 = d\tau, \qquad \nu_0\, T_0 = dt,$$

in which ν_1 and ν_0 are integrating factors, of which ν_0 is known and equal to $\sqrt{a_{11}}/\Delta$, while ν_1 satisfies the differential equation:

$$(d\log\nu_1)_{T_0} = \frac{1}{\Delta}\left(\frac{\partial}{\partial p}\frac{a_{12}}{\sqrt{a_{11}}} - \frac{\partial\sqrt{a_{11}}}{\partial t}\right).$$

A function $\mathfrak{F}$ of p and t is also a function of t and τ. However, one has the two representations for $\dfrac{\partial^2\mathfrak{F}}{\partial t\,\partial\tau}$:

$$\frac{1}{\nu_1\nu_0}\left[(d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_1}(d\log\nu_1)_{T_0}\right]$$

and

$$\frac{1}{\nu_1\nu_0}\left[(d\mathfrak{F})_{T_0T_1} - (d\mathfrak{F})_{T_0}(d\log\nu_0)_{T_1}\right],$$

such that the desired theorem will take the form:

$$(d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_0T_1} = (d\mathfrak{F})_{T_1}(d\log\nu_1)_{T_0} - (d\mathfrak{F})_{T_0}(d\log\nu_0)_{T_1}.$$

The quantities $(d\log\nu_1)_{T_0}$ and $(d\log\nu_0)_{T_1}$ will take on intuitive meanings when one replaces $\mathfrak{F}$ is the foregoing equations with x and y, in turn.

Since:

$$(dx)_{T_1} = \kappa, \qquad (dy)_{T_1} = \lambda, \qquad (dx)_{T_0} = \xi, \qquad (dy)_{T_0} = \eta,$$

one will have:

$$(d\log\nu_1)_{T_0} = -\kappa(d\xi)_{T_1} - \lambda(d\eta)_{T_1}, \qquad (d\log\nu_0)_{T_1} = -\xi(d\kappa)_{T_0} - \eta(d\lambda)_{T_0}.$$

We let ρ_1 denote the radius of curvature of the curves t = const. and let ρ_0 denote that of its orthogonal trajectories:

$$\frac{1}{\rho_1} = \xi(d\kappa)_{T_1} + \eta(d\lambda)_{T_1}, \qquad \frac{1}{\rho_0} = \kappa(d\xi)_{T_0} + \lambda(d\eta)_{T_0},$$

so as a result:

$$(d\log\nu_1)_{T_0} = \frac{1}{\rho_1}, \qquad (d\log\nu_0)_{T_1} = \frac{1}{\rho_0}$$

and

$$(d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_0T_1} = \frac{(d\mathfrak{F})_{T_1}}{\rho_1} - \frac{(d\mathfrak{F})_{T_0}}{\rho_0}.$$

That notation shows that the linear differential form:

$$a_1\,T_1 + a_0\,T_0$$

will be a complete differential when:

$$(da_1)_{T_0} - (da_0)_{T_1} = \frac{a_1}{\rho_1} - \frac{a_0}{\rho_0}.$$

One will get a differential equation between ρ_1 and ρ_0 when one replaces $\mathfrak{F}$ with κ or λ, resp., in the last equation.

Since:

$$(d\kappa)_{T_1} = -\frac{\lambda}{\rho_1}, \qquad (d\lambda)_{T_1} = \frac{\kappa}{\rho_1},$$

$$(d\kappa)_{T_0} = \frac{\lambda}{\rho_0}, \qquad (d\lambda)_{T_0} = -\frac{\kappa}{\rho_0},$$

it will follow that:

$$\left(d\frac{1}{\rho_1}\right)_{T_0} = \left(d\frac{1}{\rho_0}\right)_{T_1} = \frac{1}{\rho_1^2} + \frac{1}{\rho_0^2}.$$

One often uses the following definition of a family of curves:

$$f(x, y) = t$$

in place of the latter one. One can then understand this to mean that it arises from the general ones:

$$x = f_1(p, t), \qquad y = f_2(p, t),$$

in such a way that one takes $f_1 = p$ and solves the equation:

$$y = f_1(x, t)$$

for t. One then gets:

$$\kappa = \frac{\dfrac{\partial f}{\partial y}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}}, \qquad \lambda = -\frac{\dfrac{\partial f}{\partial x}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}},$$

$$a_{11} = 1 + \frac{\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^2}, \qquad a_{12} = -\frac{\dfrac{\partial f}{\partial x}}{\left(\dfrac{\partial f}{\partial y}\right)^2}, \qquad \Delta = \frac{1}{\dfrac{\partial f}{\partial y}},$$

$$T_1 = \frac{\dfrac{\partial f}{\partial y}dx - \dfrac{\partial f}{\partial x}dy}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}}, \qquad T_0 = \frac{\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}},$$

$$(d\mathfrak{F})_{T_1} = \frac{\dfrac{\partial f}{\partial y}\dfrac{\partial \mathfrak{F}}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial \mathfrak{F}}{\partial y}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}}, \quad (d\mathfrak{F})_{T_0} = \frac{\dfrac{\partial f}{\partial x}\dfrac{\partial \mathfrak{F}}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial \mathfrak{F}}{\partial y}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}},$$

$$\nu_0 = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

With the second way of defining a family of curves, which is accomplished with the help of a differential equation of the form:

$$dx : dy = \varphi_1(x, y) : \varphi_2(x, y),$$

we obtain:

$$\kappa = \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}}, \qquad \lambda = \frac{\varphi_2}{\sqrt{\varphi_1^2 + \varphi_2^2}},$$

$$T_1 = \kappa\,dx + \lambda\,dy, \qquad T_0 = -\lambda\,dx + \kappa\,dy,$$

$$(d\mathfrak{F})_{T_1} = \kappa\frac{\partial \mathfrak{F}}{\partial x} + \lambda\frac{\partial \mathfrak{F}}{\partial y}, \qquad (d\mathfrak{F})_{T_0} = -\lambda\frac{\partial \mathfrak{F}}{\partial x} + \kappa\frac{\partial \mathfrak{F}}{\partial y},$$

$$\frac{1}{\rho_0} = -\frac{\partial \kappa}{\partial x} - \frac{\partial \lambda}{\partial y}, \qquad \frac{1}{\rho_1} = -\frac{\partial \kappa}{\partial y} - \frac{\partial \lambda}{\partial x}.$$

The derivatives with respect to arc-length that were introduced are *invariable operations*. The meaning of that terminology is illuminated by the following argument:

If we consider a family of curves to be given by the equations:

$$x = f_1(p, t), \qquad y = f_2(p, t)$$

then that family will remain unchanged when one introduces a function of q and τ in place of p and a function of τ alone in place of t, and considers τ to be unvarying along each curve of the family.

Now, if:

$$a'_{11} = \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial x}{\partial \tau}\right)^2, \qquad a'_{12} = \frac{\partial x}{\partial q}\frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial \tau}, \qquad \Delta' = \frac{\partial x}{\partial q}\frac{\partial y}{\partial \tau} - \frac{\partial y}{\partial q}\frac{\partial x}{\partial \tau}$$

then one will have:

$$T_1 = \frac{a'_{11}\,dq + a'_{12}\,d\tau}{\sqrt{a'_{11}}}, \qquad T_0 = \frac{\Delta'}{\sqrt{a'_{11}}}\,d\tau,$$

$$(d\mathfrak{F})_{T_1} = \frac{1}{\sqrt{a'_{11}}}\frac{\partial \mathfrak{F}}{\partial q}, \qquad (d\mathfrak{F})_{T_0} = -\frac{1}{\Delta'\sqrt{a'_{11}}}\left(-a'_{12}\frac{\partial \mathfrak{F}}{\partial q} + a'_{11}\frac{\partial \mathfrak{F}}{\partial \tau}\right).$$

The family of curves will remain unchanged for the second way of defining them when one introduces another rectangular coordinate system in place of the x, y, namely, u, v. If one calls the cosines of the angles that the tangent to the curves of the family makes with the u, v axes κ', λ', resp., then one will have:

$$T_1 = \kappa'\,du + \lambda'\,dv, \qquad T_0 = -\lambda'\,du + \kappa'\,dv,$$

$$(d\mathfrak{F})_{T_1} = \kappa'\frac{\partial \mathfrak{F}}{\partial u} + \lambda'\frac{\partial \mathfrak{F}}{\partial v}, \qquad (d\mathfrak{F})_{T_0} = -\lambda'\frac{\partial \mathfrak{F}}{\partial u} + \kappa'\frac{\partial \mathfrak{F}}{\partial v}.$$

In both cases, the same way of defining things will give the same results. In the first case, the choice of independent variables has no effect, while in the second case, it is the choice of coordinate system that has no effect.

Two functions that are formed from derivatives of a function $\mathfrak{F}$ of x and y are closely connected with that function, which can appear to be important when one assumes the various viewpoints.

With **Lamé** (*Leçons sur les coordonnées curvilignes*, pp. 6), we set:

$$(\Delta_1\mathfrak{F})^2 = \left(\frac{\partial \mathfrak{F}}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{F}}{\partial y}\right)^2,$$

$$\Delta_2\mathfrak{F} = \frac{\partial^2 \mathfrak{F}}{\partial x^2} + \frac{\partial^2 \mathfrak{F}}{\partial y^2},$$

and call $\Delta_1\mathfrak{F}$ the first *differential parameter* of $\mathfrak{F}$, while $\Delta_2\mathfrak{F}$ is the second. In that way, the sign of $\Delta_1\mathfrak{F}$ will have no effect, and can be taken to be + once and for all.

What form do the differential parameters take when one bases them upon curvilinear coordinates?

One has:

$$\frac{\partial \mathfrak{F}}{\partial x} = \kappa (d\mathfrak{F})_{T_1} - \lambda (d\mathfrak{F})_{T_0},$$

$$\frac{\partial \mathfrak{F}}{\partial y} = \lambda (d\mathfrak{F})_{T_1} + \kappa (d\mathfrak{F})_{T_0},$$

$$\frac{\partial^2 \mathfrak{F}}{\partial x^2} = \kappa^2 (d\mathfrak{F})_{T_1^2} - \kappa\lambda \{(d\mathfrak{F})_{T_1 T_0} + (d\mathfrak{F})_{T_0 T_1}\} + \lambda^2 (d\mathfrak{F})_{T_0^2} - \{\lambda (d\mathfrak{F})_{T_1} + \kappa (d\mathfrak{F})_{T_0}\}\left(\frac{\kappa}{\rho_1} + \frac{\lambda}{\rho_0}\right),$$

$$\frac{\partial^2 \mathfrak{F}}{\partial y^2} = \lambda^2 (d\mathfrak{F})_{T_1^2} + \kappa\lambda \{(d\mathfrak{F})_{T_1 T_0} + (d\mathfrak{F})_{T_0 T_1}\} + \kappa^2 (d\mathfrak{F})_{T_0^2} + \{\kappa (d\mathfrak{F})_{T_1} - \lambda (d\mathfrak{F})_{T_0}\}\left(\frac{\lambda}{\rho_1} - \frac{\kappa}{\rho_0}\right),$$

and as a result, one will get:

$$(\Delta_1\mathfrak{F})^2 = (d\mathfrak{F})_{T_1}^2 + (d\mathfrak{F})_{T_0}^2,$$

$$\Delta_2\mathfrak{F} = (d\mathfrak{F})_{T_1^2} + (d\mathfrak{F})_{T_0^2} - \frac{(d\mathfrak{F})_{T_1}}{\rho_0} - \frac{(d\mathfrak{F})_{T_0}}{\rho_1}$$

as the definitions of the differential parameters of a function $\mathfrak{F}$ in curvilinear coordinates.

The differential parameters of a function are so-called *invariant* functions – i.e., they always possess the same value at the same place (x, y) – that one might have also used for the determination of T_1, T_0 for a system of coordinate lines. One easily convinces oneself of that by the following argument: In addition to the family of curves that was considered before, one also takes a second one and denotes the expressions κ, λ, ρ_1, ρ_0, T_1, T_0 that one defines for it by κ', λ', r_1, r_0, S_1, S_0, respectively. In addition, one sets:

$$\kappa\kappa' + \lambda\lambda' = \cos\varphi,$$
$$\kappa\lambda' - \lambda\kappa' = \sin\varphi.$$

Since:

$$S_1 = \kappa' dx + \lambda' dy, \qquad S_0 = -\lambda' dx + \kappa' dy,$$

one will have:

$$T_1 = -\sin\varphi\, S_0 + \cos\varphi\, S_1,$$

$$T_0 = \sin\varphi\, S_1 + \cos\varphi\, S_0,$$

$$(d\mathfrak{F})_{S_1} = \cos\varphi (d\mathfrak{F})_{T_1} + \sin\varphi (d\mathfrak{F})_{T_0},$$

$$(d\mathfrak{F})_{S_0} = -\sin\varphi (d\mathfrak{F})_{T_1} + \cos\varphi (d\mathfrak{F})_{T_0}.$$

That shows that the first differential parameter is an invariant function, since:

$$(d\mathfrak{F})_{T_1}^2 + (d\mathfrak{F})_{T_0}^2 = (d\mathfrak{F})_{S_1}^2 + (d\mathfrak{F})_{S_0}^2.$$

Furthermore, one has:

$$(d\mathfrak{F})_{S_1^2} + (d\mathfrak{F})_{S_0^2} = (d\mathfrak{F})_{T_1^2} + (d\mathfrak{F})_{T_0^2} + (d\mathfrak{F})_{T_0}(d\mathfrak{F})_{T_1} - (d\mathfrak{F})_{T_1}(d\mathfrak{F})_{T_0}.$$

In order to calculate the quantities r_0 and r_1, one can possibly employ the equations:

$$\frac{\kappa'}{r_1} = (d\lambda')_{S_1}, \qquad \frac{\kappa'}{r_0} = -(d\lambda')_{S_0}.$$

In that, one has:

$$(d\lambda')_{S_1} = \kappa' \left\{ \cos\varphi \left(\frac{1}{\rho_1} + (d\varphi)_{T_1} \right) - \sin\varphi \left(\frac{1}{\rho_0} - (d\varphi)_{T_0} \right) \right\},$$

$$(d\lambda')_{S_0} = \kappa' \left\{ -\sin\varphi \left(\frac{1}{\rho_1} + (d\varphi)_{T_1} \right) - \cos\varphi \left(\frac{1}{\rho_0} - (d\varphi)_{T_0} \right) \right\},$$

and therefore:

$$\frac{(d\mathfrak{F})_{S_1}}{r_0} + \frac{(d\mathfrak{F})_{S_0}}{r_1} = \frac{(d\mathfrak{F})_{T_1}}{\rho_0} + \frac{(d\mathfrak{F})_{T_0}}{\rho_1} - (d\mathfrak{F})_{T_1}(d\varphi)_{T_0} + (d\mathfrak{F})_{T_0}(d\varphi)_{T_1}.$$

It now follows that:

$$(d\mathfrak{F})_{S_1^2} + (d\mathfrak{F})_{S_0^2} - \frac{(d\mathfrak{F})_{S_1}}{r_0} - \frac{(d\mathfrak{F})_{S_0}}{r_1} = (d\mathfrak{F})_{T_1^2} + (d\mathfrak{F})_{T_0^2} - \frac{(d\mathfrak{F})_{T_1}}{\rho_0} - \frac{(d\mathfrak{F})_{T_0}}{\rho_1},$$

so the second differential parameter is also an invariant function.

We shall refer to the foregoing types of curve families as *parallel* and *isothermal* curves. The former possess the property that their orthogonal trajectories are straight lines, such that $1 / \rho_0 = 0$. With the representation:

$$x = f_1 (p, t), \qquad y = f_2 (p, t),$$

one will be dealing with parallel curves when:

$$\frac{\partial}{\partial p} \frac{\sqrt{a_{11}}}{\Delta} = 0,$$

with the representation:

$$f(x, y) = t,$$

that will be true when:

$$(d\Delta_1 f)_{T_1} = 0,$$

and with the representation:

$$dx : dy = \varphi_1 (x, y) : \varphi_2 (x, y),$$

it will be true when $\lambda\, dx - \kappa\, dy$ is a complete differential.

The name *parallel curves* will be justified by two properties of those lines that can be established as follows:

Let $f(x, y) = t$ be the equation of a family of curves. We fix our attention upon an individual curve of the family by assigning a special value t_0 to t, and consider x, y to be the coordinates of a point P that traverses that individual curve (t_0).

One now takes:

$$u = x + h\, \xi = x - \lambda\, h, \qquad v = y + h\, \eta = y + \kappa h.$$

The u, v are then the coordinates of a point Q on the normal to the curve (t_0) that belongs to P. When will Q traverse a second curve of the family? For that to be true, it is necessary that h must be determined from the equation:

$$f(x - \lambda\, h, y + \kappa h) = t_0 + \Delta t_0\,,$$

in which Δt_0 is an arbitrary numerical quantity. If one develops the left-hand side of that equation in powers of h then that will give:

$$\Delta t_0 = h \cdot \Delta_1 f + \frac{h^2}{2!} (d\Delta_1 f)_{T_0} + \frac{h^3}{3!} (d\Delta_1 f)_{T_0^2} + \ldots$$

When $(d\Delta_1 f)_{T_1} = 0$, $\Delta_1 f$ will no longer depend upon τ, and will be constant along the curve (t_0). However, h will then be constant for any choice of Δt_0, so one will also have $(dh)_{T_1} = 0$.

We further consider the quotient du / dv along the curve $(t_0 + \Delta t_0)$. We will get it when it is expressed in terms of dx and dy and $dx : dy$ is taken to be equal to $\kappa : \lambda$. In that way, we will get:

$$\frac{du}{dv} = \frac{\kappa\left(1-\dfrac{h}{\rho_1}\right)-\lambda(dh)_{T_1}}{\lambda\left(1-\dfrac{h}{\rho_1}\right)+\kappa(dh)_{T_1}},$$

and for $(dh)_{T_1} = 0$, we will get:

$$\frac{du}{dv} = \frac{\kappa}{\lambda}.$$

If one calls points like P and Q *corresponding* points then the foregoing will show that parallel curves at corresponding points will possess parallel tangents and that the distance between corresponding points along two parallel curves will not change.

One defines *isothermal* families of curves by **Lamé**'s process (*Leçons*, pp. 31) as follows: Let $f(x, y) = t$ be the equation of a family of curves, while $\varphi(x, y) = \tau$ is the equation of their orthogonal trajectories. The family of curves is *isothermal* when the ratio:

$$\frac{\Delta_1 t}{(\Delta_1 t)^2}$$

depends upon only t, but not τ. Although that ratio can generally be calculated in the case where the family of curves is given by a differential equation, since the parameter t can only be ascertained by integration, the demand that that ratio must be independent of τ can always be brought into a computable form.

If we keep to the previously-applied notation:

$$\nu_0\, T_0 = dt, \qquad \nu_1\, T_1 = d\tau$$

then we will have:

$$(\Delta_1 t)^2 = \nu_0^2, \quad \Delta_2 t = (d\nu_0)_{T_0} - \frac{\nu_0}{\rho_1},$$

$$\frac{\Delta_2 t}{(\Delta_1 t)^2} = \frac{1}{\nu_0}\left[(d\log \nu_0)_{T_0} - \frac{1}{\rho_1}\right].$$

The condition:

$$\left[d\,\frac{\Delta_2 t}{(\Delta_1 t)^2}\right]_{T_1} = 0$$

will then assume the form:

$$-\frac{1}{\rho_0}\left[(d\log \nu_0)_{T_0} - \frac{1}{\rho_1}\right] + (d\log \nu_0)_{T_0 T_1} - \left(d\,\frac{1}{\rho_1}\right)_{T_1} = 0,$$

so the desired result will read:

$$\left(d\frac{1}{\rho_1}\right)_{T_1} = \left(d\frac{1}{\rho_0}\right)_{T_0}.$$

One obtains the same equation as the condition for the orthogonal trajectories of a given family of curves to define an isothermal system. The property of being isothermal will always be simultaneously possessed by two mutually-perpendicular curve families then.

If we write the condition for isothermal families of curves in the form:

$$(d\log\nu_1)_{T_0T_1} = (d\log\nu_0)_{T_1T_0}$$

and replace the expression on the right with:

$$(d\log\nu_0)_{T_1T_0} + (d\log\nu_0)_{T_1}\left[(d\log\nu_1)_{T_0} - (d\log\nu_0)_{T_0}\right]$$

then it will follow that:

$$\left(d\log\frac{\nu_1}{\nu_0}\right)_{T_0T_1} - (d\log\nu_0)_{T_1}\left(d\log\frac{\nu_1}{\nu_0}\right)_{T_0} = 0.$$

However, the left-hand side of that equation will possess the value:

$$\nu_0\,\nu_1\frac{\partial^2\log\dfrac{\nu_1}{\nu_0}}{\partial t\,\partial\tau}.$$

The quotient ν_1 / ν_0 will then be equal to a function of t, multiplied by one of τ, or what amounts to the same thing, there will exist two functions $g(t)$ and $g_0(t)$ such that:

$$\nu_1\,g(\tau) = \nu_0\,g_1(t).$$

The differential forms T_1 and T_0 will then possess a common integrating factor. If we call it $1/\mu$ and take:

$$\frac{1}{\mu}T_1 = du, \qquad \frac{1}{\mu}T_0 = dv$$

then the square of the line element in the plane will become:

$$\mu^2\,(du^2 + dv^2).$$

§ 2. – Simply-infinite families of curves in space.

A simply-infinite family of curves in space will be established when one is given the surface that it defines, e.g., by equations of the form:

$$x = f(p, q), \qquad y = f_1(p, q), \qquad z = f_2(p, q),$$

with the help of a differential equation of the form:

$$(1) \qquad \alpha_{11}\, dp + \alpha_{12}\, dq = 0,$$

in which α_{11} and α_{12} mean functions of p and q. The case in which (1) is assumed to have been integrated shall not be treated here, in particular.

One must next address the way that one calculates the derivatives of a function with respect to arc-length of the curves in the family and that of their orthogonal trajectories.

As usual, one sets:

$$\sum\left(\frac{\partial x}{\partial p}\right)^2 = E, \qquad \sum\frac{\partial x}{\partial p}\frac{\partial x}{\partial q} = F, \qquad \sum\left(\frac{\partial x}{\partial q}\right)^2 = G,$$

and in addition:

$$N = \alpha_{12}^2 E - 2\alpha_{11}\,\alpha_{12}\, F + \alpha_{11}^2 G,$$

$$a_{11} = \frac{-\alpha_{11}F + \alpha_{12}E}{\sqrt{N}}, \qquad a_{12} = \frac{-\alpha_{11}G + \alpha_{12}F}{\sqrt{N}},$$

$$a_{21} = \frac{\alpha_{11}\sqrt{EG - F^2}}{\sqrt{N}}, \qquad a_{22} = \frac{\alpha_{12}\sqrt{EG - F^2}}{\sqrt{N}},$$

$$T_1 = a_{11}\, dp + a_{11}\, dq,$$

$$T_0 = a_{21}\, dp + a_{22}\, dq.$$

The differential equation $T_0 = 0$ likewise defines our family of curves, since T_0 differs from the left-hand side of (1) only by a factor. The differential of a function $\mathfrak{F}$ of p and q will take the form:

$$d\,\mathfrak{F} = \frac{a_{22}\dfrac{\partial\mathfrak{F}}{\partial p} - a_{21}\dfrac{\partial\mathfrak{F}}{\partial q}}{a_{11}\,a_{22} - a_{12}\,a_{21}}T_1 + \frac{-a_{12}\dfrac{\partial\mathfrak{F}}{\partial p} + a_{11}\dfrac{\partial\mathfrak{F}}{\partial q}}{a_{11}\,a_{22} - a_{12}\,a_{21}}T_0,$$

and one will get the following equations for the derivatives in question with respect to arc-length:

$$(d\mathfrak{F})_{T_1} = \frac{a_{22}\dfrac{\partial\mathfrak{F}}{\partial p} - a_{21}\dfrac{\partial\mathfrak{F}}{\partial q}}{a_{11}\,a_{22} - a_{12}\,a_{21}} = \frac{\alpha_{12}\dfrac{\partial\mathfrak{F}}{\partial p} - \alpha_{11}\dfrac{\partial\mathfrak{F}}{\partial q}}{\sqrt{N}},$$

$$(d\mathfrak{F})_{T_0} = \frac{-a_{12}\dfrac{\partial\mathfrak{F}}{\partial p} + a_{11}\dfrac{\partial\mathfrak{F}}{\partial q}}{a_{11}\,a_{22} - a_{12}\,a_{21}} = \frac{(\alpha_{11}G - \alpha_{12}F)\dfrac{\partial\mathfrak{F}}{\partial p} + (\alpha_{12}G - \alpha_{11}F)\dfrac{\partial\mathfrak{F}}{\partial q}}{\sqrt{N}\sqrt{EG-F^2}}.$$

It follows from this that:

$$\sum (dx)_{T_1}^2 = 1, \qquad \sum (dx)_{T_0}^2 = 1, \qquad \sum (dx)_{T_1}(dx)_{T_0} = 0.$$

This shows that $(dx)_{T_1}$, $(dy)_{T_1}$, $(dz)_{T_1}$ are the direction cosines of the tangents to the curves of the family. Furthermore, it shows that $T_1 = 0$ is the differential equation of the orthogonal family, and that the direction cosines of the tangents to the curves of that family is represented by $(dx)_{T_0}$, $(dy)_{T_0}$, $(dz)_{T_0}$.

If ν_1 denotes an integrating factor of T_1, ν_0 denotes an integrating factor of T_0, and we set:

$$\nu_1\,T_1 = d\tau, \qquad \nu_0\,T_0 = dt$$

then that will give:

$$(d\mathfrak{F})_{T_1} = \nu_1\frac{\partial\mathfrak{F}}{\partial\tau}, \qquad (d\mathfrak{F})_{T_0} = \nu_0\frac{\partial\mathfrak{F}}{\partial t},$$

(2)

$$(d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_0T_1} = (d\mathfrak{F})_{T_1}(d\log\nu_1)_{T_0} - (d\mathfrak{F})_{T_0}(d\log\nu_0)_{T_1},$$

as in the previous paragraph. In order to see the geometric meaning of the quantities $(d\log\nu_1)_{T_0}$ and $(d\log\nu_0)_{T_1}$ that appear here, we consider the fact that the axis of curvature of a curve that is drawn on a surface meets the surface normal at a point (*A*) that coincides with the center of curvature of the normal section that is laid through the tangent to the curve and the fact that it cuts the tangent plane to the surface at a point (*B*) that one calls the *center of geodetic curvature* of the curve. If we fix our attention upon a curve $T_0 = 0$ and let h_{T_1} denote the abscissa of the corresponding point (*A*) relative to the surface point (x, y, z), while r denotes the radius of the first curvature of the curve, then we will have:

$$\frac{r}{h_{T_1}} = X\cos a + Y\cos b + Z\cos c,$$

in the event that X, Y, Z are the direction cosines of the surface normal, while $\cos a$, $\cos b$, $\cos c$ are those of the principal normal to the curve. However, from the first **Frenet** formula, we will have:

$$\cos a = r\,(dx)_{T_1^2}\,, \qquad \cos b = r\,(dy)_{T_1^2}\,, \qquad \cos c = r\,(dz)_{T_1^2}\,,$$

and as a result, the following equation for the normal curvature $1/h_{T_1}$ will exist:

$$\frac{1}{h_{T_1}} = \sum X\,(dx)_{T_1^2} = -\sum (dX)_{T_1}\,(dx)_{T_1}\,.$$

If one lets R_{T_1} denote the abscissa of the point (B) relative to the surface point (x, y, z) then that will make:

$$\frac{r}{R_{T_1}} = (dx)_{T_0}\cos a + (dy)_{T_0}\cos b + (dz)_{T_0}\cos c\,.$$

As a result, one will have the following equation for the geodetic curvature $1/R_{T_1}$:

$$\frac{1}{R_{T_1}} = \sum (dx)_{T_0}\,(dx)_{T_1^2} = -\sum (dx)_{T_1}\,(dx)_{T_0 T_1}\,.$$

One will likewise get the following expressions for the normal and geodetic curvature of the curve $T_1 = 0$:

$$\frac{1}{h_{T_0}} = \sum X\,(dx)_{T_0^2} = -\sum (dX)_{T_0}\,(dx)_{T_0}\,,$$

$$\frac{1}{R_{T_0}} = \sum (dx)_{T_1}\,(dx)_{T_0^2} = -\sum (dx)_{T_0}\,(dx)_{T_1 T_0}\,.$$

However, when $\mathfrak{F}$ is replaced with x, y, z, in succession, it will follow from (2) that:

$$\sum (dx)_{T_0}\,(dx)_{T_1 T_0} = -\,(d\log \nu_0)_{T_1}\,, \qquad \sum (dx)_{T_1}\,(dx)_{T_0 T_1} = -\,(d\log \nu_1)_{T_0}\,.$$

One will then have:

(3) $$\frac{1}{R_{T_1}} = (d\log \nu_1)_{T_0}\,, \qquad \frac{1}{R_{T_0}} = (d\log \nu_0)_{T_1}\,.$$

In order to find the differential equation that exists between the quantities $1/R_{T_1}$ and $1/R_{T_0}$, one takes the function $\mathfrak{F}$ in (2) to be equal to $(dx)_{T_1}$, $(dy)_{T_1}$, $(dz)_{T_1}$, in succession. It will then follow that:

$$\sum (dx)_{T_0}\,(dx)_{T_1^2 T_0} - \sum (dx)_{T_0}\,(dx)_{T_1 T_0 T_1} = \left(\frac{1}{R_{T_1}}\right)^2 + \left(\frac{1}{R_{T_0}}\right)^2.$$

However:

$$\sum (dx)_{T_0} (dx)_{T_1 T_0} = - \sum (dx)_{T_1{}^2} (dx)_{T_0^2} + \left(d \frac{1}{R_{T_1}} \right)_{T_0} ,$$

$$\sum (dx)_{T_0} (dx)_{T_1 T_0 T_1} = - \sum (dx)_{T_1 T_0} (dx)_{T_0 T_1} - \left(d \frac{1}{R_{T_0}} \right)_{T_1} .$$

It still remains for us to discover the meaning of the expression:

$$\sum (dx)_{T_1{}^2} (dx)_{T_0^2} - \sum (dx)_{T_1 T_0} (dx)_{T_0 T_1} .$$

We take:

$$\frac{1}{h_{T_1}} = \Theta, \qquad \sum X (dx)_{T_1 T_0} = \Theta', \qquad \frac{1}{h_{T_0}} = \Theta'',$$

to abbreviate. Since, from (2), we have:

$$\sum X (dx)_{T_1 T_0} = \sum X (dx)_{T_0 T_1} ,$$

that will imply the following system of equations:

$$(dx)_{T_1{}^2} = \frac{(dx)_{T_0}}{R_{T_1}} + \Theta X, \qquad (dx)_{T_1 T_0} = - \frac{(dx)_{T_0}}{R_{T_0}} + \Theta' X,$$

$$(dx)_{T_0{}^2} = \frac{(dx)_{T_1}}{R_{T_0}} + \Theta'' X, \qquad (dx)_{T_0 T_1} = - \frac{(dx)_{T_1}}{R_{T_1}} + \Theta' X.$$

As we know, we have:

$$\frac{1}{\rho} = \frac{X\, d^2 x + Y\, d^2 y + Z\, d^2 z}{dx^2 + dy^2 + dz^2}$$

for the radius of curvature ρ of a normal section. Under an application of the previous system, it will be converted into:

$$\frac{1}{\rho} = \frac{\Theta T_1{}^2 + 2\Theta' T_1 T_2 + \Theta'' T_2{}^2}{T_1{}^2 + T_2{}^2} ,$$

and as a result, the following equation for the Gaussian curvature will exist:

$$\frac{1}{\rho_1 \rho_2} = \Theta\, \Theta'' - \Theta'^{\,2}.$$

We will then get the desired differential equation in the form:

(4) $$\left(d\frac{1}{R_{T_1}}\right)_{T_0} + \left(d\frac{1}{R_{T_0}}\right)_{T_1} = \frac{1}{\rho_1\,\rho_2} + \left(\frac{1}{R_{T_1}}\right)^2 + \left(\frac{1}{R_{T_0}}\right)^2.$$

One can the regard the family of curves considered to be the arrangement of points:

$$x = f\,(p, q), \qquad y = f_1\,(p, q), \qquad z = f_2\,(p, q)$$

on the surface (x, y, z) that is the image of the family of curves in the p-q-plane that is established by the differential equation (1).

If one chooses a second surface (x_1, y_1, z_1) and considers the arrangement of points:

$$x_1 = g\,(p, q), \quad y_1 = g_1\,(p, q), \quad z_1 = g_2\,(p, q)$$

on the surface to be the image of a planar family of curves then one will get two new differential forms T_1' and T_0' in place of T_1 and T_0 , which will emerge from the old ones when one replaces E, F, G with:

$$E_1 = \sum\left(\frac{\partial x_1}{\partial p}\right)^2, \qquad F_1 = \sum\frac{\partial x_1}{\partial p}\frac{\partial x_1}{\partial q}, \qquad G_1 = \sum\left(\frac{\partial x_1}{\partial q}\right)^2,$$

in succession. For the point (x_1, y_1, z_1) on the surface that corresponds to the point (p, q) of the p, q-plane, the geodetic curvature of the curve $T_0' = 0$ or $T_1' = 0$ will be denoted by $1/R_{T_1'}$ or $1/R_{T_0'}$, resp., and the Gaussian curvature by $1/\rho_1'\,\rho_2'$. If one now assumes the existence of the equations:

$$E = E_1, \qquad F = F_1, \qquad G = G_1$$

then one will have:

$$T_1' = T_1, \qquad T_0' = T_0, \qquad (d\mathfrak{F})_{T_1'} = (d\mathfrak{F})_{T_1}, \qquad (d\mathfrak{F})_{T_0'} = (d\mathfrak{F})_{T_0}.$$

However, it will then follow from (3) that:

$$\frac{1}{R_{T_1}} = \frac{1}{R_{T_1'}}, \qquad \frac{1}{R_{T_0}} = \frac{1}{R_{T_0'}},$$

and (4) will imply that:

$$\frac{1}{\rho_1\,\rho_2} = \frac{1}{\rho_1'\,\rho_2'}.$$

That expresses the known fact that the geodetic curvature and the Gaussian curvature are *bending invariants.*

The **Lamé** differential parameters should now be defined for a surface. If one preserves the meanings that x, y, z had up to now as coordinates of a point on the surface considered and that X, Y, Z had as the direction cosines of its normal then one will take:

$$u = x + r\,X, \qquad v = y + r\,Y, \qquad w = z + r\,Z,$$

in which r denotes a new variable. The independent variables p, q, r will be defined conversely as functions of the rectangular coordinate u, v, w in that way.

If:

$$J = \begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} & X \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial p} & Y \\ \frac{\partial w}{\partial p} & \frac{\partial w}{\partial q} & Z \end{vmatrix}$$

then one will have, as is known:

$$J\frac{\partial p}{\partial u} = \frac{\partial v}{\partial q}Z - \frac{\partial w}{\partial q}Y, \qquad J\frac{\partial q}{\partial u} = \frac{\partial w}{\partial p}Y - \frac{\partial v}{\partial p}Z.$$

Now, a function $\mathfrak{F}$ of p and q will also be a function of u, v, w. One denotes the derivatives $\frac{\partial \mathfrak{F}}{\partial u}$, $\frac{\partial \mathfrak{F}}{\partial v}$, $\frac{\partial^2 \mathfrak{F}}{\partial u^2}$, etc., that were formed under the assumption $r = 0$ by $\frac{\partial \mathfrak{F}}{\partial x}$, $\frac{\partial \mathfrak{F}}{\partial y}$, $\frac{\partial^2 \mathfrak{F}}{\partial x^2}$, etc., respectively. J takes the value $\sqrt{EG - F^2}$ for $r = 0$, and it further follows that:

$$\frac{\partial p}{\partial x} = \frac{G\frac{\partial x}{\partial p} - F\frac{\partial x}{\partial q}}{\sqrt{EG - F^2}}, \qquad \frac{\partial q}{\partial x} = \frac{E\frac{\partial x}{\partial q} - F\frac{\partial x}{\partial p}}{\sqrt{EG - F^2}}.$$

Since:

$$\frac{\partial \mathfrak{F}}{\partial p} = \frac{(\alpha_{12}E - \alpha_{11}F)(d\mathfrak{F})_{T_1} + \alpha_{11}\sqrt{EG - F^2}\,(d\mathfrak{F})_{T_0}}{\sqrt{N}},$$

$$\frac{\partial \mathfrak{F}}{\partial q} = \frac{(\alpha_{12}F - \alpha_{11}G)(d\mathfrak{F})_{T_1} + \alpha_{12}\sqrt{EG - F^2}\,(d\mathfrak{F})_{T_0}}{\sqrt{N}},$$

one has:

$$\frac{\partial p}{\partial x} = \frac{\alpha_{12}}{\sqrt{N}}(dx)_{T_1} + \frac{\alpha_{11}G - \alpha_{12}F}{\sqrt{N}\sqrt{EG - F^2}}(dx)_{T_0},$$

$$\frac{\partial q}{\partial x} = \frac{-\alpha_{12}}{\sqrt{N}}(dx)_{T_1} + \frac{\alpha_{12}E - \alpha_{11}F}{\sqrt{N}\sqrt{EG - F^2}}(dx)_{T_0},$$

or

$$\frac{\partial p}{\partial x} = (dp)_{T_1}(dx)_{T_1} + (dp)_{T_0}(dx)_{T_0}, \qquad \frac{\partial q}{\partial x} = (dq)_{T_1}(dx)_{T_1} + (dq)_{T_0}(dx)_{T_0},$$

so:

$$\frac{\partial \mathfrak{F}}{\partial x} = (d\mathfrak{F})_{T_1}(dx)_{T_1} + (d\mathfrak{F})_{T_0}(dx)_{T_0}.$$

When $r = 0$ – i.e., along the surface (x, y, z) – the **Lamé** first differential parameter:

$$\sqrt{\left(\frac{\partial \mathfrak{F}}{\partial u}\right)^2 + \left(\frac{\partial \mathfrak{F}}{\partial v}\right)^2 + \left(\frac{\partial \mathfrak{F}}{\partial w}\right)^2}$$

will have the equation:

$$(\Delta \mathfrak{F}_1)^2 = (d\mathfrak{F})_{T_1}^2 + (d\mathfrak{F})_{T_0}^2 = \nu_1^2\left(\frac{\partial \mathfrak{F}}{\partial \tau}\right)^2 + \nu_0^2\left(\frac{\partial \mathfrak{F}}{\partial t}\right)^2.$$

If we take $\mathfrak{F} = t$ here then we will have:

$$\Delta_1 t = \nu_0$$

on the one hand, and on the other hand, since:

$$\alpha_{11} : \alpha_{12} = \frac{\partial t}{\partial p} : \frac{\partial t}{\partial q},$$

it will arise that:

$$(\Delta_1 t)^2 = (dt)_{T_0}^2 = \frac{G\left(\frac{\partial t}{\partial p}\right)^2 - 2F\frac{\partial t}{\partial p}\frac{\partial t}{\partial q} + E\left(\frac{\partial t}{\partial q}\right)^2}{EG - F^2}.$$

That is the known equation for **Beltrami**'s first differential parameter. From the fact that it is equal to ν_0, one sees that the orthogonal trajectories of the family of curves $T_0 = 0$ will be geodetic lines of the surface (x, y, z) when $\Delta_1 t$ depends upon only t, since $1/R_{T_0}$ will vanish then.

In order to define the second **Lamé** differential parameter $\frac{\partial^2 \mathfrak{F}}{\partial u^2} + \frac{\partial^2 \mathfrak{F}}{\partial v^2} + \frac{\partial^2 \mathfrak{F}}{\partial w^2}$ of a function for $r = 0$, one considers that:

$$\frac{\partial^2 \mathfrak{F}}{\partial x^2} = (dx)_{T_1}\{(d\mathfrak{F})_{T_1^2}(dx)_{T_1} + (d\mathfrak{F})_{T_0 T_1}(dx)_{T_0} + (d\mathfrak{F})_{T_1}(dx)_{T_1^2} + (d\mathfrak{F})_{T_0}(dx)_{T_0 T_1}\}$$

$$+ (dx)_{T_0}\{(d\mathfrak{F})_{T_1 T_0}(dx)_{T_1} + (d\mathfrak{F})_{T_0^2}(dx)_{T_0} + (d\mathfrak{F})_{T_1}(dx)_{T_1 T_0} + (d\mathfrak{F})_{T_0}(dx)_{T_0^2}\}.$$

That implies the following equation for the second **Lamé** differential parameter along the surface (x, y, z):

$$\Delta_2\,\mathfrak{F} = (d\mathfrak{F})_{T_1^2} + (d\mathfrak{F})_{T_0^2} - \frac{(d\mathfrak{F})_{T_0}}{R_{T_1}} - \frac{(d\mathfrak{F})_{T_1}}{R_{T_0}}.$$

Since $(dx)_{T_0} = \nu_0$, one has the following relation for $\mathfrak{F} = t$:

$$\Delta_1\, t = \nu_0 (d\log \nu_0)_{T_0} - \frac{\nu_0}{R_{T_1}}.$$

Now, one has:

$$a_{11} = \frac{E\dfrac{\partial t}{\partial q} - F\dfrac{\partial t}{\partial p}}{\nu_0\sqrt{EG - F^2}}, \qquad a_{12} = \frac{F\dfrac{\partial t}{\partial q} - G\dfrac{\partial t}{\partial p}}{\nu_0\sqrt{EG - F^2}}.$$

Here, one now sets $a_{11}\,\nu_0 = \alpha$, $a_{12}\,\nu_0 = -\beta$, for brevity. One will then have, in general:

$$(d\mathfrak{F})_{T_0} = \frac{\alpha\dfrac{\partial\mathfrak{F}}{\partial q} + \beta\dfrac{\partial\mathfrak{F}}{\partial p}}{\nu_0\sqrt{EG - F^2}}.$$

Since one further has:

$$\frac{1}{R_{T_1}} = \frac{\dfrac{\partial a_{12}}{\partial p} - \dfrac{\partial a_{11}}{\partial q}}{\sqrt{EG - F^2}} = \frac{-\dfrac{\partial \beta}{\partial p} - \dfrac{\partial \alpha}{\partial q}}{\nu_0\sqrt{EG - F^2}} + \frac{\beta\dfrac{\partial \log\nu_0}{\partial p} + \alpha\dfrac{\partial \log\nu_0}{\partial q}}{\nu_0\sqrt{EG - F^2}} = \frac{\dfrac{\partial \beta}{\partial p} + \dfrac{\partial \alpha}{\partial q}}{\sqrt{EG - F^2}} + (d\log\nu_0)_{T_0},$$

it will follow that:

$$\Delta_2\, t = \frac{\dfrac{\partial \beta}{\partial p} + \dfrac{\partial \alpha}{\partial q}}{\sqrt{EG - F^2}}.$$

That is the known equation for the second **Beltrami** differential parameter.

As one did in **§ 1**, one shows that one will have the relation:

$$\left(d\frac{1}{R_{T_1}}\right)_{T_1} = \left(d\frac{1}{R_{T_0}}\right)_{T_0}$$

for planar families of curves when the quotient $\frac{\Delta_2 t}{(\Delta_1 t)^2}$ depends upon only t (so the family considered is isothermal), with which, the orthogonal family will also prove to be isothermal, and furthermore, T_1 and T_0 will now possess an integrating factor $1 / \mu$, with the help of which, one can give the square of the line element of the surface the form:

$$ds^2 = \mu^2 (dt^2 + d\tau^2).$$

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PART TWO

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Doubly-infinite families of curves defined by finite equations

§ 3. – Orthogonal trajectories. Normal family. Special family.

We consider a doubly-infinite family of curves in space and take it to be given by finite equations of the form:

$$(1) \qquad x = f(p, q, r), \qquad y = f_1(p, q, r), \qquad z = f_2(p, q, r) .$$

Here, p and q should have the meaning of parameters, such that only r changes along each individual curve, while p and q keep their values. In regard to the functions f, f_1, f_2, one assumes that the determinant:

$$J = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial p} & \frac{\partial z}{\partial q} & \frac{\partial z}{\partial r} \end{vmatrix}$$

is generally non-zero, since otherwise one would be dealing with a simply-infinite family of curves.

With that assumption, the equation $J = 0$, along with (1), will determine a surface that can be considered to be the locus of points at which a curve of the family is cut by an infinitely-close one. One calls that surface the *focal surface* of the curve family. (**Darboux**, *Leçons*, v. II, pp. 5)

We shall study the curvature behavior of the family with the help of its orthogonal trajectories.

Let a second doubly-infinite family of curves be established by equations of the form:

$$x = g(p', q', r'), \qquad y = g_1(p', q', r'), \qquad z = g_2(p', q', r'),$$

in which p', q' should mean parameters.

When is that family of curves composed of nothing but orthogonal trajectories of the first family? One denotes the direction cosines of the tangents to the curves p = const., q = const. by ξ, η, ζ. One then has:

$$(2) \qquad \xi = \frac{\frac{\partial x}{\partial r}}{\sqrt{a_{33}}}, \qquad \eta = \frac{\frac{\partial y}{\partial r}}{\sqrt{a_{33}}}, \qquad \zeta = \frac{\frac{\partial z}{\partial r}}{\sqrt{a_{33}}} .$$

in which:

$$a_{33} = \sum \left(\frac{\partial x}{\partial r}\right)^2 .$$

The curves of the second family are orthogonal trajectories of the first family in the event that one always has:

$$\xi \frac{\partial x}{\partial t} + \eta \frac{\partial y}{\partial t} + \zeta \frac{\partial z}{\partial t} = 0,$$

or, when one applies the notation:

$$a_{13} = \sum \frac{\partial x}{\partial p}\frac{\partial x}{\partial r}, \qquad a_{23} = \sum \frac{\partial x}{\partial q}\frac{\partial x}{\partial r},$$

one will have:

$$a_{13}\frac{\partial p}{\partial t} + a_{23}\frac{\partial q}{\partial t} + a_{33}\frac{\partial r}{\partial t} = 0.$$

We must then consider the relation:

$$(3) \qquad a_{13}\, dp + a_{23}\, dq + a_{33}\, dr = 0$$

to be the differential equation of the orthogonal trajectories of the given family of curves.

The left-hand side of (3) can have the property that it will go to the differential of a function of p, q, r when one multiplies it by a suitable factor. The condition for that is:

$$(4) \qquad a_{13}\left(\frac{\partial a_{23}}{\partial r} - \frac{\partial a_{33}}{\partial q}\right) + a_{23}\left(\frac{\partial a_{33}}{\partial p} - \frac{\partial a_{13}}{\partial r}\right) + a_{33}\left(\frac{\partial a_{13}}{\partial q} - \frac{\partial a_{23}}{\partial p}\right) = 0.$$

If that condition exists then one will have:

$$\mu\,(a_{13}\, dp + a_{23}\, dq + a_{33}\, dr) = d\tau$$

or

$$dr = \frac{d\tau}{\mu\, a_{33}} - \frac{a_{13}}{a_{33}}dp - \frac{a_{23}}{a_{33}}dq .$$

Here, we regard r as a function of τ, p and q and think of that function as being substituted for r in (1). Equations (1) can then be considered to be equations of a family of surfaces whose parameter is τ. Now, since:

$$\frac{\partial r}{\partial p} = -\frac{a_{13}}{a_{33}}, \qquad \frac{\partial r}{\partial q} = -\frac{a_{23}}{a_{33}},$$

one will get the following expressions for the complete partial derivatives of x with respect to p and q :

$$\left(\frac{\partial x}{\partial p}\right) = \frac{\partial x}{\partial p} - \frac{a_{13}}{a_{33}}\frac{\partial x}{\partial r}, \qquad \left(\frac{\partial x}{\partial q}\right) = \frac{\partial x}{\partial q} - \frac{a_{23}}{a_{33}}\frac{\partial x}{\partial r}.$$

However, that shows that:

$$\sum \xi \left(\frac{\partial x}{\partial p}\right) = 0, \qquad \sum \xi \left(\frac{\partial x}{\partial q}\right) = 0 ;$$

i.e., the curves of the family are the orthogonal trajectories to a family of surfaces. In the event that equation (4) is true, the family of curves is a *normal family*.

For the differentiation along an orthogonal trajectory of the family of curves, let $\mathfrak{F}$ be a function of p, q, r, such that:

$$d\,\mathfrak{F} = \frac{\partial \mathfrak{F}}{\partial p}dp + \frac{\partial \mathfrak{F}}{\partial q}dq + \frac{\partial \mathfrak{F}}{\partial r}dr .$$

Here, if (3) is assumed to be valid then one should write $\delta\,\mathfrak{F}$ instead of $d\,\mathfrak{F}$. Since the differential dr in (3) is multiplied by a coefficient that is generally non-zero, one has:

$$\delta\,\mathfrak{F} = \left(\frac{\partial \mathfrak{F}}{\partial p} - \frac{a_{13}}{a_{33}}\frac{\partial \mathfrak{F}}{\partial r}\right)dp + \left(\frac{\partial \mathfrak{F}}{\partial q} - \frac{a_{23}}{a_{33}}\frac{\partial \mathfrak{F}}{\partial r}\right)dq .$$

We denote the factors of dp and dq that appear in this by $\mathfrak{F}_p$ and $\mathfrak{F}_q$, to abbreviate. The increases in the coordinates along an orthogonal trajectory of the family are then:

$$\delta x = x_p\,dp + x_q\,dq, \qquad \delta y = y_p\,dp + y_q\,dq, \qquad \delta z = z_p\,dp + z_q\,dq,$$

in which the quotient dq / dp is regarded as a function of p, q, r.

Since:

$$\sum \xi\,x_p = 0, \qquad \sum \xi\,x_q = 0,$$

when one sets:

$$\sum x_p^2 = E, \qquad \sum x_p\,x_q = F, \quad \sum x_q^2 = G,$$

one will get the following expressions:

$$\xi = \frac{y_p\,z_q - z_p\,y_q}{\sqrt{EG - F^2}}, \qquad \eta = \frac{z_p\,x_q - x_p\,z_q}{\sqrt{EG - F^2}}, \qquad \zeta = \frac{x_p\,y_q - x_p\,y_q}{\sqrt{EG - F^2}}$$

for the direction cosines ξ, η, ζ.

The aforementioned determinant J can also be written:

$$\sqrt{a_{33}}\begin{vmatrix} x_p & x_q & \xi \\ y_p & y_q & \eta \\ z_p & z_q & \zeta \end{vmatrix},$$

such that:

$$\frac{J\,\xi}{\sqrt{a_{33}}} = y_p\,z_q - z_p\,y_q\,, \quad \frac{J\,\eta}{\sqrt{a_{33}}} = z_p\,x_q - x_p\,z_q\,, \quad \frac{J\,\zeta}{\sqrt{a_{33}}} = x_p\,y_q - y_p\,x_q\,.$$

If J is non-zero then one will also that have $EG - F^2$ does vanish.

We would next like to understand a *regular point* of the curve family to mean a point (x, y, z) at which neither a_{33} nor $EG - F^2$ vanishes. It will then be a regular point of the individual curve that it belongs to, and it will not lie on the focal surface of the family of curves.

When one advances along an orthogonal trajectory, one will meet a tangent $(\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta)$ that is close to the tangent (ξ, η, ζ), in which:

$$\delta\xi = \xi_p\,dp + \xi_q\,dq\,, \quad \delta\eta = \eta_p\,dp + \eta_q\,dq\,, \quad \delta\zeta = \zeta_p\,dp + \zeta_q\,dq\,,$$

$$\xi_p = \frac{1}{\sqrt{a_{33}}}\left(\frac{\partial^2 x}{\partial p\,\partial r} - \frac{a_{13}}{a_{33}}\frac{\partial^2 x}{\partial r^2}\right) - \frac{\xi}{2a_{33}}\left(\frac{\partial a_{33}}{\partial p} - \frac{a_{13}}{a_{33}}\frac{\partial a_{33}}{\partial r}\right),$$

$$\xi_q = \frac{1}{\sqrt{a_{33}}}\left(\frac{\partial^2 x}{\partial q\,\partial r} - \frac{a_{23}}{a_{33}}\frac{\partial^2 x}{\partial r^2}\right) - \frac{\xi}{2a_{33}}\left(\frac{\partial a_{33}}{\partial q} - \frac{a_{23}}{a_{33}}\frac{\partial a_{33}}{\partial r}\right).$$

Here, one should emphasize the case in which among the directions of advance in question, one always finds one for which the tangent (ξ, η, ζ) remains parallel, such that $\delta\xi$, $\delta\eta$, $\delta\zeta$ vanish. If we set:

$$\sum \xi_p^2 = H, \qquad \sum \xi_p\,\xi_q = \Phi, \qquad \sum \xi_q^2 = \Psi$$

then the difference $H\,\Psi - \Phi^2$ will be equal to zero for any system of values for p, q, r in the case considered. A family of curves with that property shall be called *special*, while that difference will generally be assumed to be non-zero for a *general* family of curves.

The generation of special families of curves is illuminated by the following argument: A doubly-infinite family of curves can be decomposed into a continuous sequence of simply-infinite families of curves; e.g., by assuming that q = const. Now, in a special doubly-infinite family of curves, a decomposition must be possible such that every normal plane to a curve in each individual family of the sequence is, at the same time, the normal plane to any other curve of the same individual family. Curves in space with common normal planes shall be called *parallel* curves. Let the coordinates x_0, y_0, z_0 of the given curve be functions of r. Furthermore, let $1 / \rho$ or $1 / \rho'$ be the first or second curvature of the curves, resp., and let $\cos\alpha$, $\cos\beta$, $\cos\gamma$; $\cos a$, $\cos b$, $\cos c$; $\cos\lambda$, cos

μ, cos ν be the direction cosines of the tangent, principal normal, and binormal, resp. If one chooses the positive half of that line such that:

$$\cos a = \cos\beta \cos\nu - \cos\gamma \cos\mu,$$

$$\cos b = \cos\beta \cos\lambda - \cos\alpha \cos\nu,$$

$$\cos c = \cos\alpha \cos\mu - \cos\beta \cos\lambda,$$

and sets:

$$\sigma = \sqrt{\sum \left(\frac{\partial x_0}{\partial r}\right)^2}$$

then the **Frenet** formulas (**J. Knoblauch**, *Einleitung in die allgemeine Theorie der krummen Fläche*, pp. 241) will assume the form:

$$\frac{d\cos\alpha}{dr} = \frac{\sigma\cos a}{\rho}, \qquad \frac{d\cos\lambda}{dr} = \frac{\sigma\cos a}{\rho'}, \qquad \frac{d\cos a}{dr} = -\,\sigma\left(\frac{\cos\alpha}{\rho} + \frac{\cos\lambda}{\rho'}\right).$$

The normal plane of the first curve that belongs to the point P (x_0, y_0, z_0) cuts the second curve at the point Q (x, y, z). One then has:

$$x = x_0 + m\,(\cos\varphi \cos a + \sin\varphi \cos\lambda),$$

$$y = y_0 + m\,(\cos\varphi \cos b + \sin\varphi \cos\mu),$$

$$z = z_0 + m\,(\cos\varphi \cos c + \sin\varphi \cos\nu)\,.$$

Here, m means the distance between the points P, Q, and φ is the angle that this distance subtends with the principal normal to the curve (x_0, y_0, z_0).

Since:

$$\frac{dx}{dr} = \left(1 - \frac{m\cos\varphi}{\rho}\right)\sigma\cos\alpha + \left(\frac{dm\cos\varphi}{dr} + \frac{m\sigma\cos\varphi}{\rho'}\right)\cos\alpha + \left(\frac{dm\sin\varphi}{dr} - \frac{m\sigma\cos\varphi}{\rho'}\right)\cos\lambda\,,$$

the curves in question will be parallel when:

$$\frac{dm}{dr}\cos\varphi - m\sin\varphi\left(\frac{d\varphi}{dr} - \frac{\sigma}{\rho'}\right) = 0$$

and

$$\frac{dm}{dr}\sin\varphi + m\cos\varphi\left(\frac{d\varphi}{dr} - \frac{\sigma}{\rho'}\right) = 0\,;$$

i.e., when:

$$\frac{dm}{dr} = 0 \qquad \text{and} \qquad \frac{d\varphi}{dr} = \frac{\sigma}{\rho'} .$$

If r_0 denotes a fixed value of r then one will have:

$$\varphi = \int_{r_0}^{r} \frac{\sigma}{\rho'} dr + p .$$

One now considers p to be variable and m to be a function of p. x, y, z will then represent the coordinates of a surface. The curves p = const., r = const. are the lines of curvature of the surface, and in addition the curves r = const. are the planar geodetic lines on it.

The curves p = const. define an individual family in a special family of curves when one takes x_0, y_0, z_0 to be functions of r and the parameters q, m to be functions of p and q, and φ to be the integral $\int_{r_0}^{r} \frac{\sigma}{\rho'} dr$, plus a function of p and q. Now, x, y, z are the coordinates of a doubly-infinite family of curves for which the quantities:

$$\xi = \cos\alpha, \qquad \eta = \cos\beta, \qquad \zeta = \cos\gamma$$

depend upon only q and r. However, since:

$$a_{13} = \left(1 - \frac{m\cos\varphi}{\rho}\right)\sigma \sum \cos\alpha \left(\frac{\partial m\cos\varphi}{\partial p}\cos a + \frac{\partial m\sin\varphi}{\partial p}\cos\lambda\right) = 0,$$

ξ_p , η_p , ζ_p will vanish, and therefore $H\,\Psi - \Phi^2$.

The normal planes of the curves of a special family define a doubly-infinite manifold and envelop a surface.

§ 4. – Normal curvature of orthogonal trajectories. Isotropic curve families. Lines of curvature of the first and second kind.

As always, we fix our attention upon a regular point P (x, y, z) of the curve family in what follows. An orthogonal trajectory of the family that goes through it will possess a curvature axis that belongs to P that meets the center Q of its first curvature and is parallel to the binormal of the trajectory. It cuts the tangent (ξ, η, ζ) at the point R. If ρ is once more the radius of the first curvature, ρ_1 is the abscissa of R relative to Q, and h is the abscissa of R relative to P then when one applies the same notations as in the previous paragraphs, one will have:

$$h\,\xi = \rho\cos a + \rho_1\cos\lambda, \qquad h\,\xi = \rho\cos b + \rho_1\cos\mu, \qquad h\,\zeta = \rho\cos a + \rho_1\cos\nu\,;$$

i.e.:

$$\frac{1}{h} = \sum \xi \frac{\cos a}{\rho}.$$

However, the first **Frenet** formula implies that:

$$\frac{\cos a}{\rho} = \frac{\delta \cos \alpha}{\delta s},$$

in the event that:

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2.$$

Since one further has:

$$\sum \xi\, \delta \cos \alpha = - \sum \cos \alpha\, \delta \xi,$$

that will imply that:

(1)
$$\frac{1}{h} = -\frac{\delta x\, \delta\xi + \delta y\, \delta\eta + \delta z\, \delta\zeta}{\delta x^2 + \delta y^2 + \delta z^2}.$$

We would like to call the quantity 1 / h the *normal curvature* of the trajectory considered. We can also define it to be the radius of curvature of a planar, orthogonal trajectory at the point P whose plane includes the tangent (ξ, η, ζ), as well as the tangent (cos α, cos β, cos γ).

When one introduces the notations:

$$e = \sum \xi_p\, x_p, \quad f = \sum \xi_p\, x_q, \quad f' = \sum \xi_q\, x_p, \quad g = \sum \xi_q\, x_q,$$

one will get:

(2)
$$\frac{1}{h} = -\frac{e\, dp^2 + (f + f')\, dp\, dq + g\, dq^2}{E\, dx^2 + 2F\, dp\, dq + G\, dq^2}.$$

We shall now verify some relations between the coefficients that appear above. Since:

$$\xi = \frac{1}{\sqrt{a_{33}}} \frac{\partial x}{\partial r},$$

it will follow that:

$$\xi_p = \frac{1}{\sqrt{a_{33}}}\left(\frac{\partial^2 x}{\partial r\, \partial p} - \frac{a_{13}}{a_{33}} \frac{\partial^2 x}{\partial r^2}\right) + \frac{\partial x}{\partial r}\left(\frac{1}{\sqrt{a_{33}}}\right)_p,$$

$$\xi_q = \frac{1}{\sqrt{a_{33}}}\left(\frac{\partial^2 x}{\partial r\, \partial q} - \frac{a_{23}}{a_{33}} \frac{\partial^2 x}{\partial r^2}\right) + \frac{\partial x}{\partial r}\left(\frac{1}{\sqrt{a_{33}}}\right)_q.$$

However:

$$\frac{\partial x_p}{\partial r} = \frac{\partial^2 x}{\partial r\, \partial p} - \frac{\partial x}{\partial r} \frac{\partial \dfrac{a_{13}}{a_{33}}}{\partial r} - \frac{a_{13}}{a_{33}} \frac{\partial^2 x}{\partial r^2},$$

$$\frac{\partial x_q}{\partial r} = \frac{\partial^2 x}{\partial r\,\partial q} - \frac{\partial x}{\partial r}\frac{\partial \frac{a_{23}}{a_{33}}}{\partial r} - \frac{a_{23}}{a_{33}}\frac{\partial^2 x}{\partial r^2}.$$

If one now generally sets:

$$g_0(\varphi) = \frac{1}{\sqrt{a_{33}}}\frac{\partial \varphi}{\partial r}$$

for an arbitrary function φ then that will imply:

$$(3)\qquad \begin{cases} \xi_p = g_0(x_p) + \xi\left[\dfrac{\partial \frac{a_{13}}{a_{33}}}{\partial r} - \dfrac{1}{2}(\log a_{33})_p\right], \\[2ex] \xi_q = g_0(x_q) + \xi\left[\dfrac{\partial \frac{a_{23}}{a_{33}}}{\partial r} - \dfrac{1}{2}(\log a_{33})_q\right], \end{cases}$$

and as a result:

$$(4)\qquad e = \tfrac{1}{2} g_0(E), \quad f + f' = g_0(F), \quad g = g_0(G).$$

One further has:

$$f = \frac{1}{\sqrt{a_{33}}}\left\{\frac{\partial a_{23}}{\partial p} - \sum \frac{\partial^2 x}{\partial p\,\partial q}\frac{\partial x}{\partial r} - \frac{a_{13}}{a_{33}}\left(\frac{\partial a_{23}}{\partial r} - \frac{1}{2}\frac{\partial a_{33}}{\partial q}\right) - \frac{a_{23}}{2a_{33}}\left(\frac{\partial a_{23}}{\partial p} - \frac{a_{13}}{a_{33}}\frac{\partial a_{33}}{\partial q}\right)\right\},$$

$$f' = \frac{1}{\sqrt{a_{33}}}\left\{\frac{\partial a_{13}}{\partial q} - \sum \frac{\partial^2 x}{\partial p\,\partial q}\frac{\partial x}{\partial r} - \frac{a_{23}}{2a_{33}}\left(\frac{\partial a_{33}}{\partial q} - \frac{a_{13}}{a_{33}}\frac{\partial a_{33}}{\partial r}\right) - \frac{a_{23}}{a_{33}}\left(\frac{\partial a_{13}}{\partial r} - \frac{1}{2}\frac{\partial a_{33}}{\partial p}\right)\right\};$$

i.e.:

$$(4a)\qquad f - f' = -\frac{1}{a_{23}\sqrt{a_{33}}}\left\{a_{13}\left(\frac{\partial a_{23}}{\partial r} - \frac{\partial a_{33}}{\partial q}\right) + a_{23}\left(\frac{\partial a_{33}}{\partial r} - \frac{\partial a_{13}}{\partial r}\right) + a_{33}\left(\frac{\partial a_{13}}{\partial q} - \frac{\partial a_{23}}{\partial p}\right)\right\}.$$

The equation $f = f'$ then says the same thing as equation (4) in the previous paragraph, namely, that the curve family in question is a normal family.

The quantity $1 / h$ can have the property that it does not change for all of the orthogonal trajectories that go through the point P. The necessary conditions for this are:

$$(5)\qquad e : f + f' : g = E : 2F : G.$$

If those conditions are fulfilled continually then we call the family of curves *isotropic*. For instance, the curves whose normals coincide with the lines of a linear complex define an isotropic family. If one takes the axis of the complex to be the z-axis then when the complex is special, the curves in question will be nothing but circles whose planes are perpendicular to the z-axis and whose centers lie along the z-axis. Those curves are the orthogonal trajectories to the pencil of planes that is laid through the z-axis.

However, if the complex is not special then the curves in question will be isogonal trajectories to the generators of circular cylinders, and one will have:

$$x = p\sin\frac{r}{m} + q\cos\frac{r}{m}, \qquad y = -p\cos\frac{r}{m} + q\sin\frac{r}{m}, \qquad z = r,$$

in which m means an arbitrary constant. Here, a simple calculation will show that:

$$e = f + f' = g = 0.$$

We exclude isotropic families of curves from consideration and impose the further condition on a regular point of a non-isotropic family of curves that equations (5) must not be fulfilled for it. The quantity $1 / h$ will then possess a greatest value $1 / h_1$ and a smallest value $1 / h_2$. Those values (viz., *principal normal curvatures*) are the roots of the equation:

$$\frac{EG - F^2}{h^2} + \frac{1}{h}[eG - (f + f')F + gE] + eg - \left(\frac{f + f'}{2}\right)^2 = 0, \tag{6}$$

and they belong to the orthogonal trajectories that are determined by the equation:

$$\left(\frac{f + f'}{2}E - eF\right)dp^2 - (eG - gE)\,dp\,dq + \left(gF - \frac{f + f'}{2}G\right)dq^2 = 0. \tag{7}$$

We shall discuss only the case in which the coefficient of dq^2 in (7) does not vanish, since the opposite case poses no difficulty.

One takes $dq / dp = t$ and denotes the roots of (7) by t_1 and t_2 in such a way that:

$$\frac{1}{h_1} = -\frac{e + (f + f')t_1 + gt_1^2}{E + 2Ft_1 + Gt_1^2}.$$

The following relations exist between the roots t_1 and t_2 :

$$\begin{cases} e + \dfrac{f + f'}{2}(t_1 + t_2) + g\,t_1\,t_2 = 0, \\ E + F(t_1 + t_2) + G\,t_1\,t_2 = 0. \end{cases} \tag{8}$$

With the help of the latter, one finds that:

$$(F + G\,t_1)\,(F + G\,t_2) = -\,(EG - F^2).$$

That equation shows that t_1 and t_2 are always real, since the quantity $EG - F^2$ is always positive.

The direction cosines of the tangents to the orthogonal trajectories whose normal curvature is $1 / h_1$ or $1 / h_2$ shall be denoted by $\kappa_1, \lambda_1, \mu_1$ or $\kappa_2, \lambda_2, \mu_2$, resp.

If one sets:

$$V_1^2 = E + 2\,F\,t_1 + G\,t_1^2, \qquad V_2^2 = E + 2\,F\,t_2 + G\,t_2^2$$

then one will have:

$$(9) \qquad \begin{cases} \kappa_1 = \dfrac{x_p + x_q t_1}{V_1}, \quad \lambda_1 = \dfrac{y_p + y_q t_1}{V_1}, \quad \mu_1 = \dfrac{z_p + z_q t_1}{V_1}, \\[2ex] \kappa_2 = \dfrac{x_p + x_q t_2}{V_2}, \quad \lambda_2 = \dfrac{y_p + y_q t_2}{V_2}, \quad \mu_2 = \dfrac{z_p + z_q t_2}{V_2}. \end{cases}$$

The second equation in (8) shows that:

$$\kappa_1\,\kappa_2 + \lambda_1\,\lambda_2 + \mu_1\,\mu_2 = 0.$$

We have then found two doubly-infinite families of orthogonal trajectories to a family of curves that are mutually perpendicular, in addition. They shall be called *lines of curvature of the first kind* of the family of curves.

The lines of curvature, as well as the quantities $1 / h_1$ and $1 / h_2$, are independent of the choice of variables and the parameter with the aid of which one establishes the family of curves considered. Namely, the latter will not change under the substitution:

$$p = \psi(p_1, q_1), \qquad q = \psi_1(p_1, q_1), \qquad r = \psi_2(p_1, q_1),$$

as long as r_1 is the new variable, while p_1, q_1 appear as the new parameters, and the determinant:

$$\frac{\partial p}{\partial p_1}\frac{\partial q}{\partial q_1} - \frac{\partial p}{\partial q_1}\frac{\partial q}{\partial p_1}$$

does not vanish. The coordinates x, y, z of the point of the family will then be functions of p_1, q_1, r_1. If one calculates the expressions for $1 / h_1$, $1 / h_2$, κ_1, κ_2, λ_1, λ_2, μ_1, μ_2, then they will denote the expressions that correspond to the ones that are formed in the system of variables p, q, r. The basis for the calculations that are necessary in order to corroborate these assertions (which shall not be specified, due to their simplicity) is defined by the following equations, which are valid for an arbitrary function $\mathfrak{F}$ of p, q, and r :

$$\mathfrak{F}_{p_1} = \mathfrak{F}_p \frac{\partial p}{\partial p_1} + \mathfrak{F}_q \frac{\partial q}{\partial p_1}, \qquad \mathfrak{F}_{q_1} = \mathfrak{F}_p \frac{\partial p}{\partial q_1} + \mathfrak{F}_q \frac{\partial q}{\partial q_1}, \qquad \frac{\partial \mathfrak{F}_r}{\partial r_1} = \frac{\partial \mathfrak{F}}{\partial r}\frac{\partial r}{\partial r_1}.$$

The given definition of the lines of curvature of the first kind assumes only that the family of curves considered is not isotropic. If one assumes, in addition, that is it not special, so $H\,\Psi - \Phi^2 > 0$ (in which case, the family will be called *general*), then one can give yet a second definition for the lines of curvature in question, which will now be considered.

The shortest distance between neighboring tangents (x, ξ) and $(x + \delta x, \xi + \delta\xi)$ meets the tangent (x, ξ) at a point whose abscissa relative to the point (x, y, z) shall be $\mathfrak{r}$.

One will then have:

$$\mathfrak{r} = -\frac{\sum \delta x\,\delta\xi}{\sum \delta\xi^2} = -\frac{e\,dp^2 + (f + f')\,dp\,dq + g\,dq^2}{H\,dp^2 + 2\Phi\,dp\,dq + \Psi\,dq^2}.$$

Since the determinant $H\,\Psi - \Phi^2$ is non-zero, $\mathfrak{r}$ will possess a greatest value $\mathfrak{r}_1$ and a smallest one $\mathfrak{r}_2$. The values are the roots of the equation:

$$(10) \qquad (H\,\Psi - \Phi^2)\,\mathfrak{r}^2 + [g\,H + e\,\Psi - (f + f')\,\Phi]\,\mathfrak{r} + e\,g - \left(\frac{f + f'}{4}\right)^2 = 0,$$

and shall be called the *abscissas of the endpoints of the shortest distance*.

The values τ_1 and τ_2 of the ratio $dq / dp = t$, which determine the neighboring tangents that yield the maximum and minimum, resp., are the roots of the equation:

$$(11) \qquad \left(g\Phi - \frac{f + f'}{2}\Psi\right)t^2 - (e\Psi - gH)\,t + \frac{f + f'}{2}H - e\Phi = 0.$$

We choose the notation for the roots such that:

$$\mathfrak{r}_1 = -\frac{e + (f + f')\,\tau_1 + g\,\tau_1^2}{H + 2\Phi\,\tau_1 + \Psi\,\tau_1^2}.$$

The equation:

$$H + \Phi\,(\tau_1 + \tau_2) + \Psi\,\tau_1\,\tau_2 = 0$$

implies that the direction cosines of the shortest distance that belong to $\mathfrak{r}_2$ ($\mathfrak{r}_1$, resp.) are:

$$\kappa_1' = \frac{\xi_p + \xi_q\,\tau_1}{W_1}, \qquad \lambda_1' = \frac{\eta_p + \eta_q\,\tau_1}{W_1}, \qquad \mu_1' = \frac{\zeta_p + \zeta_q\,\tau_1}{W_1},$$

or

$$\kappa_2' = \frac{\xi_p + \xi_q\,\tau_2}{W_2}, \qquad \lambda_2' = \frac{\eta_p + \eta_q\,\tau_2}{W_2}, \qquad \mu_2' = \frac{\zeta_p + \zeta_q\,\tau_2}{W_2},$$

resp., in which:

$$W_1^2 = H + 2\,\Phi\,\tau_1 + \Psi\tau_1^2, \qquad W_2^2 = H + 2\,\Phi\,\tau_2 + \Psi\tau_2^2.$$

It shall now be shown that:

$$\kappa_1' = \kappa_1, \qquad \lambda_1' = \lambda_1, \qquad \mu_1' = \mu_1, \qquad \kappa_2' = \kappa_2, \qquad \lambda_2' = \lambda_2, \qquad \mu_2' = \mu_2,$$

such that the lines of curvature of the first kind run parallel to the shortest distance at the endpoints. Before we do that, we shall develop some formulas.

If one solves the equations:

$$\sum \xi_p\, x_p = e, \quad \sum \xi_p\, x_q = f, \quad \sum \xi_p\, \xi = 0$$

for ξ_p, η_p, ζ_p, and likewise solves the equations:

$$\sum \xi_q\, x_p = f', \quad \sum \xi_q\, x_q = g, \quad \sum \xi_q\, \xi = 0$$

for ξ_q, η_q, ζ_q then it will follow that:

$$\xi_p = \frac{x_p(eG - f\,F) + x_q(f\,E - e\,F)}{EG - F^2}, \quad \xi_q = \frac{x_p(f'G - g\,F) + x_q(g\,E - f'F)}{EG - F^2},$$

and therefore:

$$H = \frac{e^2\,G - 2e\,f\,F + f^2 E}{EG - F^2}, \qquad \Phi = \frac{e\,f'G - (e\,g + f\,f')F + f\,g\,E}{EG - F^2},$$

$$\Psi = \frac{f'^2\,G - 2\,f'\,g\,F + g^2 E}{EG - F^2}, \qquad H\,\Psi - \Phi^2 = \frac{(e\,g - f\,f')^2}{EG - F^2}.$$

The quantity $e\,g - f\,f'$ is non-zero then. One now introduces two values t' and t'' with the help of the equations:

$$t' = -\frac{e + f'\tau_1}{f + g\,\tau_1}, \qquad t'' = -\frac{e + f'\tau_2}{f + g\,\tau_2}.$$

One then has:

$$e + \frac{f + f'}{2}(t' + t) + g\,t't'' = (e\,g - f\,f')\,\frac{e + \dfrac{f + f'}{2}(\tau_1 + \tau_2) + g\,\tau_1\tau_2}{f^2 + f\,g\,(\tau_1 + \tau_2) + g^2\,\tau_1\,\tau_2},$$

$$E + F\,(t' + t) + G\,t't'' = \frac{(EG - F^2)[H + \Phi(\tau_1 + \tau_2) + \Psi\,\tau_1\tau_2]}{f^2 + f\,g\,(\tau_1 + \tau_2) + g^2\,\tau_1\,\tau_2}.$$

The right-hand side of these relations vanishes because of (11). In that way, however, t' and t'' will prove to be roots of equation (7). Furthermore, one will have:

$$\xi_p + \xi_q \, \tau_1 = \frac{(e g - f f')\{x_p(F + G t') - x_q(E + F t')\}}{(EG - F^2)(f' + g t')} = \frac{(e g - f f')(F + G t')(x_p + x_q t'')}{(EG - F^2)(f' + g t')}.$$

If one sets $t'' = t_1$, $t' = t_2$ then it will follow that:

$$\kappa_1' = \kappa_1, \qquad \kappa_2' = \kappa_2, \qquad \text{etc.}$$

In addition, one has:

$$e + (f + f')\,\tau_1 + g\,\tau_1^2 = \frac{e g - f f'}{(f' + g t_2)^2}[e + (f + f')t_2 + g t_2^2],$$

$$H + 2\Phi\,\tau_1 + \Psi\,\tau_1^2 = \frac{(e g - f f')^2 (E + 2F t_2 + G t_2^2)}{(EG - F^2)(f' + g t_2)^2}.$$

The following relations then exist between the quantities $\mathfrak{r}_1$, $\mathfrak{r}_2$, on the one hand, and h_1, h_2, on the other:

(12) $$\mathfrak{r}_1 = \frac{EG - F^2}{e g - f f'} \cdot \frac{1}{h_2}, \qquad \mathfrak{r}_2 = \frac{EG - F^2}{e g - f f'} \cdot \frac{1}{h_1}.$$

The name of *lines of curvature of the second kind* shall be applied to those orthogonal trajectories of the family of curves whose tangents (ξ, η, ζ) define developable surfaces. The tangent (x, ξ) will be cut by the neighboring tangent ($x + \delta x$, $\xi + \delta \xi$) in the event that:

(13) $$\delta x\,(\eta\,\delta\zeta - \zeta\,\delta\eta) + \delta y\,(\zeta\,\delta\xi - \xi\,\delta\zeta) + \delta z\,(\xi\,\delta\eta - \eta\,\delta\xi) = 0.$$

Let the first of the families of curves be special. Here, we take:

$$\xi_q = n\,\xi_p, \qquad \eta_q = n\,\eta_p, \qquad \zeta_q = n\,\zeta_p,$$

in which n means a finite function of p, q, r, which can also vanish. The case that was excluded here $\xi_p = \eta_p = \zeta_p = 0$ is treated analogously.

If one sets:

$$k = \sum x_p(\eta\,\zeta_p - \zeta\,\eta_p), \qquad k' = \sum x_q(\eta\,\zeta_p - \zeta\,\eta_p)$$

then the condition (13) will assume the form:

$$(k + k' t)(1 + n\,t) = 0.$$

Secondly, if the family of curves is general then one will have:

$$\xi = \frac{\eta_p\,\zeta_q - \zeta_p\,\eta_q}{\sqrt{H\,\Psi - \Phi^2}}, \qquad \eta = \frac{\zeta_p\,\xi_q - \xi_p\,\zeta_q}{\sqrt{H\,\Psi - \Phi^2}}, \qquad \zeta = \frac{\xi_p\,\eta_q - \xi_p\,\eta_q}{\sqrt{H\,\Psi - \Phi^2}},$$

and the condition (13) will become:

$$(g\,\Phi - f\,\Psi)\,dq^2 + [g\,H - (f - f')\,\Phi - e\,\Psi]\,dp\,dq + (f'H - e\,\Phi)\,dp^2 = 0.$$

The abscissa of the intersection point of the two neighboring tangents relative to the point (x, y, z) possesses the expression:

$$-\frac{\delta x\,\delta\xi + \delta y\,\delta\eta + \delta z\,\delta\zeta}{\delta\xi^2 + \delta\eta^2 + \delta\zeta^2}.$$

For a special family of curves with $1 + n\,t = 0$, that expression will always be infinite. The other value that one gets for $k + k'\,t = 0$ might be denoted by ρ_1, such that:

$$\rho_1 = -\frac{e\,k' - f\,k}{H(k' - nk)}.$$

In the case considered, one will have:

$$x_p = \frac{e\,\xi_p + k\,(\eta\,\zeta_p - \zeta\,\eta_p)}{H}, \qquad x_q = \frac{f\,\xi_p + k'\,(\eta\,\zeta_p - \zeta\,\eta_p)}{H},$$

and as a result:

$$E = \frac{e^2 + k^2}{H}, \qquad F = \frac{e\,f + k\,k'}{H}, \qquad G = \frac{f^2 + k'^2}{H}.$$

Since one has:

$$g = n\,f, \qquad f' = n\,e$$

here, in addition, one will find that:

$$\frac{eG + Eg - (f + f')F}{EG - F^2} = \frac{H(k' - n\,k)}{e\,k' - f\,k}, \qquad \frac{e\,g - \frac{1}{4}(f + f')^2}{EG - F^2} = -\frac{(f - f')^2}{4(EG - F^2)}.$$

We set $\frac{(f - f')^2}{4(EG - F^2)} = \varepsilon^2$, in general, and get from (6) that:

$$\frac{1}{h_1} + \frac{1}{h_2} = \frac{1}{\rho_1}, \qquad \frac{1}{h_1\,h_2} = \varepsilon^2.$$

For a general family of curves, the abscissas ρ_1 and ρ_2 of the intersection point in question will be roots of the equation:

(14) $$(H\,\Psi - \Phi^2)\,\rho^2 + [e\,\Psi - (f + f')\,\Phi + g\,H]\,\rho + e\,g - f\,f' = 0,$$

such that:

$$\frac{e\,g - f\,f'}{H\,\Psi - \Phi^2} = \frac{E\,G - F^2}{e\,g - f\,f'} = \rho_1\,\rho_2\,.$$

A comparison of (10) and (14) yields the relations between ρ_1 , ρ_2 and $\mathfrak{r}_1$, $\mathfrak{r}_2$ in the form:

(15) $$\mathfrak{r}_1 + \mathfrak{r}_2 = \rho_1 + \rho_2\,, \qquad \mathfrak{r}_1\,\mathfrak{r}_2 = \rho_1\,\rho_2\,(1 - \varepsilon^2\,\rho_1\,\rho_2)\,,$$

and a comparison of (12) and (14) will imply relations between ρ_1 , ρ_2 and h_1 , h_2 in the form:

(16) $$\begin{cases} \dfrac{1}{h_1} + \dfrac{1}{h_2} = \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}, \\[2ex] \dfrac{1}{h_1\,h_2} = \dfrac{1}{\rho_1\,\rho_2} - \varepsilon^2. \end{cases}$$

The lines of curvature of the second kind define a single family, or two separate families, or they will be imaginary according to whether equation (14) possesses equal, distinct real, or imaginary roots, resp. For any normal family ($\varepsilon = 0$), the lines of curvature of the second kind will coincide with the lines of curvature of the first kind.

§ 5. – On the theory of rectilinear ray systems.

One calls a doubly-infinite family of curves that consists of nothing but straight lines a *ray system*. The study of the curvature of those systems was first addressed by **Kummer** (J. reine angew. Math., Bd. **57**, pp. 189), which has been followed as a basis up to now, in essence. (Cf., **Bianchi**, *Lezioni di Geometria differenziale*, pp. 24, *et seq.*), although **Königs** (Ann. sci. de l'École Normale, 1882, pp. 219) has enriched the study of focal points and focal planes by basing it upon projective geometry. **Kummer** considered only general ray systems. In that way, the introduction of composed differentiations x_p , x_q , etc., would be superfluous, and the definition of the principal planes would break down for a special ray system. On that basis, an introduction to the study of ray systems shall follow here that starts from the **Königs** viewpoint and represents an application of the developments that were given in the previous paragraphs.

Let a line with the direction cosines ξ, η, ζ be laid through a point P_0 with the coordinates x_0 , y_0 , z_0 . An arbitrary point P of the line will have the coordinates:

(1) $$x = x_0 + l\,\xi, \qquad y = y_0 + l\,\eta, \qquad z = z_0 + l\,\zeta\,.$$

A second line, which should not, however, be perpendicular to the first one, possesses the direction cosines ξ', η', ζ'. The normal plane to the first line that is laid through P_0 cuts the second one at the point P_0' with the coordinates x_0', y_0', z_0'. If one lays a normal plane to the first line through P then the second line will intersect it at a point P' whose coordinates are:

$$(2) \qquad x' = x_0' + \frac{l\,\xi'}{\cos\varphi}, \qquad y' = y_0' + \frac{l\,\eta'}{\cos\varphi}, \qquad z' = z_0' + \frac{l\,\zeta'}{\cos\varphi}.$$

We now imagine that we have chosen two mutually-perpendicular planes that intersect in the first line. Let the direction cosines of the normal to the first of those planes (E_1) be α_x, α_y, α_z ; let those of the normal to the second one (E_2) be β_x, β_y, β_z. If the plane that goes through the first line and the point P' makes the angle λ with the plane (E_1) then that will yield:

$$(3) \qquad \tan\lambda = \frac{\cos\varphi\sum(x_0'-x_0)\,\alpha_x + l\sum\xi'\,\alpha_x}{\cos\varphi\sum(x_0'-x_0)\,\beta_x + l\sum\xi'\,\beta_x}.$$

We would like to consider the plane in question to be associated with the point P. If one regards the abscissa l in (3) as varying then that equation will represent a pencil of planes, whose axis is the first line and whose planes are projectively associated with that line. That association will be special when:

$$\sum(x_0'-x_0)(\eta\,\zeta'-\zeta\,\eta') = 0.$$

Here, the lines are either parallel or they will intersect at a point whose abscissa relative to the P_0 will be represented by:

$$\sigma = -\cos\varphi\,\frac{\sum(x_0'-x_0)\xi'}{\sin^2\varphi}.$$

If we exclude this case then the point at infinity on the first line will correspond to the plane (E_3) that is parallel to the second line. The endpoint of the shortest distance from the first line to the second one that lies on the first line corresponds to the plane that is perpendicular to E_3.

In addition to the point P, consider a second point Q of the first line for which the quantities l and λ will be denoted by l' and λ', respectively. If one gives equation (3) the form:

$$a_1\,l\tan\lambda + a_2\,l + a_3\,l\tan\lambda + a_4 = 0,$$

to abbreviate, then one will have:

$$\frac{1}{l'-l} = -\frac{(a_1\,a_4-a_2\,a_3)\cot(\lambda-\lambda')+(a_1^2+a_2^2)\,l+a_1\,a_3+a_2\,a_4}{(a_1\,l+a_3)^2+(a_2\,l+a_4)^2}$$

here, in which $l' - l$ means the abscissa of the point Q relative to P.

Kummer gave the quantity $\lambda - \lambda'$ the name of *rotational angle of the second line with respect to the first for the line segment PQ*.

If the angle of rotation is a right angle then it will follow that:

(4) $$\frac{1}{l'-l} = -\frac{(a_1^2+a_2^2)\,l + a_1\,a_3 + a_2\,a_4}{(a_1\,l+a_3)^2+(a_2\,l+a_4)^2}.$$

One now considers x_0, y_0, z_0, ξ, η, ζ to be functions of p and q and gives the increments Δp, Δq to those variables, with which, x, ξ, etc., might go to $x_0 + \Delta x_0$, $\xi + \Delta\xi$, etc. One will then have:

$$x_0' = x_0 + \Delta x_0 - \frac{(\xi+\Delta\xi)\sum\xi\,\Delta x_0}{\cos\varphi},$$

and since:

$$\Delta x = \Delta x_0 + l\,\Delta\xi,$$

one will also have:

$$a_1 = \sum\Delta\xi\cdot\beta_x, \qquad a_2 = -\sum\Delta\xi\cdot\alpha_x,$$

$$a_1\,l + a_3 = \cos\varphi\sum\beta_x\,\Delta_x', \qquad a_2\,l + a_4 = -\cos\varphi\sum\alpha_x\,\Delta_x'.$$

Equation (3) goes to:

$$\tan\lambda = \frac{\cos\varphi\sum\alpha_x\,\Delta x_0 + l\sum\alpha_x\,\Delta\xi - \sum\alpha_x\,\Delta\xi\cdot\sum\xi\,\Delta x_0}{\cos\varphi\sum\beta_x\,\Delta x_0 + l\sum\beta_x\,\Delta\xi - \sum\beta_x\,\Delta\xi\cdot\sum\xi\,\Delta x_0},$$

and when one takes $l' - l = h$, (4) will be replaced with:

$$\frac{1}{h} = -\frac{\sum\Delta\xi\,\Delta' x}{\cos\varphi\sum\Delta' x^2}.$$

$\Delta' x$, $\Delta' y$, $\Delta' z$ will be replaced with δx, δy, δz when one passes to the limit. One will then have:

(5) $$\tan\lambda = \frac{\sum\alpha_x\,dx_0 + l\sum\alpha_x\,d\xi}{\sum\beta_x\,dx_0 + l\sum\beta_x\,d\xi},$$

(6) $$\frac{1}{h} = -\frac{\sum d\xi\,\delta x}{\sum\delta x^2}$$

in the limit.

In that way, one will have:

$$\delta x = x_p\,dp + x_q\,dq = \left(\frac{\partial x_0}{\partial p} + l\frac{\partial\xi}{\partial p} - \xi\sum\xi\frac{\partial x_0}{\partial p}\right)dp + \left(\frac{\partial x_0}{\partial q} + l\frac{\partial\xi}{\partial q} - \xi\sum\xi\frac{\partial x_0}{\partial q}\right)dq.$$

Equation (5) represents a *linear sequence of pencils of planes*, in which each individual pencil will be established when one determines the ratio dq / dp.

The *singular* individual pencils of the sequence, at which all points of the line (x_0, ξ) are associated with one and the same plane (viz., a special projectivity), are the *focal planes* of the ray (x_0, ξ).

One sets:

$$e_0 = \sum \frac{\partial x_0}{\partial p}\frac{\partial \xi}{\partial p}, \qquad f_0 = \sum \frac{\partial x_0}{\partial q}\frac{\partial \xi}{\partial p}, \qquad f_0' = \sum \frac{\partial x_0}{\partial p}\frac{\partial \xi}{\partial q}, \qquad g_0 = \sum \frac{\partial x_0}{\partial q}\frac{\partial \xi}{\partial q},$$

$$k_0 = \sum \frac{\partial x_0}{\partial p}\left(\eta\frac{\partial \zeta}{\partial p} - \zeta\frac{\partial \eta}{\partial p}\right), \qquad k_0' = \sum \frac{\partial x_0}{\partial q}\left(\eta\frac{\partial \zeta}{\partial q} - \zeta\frac{\partial \eta}{\partial q}\right),$$

to abbreviate. If one is dealing with a special ray system, in which:

$$\frac{\partial \xi}{\partial q} = n\frac{\partial \xi}{\partial p}, \quad \frac{\partial \eta}{\partial q} = n\frac{\partial \eta}{\partial p}, \quad \frac{\partial \zeta}{\partial q} = n\frac{\partial \zeta}{\partial p},$$

then the focal planes will be established by the equations:

$$k_0\, dp + k_0'\, dq = 0 \qquad \text{and} \qquad dp + n\, dq = 0.$$

One focal point lies at infinity, while the abscissa ρ of the other one will satisfy the equation:

(7)
$$\frac{1}{\rho} = -\frac{H(k_0' - n\,k_0)}{e_0\,k_0' - k_0\,f_0}.$$

For a general ray system, the focal planes will be determined with the help of the equation:

$$(H\,f_0' - \Phi\,e_0)\,dq^2 + [H\,g_0 - \Phi(f_0 - f_0') - \Psi\,e_0]\,dp\,dq + (\Phi g_0 - \Psi f_0)\,dq^2 = 0,$$

while the abscissas of the focal points will satisfy the relations:

(8)
$$(H\,\Psi - \Phi^2)\,\rho^2 + [e_0\,\Psi - (f_0 + f_0')\,\Phi + g_0\,H]\,\rho + e_0\,g_0 - f_0\,f_0' = 0.$$

One way of generating a ray system with imaginary focal points can be obtained as follows:

Take three functions U, V, W of the complex variables $p + i\,q$ and set:

$$U = x_0 + i\,x_1, \quad V = y_0 + i\,y_1, \quad W = z_0 + i\,z_1,$$

in which the real and imaginary parts have been separated. x_0, y_0, z_0 are then the coordinates of one surface, while x_1, y_1, z_1 are the coordinates of the other. A pair of values (p, q) associates a point of the one surface with a point of the other one, and the connecting lines from one point to its associated point will define a ray system.

The direction cosines for the rays are determined by the equations:

$$\xi = \frac{x_1 - x_0}{R}, \quad \eta = \frac{y_1 - y_0}{R}, \quad \zeta = \frac{z_1 - z_0}{R},$$

in which:

$$R = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}\,.$$

One then obtains:

$$H = \frac{1}{R}\sum \frac{\partial(x_1 - x_0)}{\partial p}\frac{\partial \xi}{\partial p} = -\frac{e_0 + f_0}{R},$$

$$\Phi = \frac{1}{R}\sum \frac{\partial(x_1 - x_0)}{\partial p}\frac{\partial \xi}{\partial q} = -\frac{g_0 + f_0'}{R},$$

$$\Phi = \frac{1}{R}\sum \frac{\partial(x_1 - x_0)}{\partial q}\frac{\partial \xi}{\partial p} = \frac{e_0 - f_0}{R},$$

$$\Psi = \frac{1}{R}\sum \frac{\partial(x_1 - x_0)}{\partial q}\frac{\partial \xi}{\partial q} = \frac{f_0' - g_0}{R},$$

$$H\,\Psi - \Phi^2 = 2\,\frac{e_0\,g_0 - f_0\,f_0'}{R^2}.$$

Equation (8) takes on the form:

$$2\rho^2 - 2\rho\,R + R^2 = 0$$

and possesses imaginary roots.

We now consider equation (6). When one employs the notations of the previous paragraphs, it will assume the form:

$$(9) \qquad \frac{1}{h} = -\frac{e\,dp^2 + (f + f')\,dp\,dq + g\,dq^2}{E\,dp^2 + 2F\,dp\,dq + G\,dq^2}.$$

In this, one has:

$$e = e_0 + l\,H, \quad f = f_0 + l\,\Phi, \quad f' = f_0' + l\,\Phi, \quad g = g_0 + l\,\Psi,$$

and when:

$$k = \sum x_p \left(\eta \frac{\partial \zeta}{\partial p} - \zeta \frac{\partial \eta}{\partial p}\right), \qquad k' = \sum x_q \left(\eta \frac{\partial \zeta}{\partial p} - \zeta \frac{\partial \eta}{\partial p}\right),$$

one will further have:

$$x_p = \frac{1}{H}\left[e\frac{\partial\xi}{\partial p} + k\left(\eta\frac{\partial\zeta}{\partial p} - \zeta\frac{\partial\eta}{\partial p}\right)\right], \quad x_q = \frac{1}{H}\left[f\frac{\partial\xi}{\partial p} + k'\left(\eta\frac{\partial\zeta}{\partial p} - \zeta\frac{\partial\eta}{\partial p}\right)\right],$$

so one will also have:

$$E = \frac{e^2 + k^2}{H}, \quad F = \frac{e\,f + k\,k'}{H}, \quad G = \frac{f^2 + k'^2}{H}, \quad E\,G - F^2 = \left(\frac{e\,k' - f\,k}{H}\right)^2.$$

The difference $E\,G - F^2$ vanishes only when the point is a focal point with the abscissa l. Namely, for a special ray system:

$$e\,k' - f\,k = e_0\,k_0' - f_0\,k_0 + H\,l\,(k_0' - n\,k_0),$$

but for a general one, one will have:

$$\xi = \frac{\dfrac{\partial\eta}{\partial p}\dfrac{\partial\zeta}{\partial q} - \dfrac{\partial\eta}{\partial q}\dfrac{\partial\zeta}{\partial p}}{\sqrt{H\Psi - \Phi^2}}, \text{ etc.,}$$

and as a result:

(10) $$k = \frac{H\,f' - \Phi\,e}{\sqrt{H\Psi - \Phi^2}}, \qquad k' = \frac{H\,g - \Phi\,f}{\sqrt{H\Psi - \Phi^2}},$$

$$e\,k' - f\,k = \frac{H\,(eg - f\,f')}{\sqrt{H\Psi - \Phi^2}},$$

$$e\,g - f\,f' = l^2\,(H\,\Psi - \Phi^2) + l\,[e_0\,\Psi - (f_0 + f_0')\,\Phi + g_0\,H] + e_0\,g_0 - f_0\,f_0'.$$

If the point (x, y, z) is not a focal point then that will imply the two normal principal curvatures and the lines of the first kind, as in the previous paragraphs. In order to arrive at **Kummer**'s principal planes, one must show that the tangents to one and the same family of lines of curvature of the first kind are parallel to a ray, or what amounts to the same thing, that the direction cosines κ_1, κ_2, etc., do not depend upon l.

When one employs the expressions for x_p, x_q above, one will have:

$$x_\nu = \frac{(e + f\,t_\nu)\dfrac{\partial\xi}{\partial p} + (k + k'\,t_\nu)\left(\eta\dfrac{\partial\zeta}{\partial p} - \zeta\dfrac{\partial\eta}{\partial p}\right)}{\sqrt{H}\,\sqrt{(e + f\,t_\nu)^2 + (k + k'\,t_\nu)^2}} \qquad (\nu = 1, 2).$$

The two quotients:

$$u_\nu = \frac{k + k'\,t_\nu}{e + f\,t_\nu}$$

must then be independent of l.

In order to prove that, one takes:

$$u = \frac{k + k' t}{e + f t}$$

and replaces the quantity $t = dq / dp$ with:

$$\frac{e u - k}{k' - f u}$$

in the expression for $1 / h$. If one lets J denote the determinant:

$$\begin{vmatrix} \xi & \eta & \zeta \\ \frac{\partial \xi}{\partial p} & \frac{\partial \eta}{\partial p} & \frac{\partial \zeta}{\partial p} \\ \frac{\partial \xi}{\partial q} & \frac{\partial \eta}{\partial q} & \frac{\partial \zeta}{\partial q} \end{vmatrix}$$

then one will have:

$$f = \frac{e \Phi + k J}{H}, \qquad g = \frac{f \Phi + k' J}{H},$$

and

$$e + (f + f') t + g t^2 = \frac{e k' - f k}{H (k' - f u)^2} [k' H - k \Phi + (e \Phi - f H - k J) u + e J u^2],$$

$$E + 2F t + G t^2 = \frac{(e k' - f k)^2}{H (k' - f u)^2} (1 + u^2),$$

so

$$\frac{1}{h} = \frac{k' H - k \Phi + (e \Phi - f H - k J) u + e J u^2}{(f k - e k')(1 + u^2)}.$$

The quantities u_1 and u_2 are the values of u that come from the maximum and minimum of $1 / h$, resp., and as a result, they will be the roots of the equation:

$$u^2 + \frac{k' H - k \Phi - e J}{e \Phi - f H - k J} - 1 = 0.$$

Now since:

$$k = k_0, \qquad k' = k_0' + l J,$$

one has:

$$u_1 + u_2 = -\, 2 \frac{k_0' H - k_0 \Phi - e_0 J}{e_0 \Phi - f_0 H - k_0 J};$$

i.e., the quantities u_1 and u_2 are, in fact, independent of l.

The *spherical image* plays a key role for the general ray system. One draws radii of the unit sphere that are parallel to the positive halves of the rays of the system and considers the endpoint of a radius that lies on the surface of the sphere to be the spherical image of the ray that is parallel to the radius. In that way, any surface that is generated by rays of the system will be mapped to a curve on the unit sphere.

The lines of curvature of the second kind run through developable surfaces. Such a line will be mapped to a spherical curve whose tangents are parallel to the tangents to the lines of curvature of the second kind that run through the developable surface considered. We call the direction cosines of those tangents κ_3, λ_3, μ_3; κ_4, λ_4, μ_4. We further let:

(10)[*sic*] $$d\xi = \kappa_3 S_1 + \kappa_4 S_2, \quad d\eta = \lambda_3 S_1 + \lambda_4 S_2, \quad d\zeta = \mu_3 S_1 + \mu_4 S_2 .$$

The symbols S_1 and S_2 mean linear differential forms that will yield the differential equations of the spherical images of the lines of curvature when they are set equal to zero.

There must be a representation of the differential forms δx_0, δy_0, δz_0 that takes the form:

$$\delta x_0 = \kappa_3 (a S_1 + b S_2) + \kappa_4 (c S_1 + d S_2), \qquad \text{etc.}$$

Now, one has:

$$\sum \delta x_0 (\eta\, d\zeta - \zeta\, d\eta) = 0$$

or

$$c S_1^2 - (a - d) S_1 S_2 - b S_2^2 = 0.$$

In order for that relation to be satisfied by $S_1 = 0$ and $S_2 = 0$, c and b must vanish. One now has:

$$\frac{1}{h} = -\frac{a S_1^2 + (a+d)\sum \kappa_3 \kappa_4 S_1 S_2 + d S_2^2}{a^2 S_1^2 + 2a d \sum \kappa_3 \kappa_4 S_1 S_2 + d^2 S_2^2}$$

for $l = 0$. Since the quantity h possesses the value ρ_1 for $S_2 = 0$ and the value ρ_2 for $S_1 = 0$, it will follow that:

$$a = -\rho_1, \qquad d = -\rho_2,$$

and

(11) $$\delta x_0 = -\rho_1 \kappa_3 S_1 - \rho_2 \kappa_4 S_2 .$$

We employ the representations (10) and (11) in order to convert equation (5).

One has:

$$\sum \alpha_x dx_0 = \sum \alpha_x \delta x_0, \qquad \sum \beta_x dx_0 = \sum \beta_x \delta x_0,$$

which will then imply that:

$$\tan \lambda = \frac{(l-\rho_1) S_1 \sum \alpha_x \kappa_3 + (l-\rho_2) S_2 \sum \alpha_x \kappa_4}{(l-\rho_1) S_1 \sum \beta_x \kappa_3 + (l-\rho_2) S_2 \sum \beta_x \kappa_4}.$$

The values of λ that belong to $S_2 = 0$ and $S_1 = 0$ determine the two focal planes. If we denote those values by λ_1 and λ_2 then we will have:

$$\tan\lambda = \frac{\tan\lambda_1\,(l-\rho_1)\sum\beta_x\kappa_3 + \tan\lambda_2(l-\rho_2)\sum\beta_x\kappa_4\cdot\tau}{(l-\rho_1)\sum\beta_x\kappa_3 + (l-\rho_2)\sum\beta_x\kappa_4\cdot\tau}, \tag{12}$$

in which one takes:

$$\frac{S_2}{S_1} = \tau.$$

One now fixes one's attention on two non-singular individual pencils of the linear sequence that is established by (12). They might belong to the values τ and τ', respectively. If one considers every plane that is laid through the ray (x_0 , ξ) to first belong to the individual pencil (τ) and then to the individual pencil (τ') then one will get a point in the pencil (τ) that corresponds to it whose abscissa is l, and a point in the pencil (τ') that corresponds to it whose abscissa is l'. Now, since the relation:

$$\tan\lambda = \frac{\tan\lambda_1\,(l'-\rho_1)\sum\beta_x\kappa_3 + \tan\lambda_2(l'-\rho_2)\sum\beta_x\kappa_4\cdot\tau'}{(l'-\rho_1)\sum\beta_x\kappa_3 + (l'-\rho_2)\sum\beta_x\kappa_4\cdot\tau}$$

exists, along with (12), the two individual pencils (τ) and (τ') will determine a projective point association on the ray (x_0 , ξ) whose equation is:

$$\frac{(l-\rho_1)(l'-\rho_2)}{(l-\rho_2)(l'-\rho_1)} = \frac{\tau}{\tau'}. \tag{13}$$

That will then show that the focal points of the ray (x_0 , ξ) are the double elements of each point-association of the type considered that one obtains.

The two focal planes determine two spherical tangents that go through the spherical image of the ray (x_0 , ξ) that might be called *focal tangents*. Likewise, τ and τ' determine two spherical tangents that go through the same point of the sphere, and τ / τ' is their double ratio with the pair of focal tangents. From (13), that is equal to the double ratio of every of points (l, l') that correspond to the pair of focal points.

The point-association (13) will be an involution when the double ratio τ / τ' is harmonic.

The focal planes prove to be double elements of a projective association in a similar way. Any point on the line (x_0 , ξ) corresponds to a plane in the pencil (τ), as well as one in the pencil (τ'). The former is determined by the angle λ, while the latter will be established by the angle λ'.

Now, since one has:

$$l = \frac{\rho_1(\tan\lambda - \tan\lambda_1)\sum\beta_x\kappa_3 + \rho_2(\tan\lambda - \tan\lambda_2)\sum\beta_x\kappa_4\cdot\tau'}{(\tan\lambda - \tan\lambda_1)\sum\beta_x\kappa_3 + (\tan\lambda - \tan\lambda_2)\sum\beta_x\kappa_4\cdot\tau},$$

as well as:

$$l = \frac{\rho_1(\tan\lambda' - \tan\lambda_1)\sum\beta_x\kappa_3 + \rho_2(\tan\lambda' - \tan\lambda_2)\sum\beta_x\kappa_4\cdot\tau'}{(\tan\lambda' - \tan\lambda_1)\sum\beta_x\kappa_3 + (\tan\lambda' - \tan\lambda_2)\sum\beta_x\kappa_4\cdot\tau},$$

one will then have:

$$(14) \qquad \frac{(\tan\lambda - \tan\lambda_1)(\tan\lambda' - \tan\lambda_2)}{(\tan\lambda - \tan\lambda_1)(\tan\lambda' - \tan\lambda_2)} = \frac{\tau}{\tau'}.$$

According to the choice of τ and τ', that equation will assign every plane that goes through the ray (x_0 , ξ) with a second one projectively. The double elements of that type of assignment are the two focal planes.

In connection with those remarks on ray systems, **Guichard**'s theorems (Ann. de l'Éc. Norm., 1889, pp. 333; cf., **Bianchi**, pp. 261) shall be established from the viewpoint that has been assumed here, in order to show that the introduction of the derivatives with respect to arc-lengths have a simplification of the calculations as a consequence, and then in order to communicate an, as it seems, as-yet-unnoticed surface-theoretic theorem that completes **Guichard**'s result. In order to explain the theorem in question, it is necessary to preface some remarks that can be regarded as an extension of the developments in § **2**.

We choose two linear differential forms:

$$T_1 = \alpha_{11}\,dp + \alpha_{12}\,dq, \qquad T_2 = \alpha_{21}\,dp + \alpha_{22}\,dq,$$

and consider $T_1 = 0$ and $T_2 = 0$ to be the differential equations of two families of curves on a surface (x, y, z). In that way, we shall assume that:

$$\sum (dx)_{T_1}^2 = \sum (dx)_{T_2}^2 = 1, \qquad \sum (dx)_{T_1}(dx)_{T_2} = \cos\varphi \neq 0,$$

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} = \Delta \neq 0.$$

If ν_1 means an integrating factor for T_1 and ν_2 means one for T_2 then one will have:

$$(d\log\nu_1)_{T_2} = \frac{1}{\Delta}\left(\frac{\partial\alpha_{12}}{\partial p} - \frac{\partial\alpha_{11}}{\partial q}\right), \qquad (d\log\nu_2)_{T_1} = \frac{1}{\Delta}\left(\frac{\partial\alpha_{21}}{\partial q} - \frac{\partial\alpha_{22}}{\partial p}\right).$$

As a consequence, T_1 and T_2 will be complete differentials when:

$$(d\log\nu_1)_{T_2} = (d\log\nu_2)_{T_1} = 0.$$

For an arbitrary function $\mathfrak{F}$ of p and q, one will get:

$$(15) \qquad (d\mathfrak{F})_{T_1T_2} - (d\mathfrak{F})_{T_2T_1} = (d\mathfrak{F})_{T_1}(d\log\nu_1)_{T_2} - (d\mathfrak{F})_{T_2}(d\log\nu_2)_{T_1},$$

as in § **2**. One must now address the problem of understanding the geometric meaning of the quantities $(d\log\nu_1)_{T_2}$ and $(d\log\nu_2)_{T_1}$. To that end, we introduce two further differential forms by the equations:

$$T_1' = \frac{T_2 + T_1\cos\varphi}{\sin\varphi}, \qquad T_2' = \frac{T_1 + T_2\cos\varphi}{\sin\varphi}.$$

$T_1' = 0$, $T_2' = 0$ are then the differential equations for the orthogonal trajectories of the curves $T_1 = 0$, $T_2 = 0$. Four individual curves of the four families considered go through a point on the surface. One gets the following expressions for its geodetic curvatures:

$$\frac{1}{R_{T_1}} = \sum (dx)_{T_1'}(dx)_{T_1^2}, \qquad \frac{1}{R_{T_2}} = \sum (dx)_{T_2'}(dx)_{T_2^2},$$

$$\frac{1}{R_{T_1'}} = -\sum (dx)_{T_1'}(dx)_{T_1T_1'}, \qquad \frac{1}{R_{T_2'}} = -\sum (dx)_{T_2'}(dx)_{T_2T_2'}.$$

Since:

$$(d\mathfrak{F})_{T_1'} = \frac{-\cos\varphi(d\mathfrak{F})_{T_1} + (d\mathfrak{F})_{T_2}}{\sin\varphi}, \qquad (d\mathfrak{F})_{T_2'} = \frac{(d\mathfrak{F})_{T_1} - \cos\varphi(d\mathfrak{F})_{T_2}}{\sin\varphi},$$

one will have:

$$(dx)_{T_1T_1'} = \frac{-\cos\varphi(dx)_{T_1^2} + (dx)_{T_1T_2}}{\sin\varphi}, \qquad (dx)_{T_2T_2'} = \frac{(dx)_{T_2T_1} - \cos\varphi(dx)_{T_2^2}}{\sin\varphi},$$

and

$$\frac{1}{R_{T_1'}} = \frac{\cot\varphi}{R_{T_1}} - \frac{\sum (dx)_{T_2}(dx)_{T_1T_2}}{\sin^2\varphi}, \qquad \frac{1}{R_{T_2'}} = \frac{\cot\varphi}{R_{T_2}} - \frac{\sum (dx)_{T_1}(dx)_{T_2T_1}}{\sin^2\varphi}.$$

If one replaces $\mathfrak{F}$ in (15) with the quantities x, y, z, in succession, multiplied in the first case by $(dx)_{T_1'}$, $(dy)_{T_1'}$, $(dz)_{T_1'}$, then by $(dx)_{T_2'}$, $(dy)_{T_2'}$, $(dz)_{T_2'}$, and then adds them together then:

$$(16) \qquad \left\{\begin{aligned} (d\log\nu_1)_{T_2} &= \cos\varphi\left(\frac{1}{R_{T_1'}} - \frac{\cot\varphi}{R_{T_1}}\right) + \frac{1}{R_{T_2'}} - \frac{\cot\varphi}{R_{T_2}}, \\ (d\log\nu_2)_{T_1} &= \frac{1}{R_{T_1'}} - \frac{\cot\varphi}{R_{T_1}} + \cos\varphi\left(\frac{1}{R_{T_2'}} - \frac{\cot\varphi}{R_{T_2}}\right). \end{aligned}\right.$$

On the basis of those equations, the theorem to be proved can be expressed as follows:

The differential forms T_1 and T_2 will be complete differentials when:

$$\frac{1}{R_{T_2'}} - \frac{\cot\varphi}{R_{T_2}} = 0 \qquad \text{and} \qquad \frac{1}{R_{T_1'}} - \frac{\cot\varphi}{R_{T_1}} = 0 \, ;$$

i. e., when the tangents to the curves $T_2 = 0$ ($T_1 = 0$, *resp.*) *are perpendicular to the connecting lines of the centers of geodetic curvature for the curves* $T_1 = 0$, $T_1' = 0$ ($T_2 = 0$, $T_2' = 0$, *resp.*).

If one takes $T_1 = du$, $T_2 = dv$ then the square of the line element will take the form:

$$ds^2 = du^2 + dv^2 + 2 \cos\varphi \, du \, dv \, .$$

Now, let two families of curves on the unit sphere be given that are not mutually-perpendicular. We shall next answer the question that **Guichard** treated of finding the condition under which those families will be the spherical images of the asymptotic lines of a surface.

If one considers the curves $T_2 = 0$, $T_1 = 0$ on the unit sphere (ξ, η, ζ) to be the spherical images of two families of curves on a surface (x, y, z) then one will have:

$$dx = (dx)_{T_1} T_1 + (dx)_{T_2} T_2 \, ,$$

in which $(dx)_{T_1}$, $(dx)_{T_2}$, etc., are not direction cosines, however.

Since the asymptotic lines of a surface run perpendicular to their spherical images when the curves $T_2 = 0$, $T_1 = 0$ on the surface coincide with the asymptotic lines of the surface, one will have:

$$dx = \lambda (d\xi)_{T_1} T_1 + \mu (d\xi)_{T_2} T_2 \, ,$$

in which λ and μ denote proportionality factors. The direction cosines of the normals to the surface are ξ, η, ζ, and since it would generally emerge from (15) that:

$$\sum \xi (dx)_{T_1 T_2} = \sum \xi (dx)_{T_2 T_1}$$

or

$$\sum (d\xi)_{T_1} (dx)_{T_2} = \sum (d\xi)_{T_2} (dx)_{T_1}$$

then it would follow that:

$$\lambda = \mu \, .$$

In order for the differential form dx to be a complete differential, the relation:

$$\begin{aligned} \lambda[(d\xi)_{T_1' T_1} - (dx)_{T_2' T_1}] + (d\lambda)_{T_2} (d\xi)_{T_1'} - (d\lambda)_{T_1} (d\xi)_{T_2'} \\ = \lambda[(d\xi)_{T_1'} (d \log \nu_1)_{T_2} - (d\xi)_{T_2'} (d \log \nu_2)_{T_1}] \end{aligned}$$

must exist. If one poses the corresponding conditions for dy, dz then it will follow that:

$$(17)\qquad \begin{cases} \lambda\sin\varphi(d\log\nu_1)_{T_2} = (d\lambda)_{T_2}\sin\varphi + \lambda\left\{\sum(d\xi)_{T_2}(d\xi)_{T_1'T_2} - \sum(d\xi)_{T_2}(d\xi)_{T_2'T_1}\right\}, \\ \lambda\sin\varphi(d\log\nu_2)_{T_1} = (d\lambda)_{T_1}\sin\varphi - \lambda\left\{\sum(d\xi)_{T_1}(d\xi)_{T_1'T_2} - \sum(d\xi)_{T_1}(d\xi)_{T_2'T_1}\right\}. \end{cases}$$

However:

$$\sum(d\xi)_{T_2}(d\xi)_{T_1'T_2} = \cot(d\varphi)_{T_2} + \cot\varphi\sum(d\xi)_{T_1}(d\xi)_{T_2^2}$$

and

$$\sum(d\xi)_{T_1}(d\xi)_{T_2^2} = -\sin(d\varphi)_{T_2} - \sum(d\xi)_{T_2}(d\xi)_{T_1T_2}\ .$$

When one employs the abbreviations:

$$\frac{1}{R_{T_1'}} - \frac{\cot\varphi}{R_{T_1}} = A, \qquad \frac{1}{R_{T_2'}} - \frac{\cot\varphi}{R_{T_2}} = B,$$

the first equation (17) will go to:

$$(d\lambda)_{T_2} = 2\,\lambda\,B,$$

and one will correspondingly find:

$$(d\lambda)_{T_1} = 2\,\lambda\,A,$$

in place of the second one. The condition in question will then demand that the expression:

$$A\,T_1 + B\,T_2$$

must be a complete differential, or that the following relation must exist:

$$(dA)_{T_2} - (dB)_{T_1} = (A^2 - B^2)\cos\varphi,$$

in which case, λ, and then x, y, z, in turn, will be determined by quadratures. In order to find the geometric meaning of λ, one observes that the reciprocal values of the radius of curvature of a normal section of the surface (x, y, z) are provided by the expression:

$$-\frac{2\sin\varphi T_1T_2}{\lambda(T_1{}^2 - 2\cos\varphi T_1T_2 + T_2{}^2)}.$$

As a result, $-1/\lambda^2$ will be the Gaussian curvature of the surface. If the previous condition equation exists in such a way that $A = B = 0$ then λ will be constant. The spherical image of the asymptotic lines of a surface of constant negative Gaussian curvature will then possess the properties that were stated in the theorem above.

The focal points of a ray system lie on two surfaces that one calls the *focal surfaces* of the system. The midpoint of the line segment that links the two focal points of a ray lies on the so-called *midpoint surface* of the system. We next represent the differentials of the coordinates of the latter by linear forms of S_1 and S_2 .

Since:

$$dx_0 = \delta x_0 + \xi \sum \xi\, dx_0\,,$$

when one sets:

$$\sum \xi\, dx_0 = p_1\, S_1 + p_2\, S_2\,,$$

one will have:

$$dx_0 = S_1\,(-\rho_1\, \kappa_3 + p_1\, \xi) + S_1\,(-\rho_2\, \kappa_4 + p_2\, \xi)\,.$$

Should x_0 , y_0 , z_0 be the coordinates of the point of the midpoint surface then one would have to replace ρ_2 with $-\rho_1$. One writes ρ, instead of ρ_1, and gets:

$$dx_0 = S_1\,(-\rho\, \kappa_3 + p_1\, \xi) + S_2\,(\rho\, \kappa_4 + p_2\, \xi)\,.$$

One now takes $T_1 = S_1$, $T_2 = S_2$. The quantities φ, A, and B are then determined. Furthermore, let:

$$\begin{aligned} (dx)_{T_1'} &= \kappa_3', & (dy)_{T_1'} &= \lambda_3', & (dz)_{T_1'} &= \mu_3', \\ (dx)_{T_2'} &= \kappa_4', & (dy)_{T_2'} &= \lambda_4', & (dz)_{T_2'} &= \mu_4'. \end{aligned}$$

The differential form dx_0 is a complete differential. As a result, one will have:

$$-(d\rho)_{S_2}\, \kappa_3 - \rho(d\kappa_3)_{S_2} + (dp_1)_{S_2}\, \xi + p_1\, \kappa_4 - (d\rho)_{S_1}\, \kappa_4 - \rho(d\kappa_4)_{S_1} - (dp_2)_{S_1}\, \xi - p_2\, \kappa_3$$

$$= (A \cos \varphi + B)\,(-\rho\, \kappa_3 + p_1\, \xi) - (A + B \cos \varphi)\,(\rho\, \kappa_4 + p_2\, \xi).$$

If one multiplies that equation and the corresponding ones that are derived for dy_0 and dz_0, in succession, by κ_3', λ_3', μ_3', and then by κ_4', λ_4', μ_4', and finally, by ξ, η, ζ, and adds them each time, then since:

$$\sum \kappa_3'\,(d\kappa_3)_{S_2} = -A \sin \varphi, \qquad \sum \kappa_3'\,(d\kappa_4)_{S_1} = \cos \varphi \sin \varphi\, B,$$

$$\sum \kappa_4'\,(d\kappa_3)_{S_2} = A \cos \varphi \sin \varphi, \qquad \sum \kappa_4'\,(d\kappa_4)_{S_1} = -B \sin \varphi\,,$$

that will give the system of equations:

$$\begin{aligned} (d\rho)_{S_1} - p_1 - 2\rho A &= 0, \\ (d\rho)_{S_2} - p_2 - 2\rho A &= 0, \\ (dp_1)_{S_2} - (dp_2)_{S_1} + 2\rho \cos \varphi &= p_1\,(A \cos \varphi + B) - p_2\,(A + B \cos \varphi). \end{aligned}$$

If one replaces the quantities p_1 and p_2 in the third of those equations with their values that one infers from the first two equations then that will produce a **Laplace** differential equation for ρ . After integrating it, the determination of x_0, y_0, z_0 , and therefore the determination of a ray system with a prescribed spherical image of its developable surface, will require only quadratures. (**Guichard**, *loc. cit.*, pp. 344)

We now make the assumption that A and B vanish. We then have:

$$(d\rho)_{S_1 S_2} = (d\rho)_{S_2 S_1},$$

and the third equation of the system above will assume the form:

$$(d\rho)_{S_1 S_2} + \rho \cos \varphi = 0.$$

If X, Y, Z are the direction cosines of a normal to the midpoint surface then one will have:

$$X : Y : Z$$

$$= -p_2\,\kappa_3' + p_1\,\kappa_4' + \rho \sin \varphi\,\xi : -p_2\,\lambda_3' + p_1\,\lambda_4' + \rho \sin \varphi\,\eta : -p_2\,\mu_3' + p_1\,\mu_4' + \rho \sin \varphi\,\zeta.$$

Now:

$$\sum (-p_2\,\kappa_3' + p_1\,\kappa_4' + \rho \sin \varphi\,\xi)[-\kappa_3 (d\rho)_{S_2} - \rho\,(d\kappa_3)_{S_2} + \xi\,(dp_1)_{S_2} + p_1\,\kappa_4] = 0,$$

and as a result, $\sum X\,(dx_0)_{S_1 S_2}$ will also vanish. However, that says that the curves $S_1 = 0$, $S_2 = 0$ – i.e., the lines of intersection of the surface with the developable surface of the ray system – are conjugate curves. (**Guichard**, *loc. cit.*, pp. 345)

The coordinates of the points on the two focal surfaces will be:

$$x_1 = x_0 + \rho\,\xi, \qquad y_1 = y_0 + \rho\,\eta, \qquad z_1 = z_0 + \rho\,\zeta,$$
$$x_2 = x_0 - \rho\,\xi, \qquad y_2 = y_0 - \rho\,\eta, \qquad z_2 = z_0 - \rho\,\zeta.$$

One then has:

$$dx_1 = 2p_1\,\xi\,S_1 + 2\rho\,\kappa_4\,S_2, \qquad dx_2 = -2\rho\,\kappa_3\,S_1 + 2p_2\,\xi\,S_2.$$

The direction cosines of a normal to the first focal surface (x_1, y_1, z_1) are then κ_4', λ_4', μ_4', while those of the second one are κ_3', λ_3', μ_3'. As a result, as is known, the first of the focal planes, which has the equation:

$$\sum (x - x_1)\,\kappa_4' = 0,$$

will contact the second of the focal planes, which has the equation:

$$\sum (x - x_1)\,\kappa_4' = 0.$$

The lines of intersection of the developable surface of the ray system with the two focal surfaces are perpendicular to each other. Since one has:

$$\sum \kappa_4' \,(dp_1\, \xi)_{S_2} = 0, \qquad \sum \kappa_3' \,(dp_2\, \xi)_{S_1} = 0,$$

in addition, those lines of intersection will be, at the same time, the lines of curvature of the focal surfaces. (**Guichard**, *loc. cit.*, pp. 346)

§ 6. – First and second derivatives with respect to arc-length. Representing the second derivatives of the coordinates with respect to the first. Distinguished types of orthogonal trajectories.

A family of orthogonal trajectories of the given family of curves is established by two differential equations of the form:

$$(1) \qquad \begin{cases} a_{13}\, dp + a_{23}\, dq + a_{33}\, dr = 0, \\ m\, dp + n\, dq = 0, \end{cases}$$

in which m and n mean functions of p, q, r. Since the first of those equations must be true for any orthogonal trajectory, the lines that are determined by (1) can be called simply "orthogonal trajectories (or curves) $m\, dp + n\, dq = 0$." The shift components:

$$\delta x = x_p\, dp + x_q\, dq, \qquad \delta y = y_p\, dp + y_q\, dq, \qquad \delta z = z_p\, dp + z_q\, dq$$

that were introduced in § **3** are then referred to the orthogonal trajectories $dp = 0$ and $dq = 0$, or in other words: The curves $dp = 0$, $dq = 0$ are considered to be coordinate lines. We would like to introduce two arbitrarily-chosen families of curves of coordinate lines that are determined by the equations:

$$m_1\, dp + n_1\, dq = 0, \qquad m_2\, dp + n_2\, dq = 0$$

in place of them. The coefficients m_1 , n_1 , m_2 , n_2 will then have to satisfy only the condition that their determinant must not vanish. We let ν_1 and ν_2 be two temporarily-undetermined functions of p, q, r, and set:

$$\begin{aligned} \nu_1\, (m_1\, dp + n_1\, dq) &= \alpha_{11}\, dp + \alpha_{12}\, dq = T_1\,, \\ \nu_2\, (m_2\, dp + n_2\, dq) &= \alpha_{21}\, dp + \alpha_{22}\, dq = T_2\,, \\ a_{13}\, dp + a_{23}\, dq + a_{33}\, dr &= \sqrt{a_{23}}\, T_0\,. \end{aligned}$$

The differential of a function $\mathfrak{F}$ of p, q, r will be a linear form in T_1 , T_2 , T_3 . If we give it the form:

$$d\,\mathfrak{F} = (d\mathfrak{F})_{T_1} T_1 + (d\mathfrak{F})_{T_2} T_2 + (d\mathfrak{F})_{T_0} T_0$$

then we will have:

$$(2) \qquad (d\mathfrak{F})_{T_1} = \frac{\alpha_{22}\,\mathfrak{F}_p - \alpha_{21}\,\mathfrak{F}_q}{\alpha_{11}\alpha_{22} - \alpha_{12}\,\alpha_{21}}, \qquad (d\mathfrak{F})_{T_2} = \frac{-\alpha_{12}\,\mathfrak{F}_p + \alpha_{11}\,\mathfrak{F}_q}{\alpha_{11}\alpha_{22} - \alpha_{12}\,\alpha_{21}}, \qquad (d\mathfrak{F})_{T_0} = \frac{1}{\sqrt{a_{33}}}\frac{\partial \mathfrak{F}}{\partial r}.$$

Since:

$$(dx)_{T_0} = \xi, \qquad (dy)_{T_0} = \eta, \qquad (dz)_{T_0} = \zeta,$$

$(d\mathfrak{F})_{T_0}$ can be considered to be the derivative of $\mathfrak{F}$ with respect to arc-length of the curves of the given family. If one sets:

$$\nu_1 = \frac{\sqrt{n_2^2 E - 2n_2\,m_2\,F + m_2^2 G}}{m_1\,n_2 - m_2\,n_1}, \qquad \nu_2 = \frac{\sqrt{n_1^2 E - 2n_1\,m_1\,F + m_1^2 G}}{m_1\,n_2 - m_2\,n_1}$$

then $(dx)_{T_1}$, $(dy)_{T_1}$, $(dz)_{T_1}$ will be the direction cosines of the tangents to the curves $T_2 = 0$, and $(dx)_{T_2}$, $(dy)_{T_2}$, $(dz)_{T_2}$ will be the tangents to the curves $T_1 = 0$. In that way, $(d\mathfrak{F})_{T_1}$ or $(d\mathfrak{F})_{T_2}$ will be the derivatives of $\mathfrak{F}$ with respect to the arc-length of the curves $T_2 = 0$ or $T_1 = 0$, resp. If one replaces dp and dq with T_2 and T_1 in the partial differential:

$$\delta\mathfrak{F} = \mathfrak{F}_p\,dp + \mathfrak{F}_q\,dq$$

then one will get:

$$\delta\mathfrak{F} = (d\mathfrak{F})_{T_1} T_1 + (d\mathfrak{F})_{T_2} T_2,$$

so:

$$d\,\mathfrak{F} = \delta\mathfrak{F} + (d\mathfrak{F})_{T_0} T_0.$$

We introduce the following notations for the second differential of a function $\mathfrak{F}$ that is formed under the condition $T_0 = 0$:

$$\begin{aligned}\delta^2\mathfrak{F} &= \mathfrak{F}_{pp}dp^2 + (\mathfrak{F}_{pq} + \mathfrak{F}_{qp})\,dp\,dq + \mathfrak{F}_{qq}\,dq^2 + \mathfrak{F}_p\,d^2p + \mathfrak{F}_q\,d^2q \\ &= (\mathfrak{F})_{T_1^2} T_1^2 + [(d\mathfrak{F})_{T_1T_2} + (d\mathfrak{F})_{T_2T_1}]\,T_1\,T_2 + (\mathfrak{F})_{T_2^2} T_2^2 + (d\mathfrak{F})_{T_1}\delta T_1 + (d\mathfrak{F})_{T_2}\delta T_2.\end{aligned}$$

Here, we have set:

$$((d\mathfrak{F})_{T_\alpha})_{T_\beta} = (d\mathfrak{F})_{T_\alpha T_\beta}, \qquad (d\mathfrak{F})_{T_\alpha T_\alpha} = (d\mathfrak{F})_{T_\alpha^2},$$

while the following equation will be true for δT_1 and δT_2:

$$\delta T_\nu = (\alpha_{\nu 1})_p\,dp^2 + [(\alpha_{\nu 1})_q + (\alpha_{\nu 2})_p]\,dp\,dq + (\alpha_{\nu 2})_q\,dq^2 + \alpha_{\nu 1}d^2p + \alpha_{\nu 2}d^2q \qquad (\nu = 1, 2).$$

We shall now address the problem of expressing the second derivatives of the coordinates with respect to arc-length of the curves of the given family and the lines $T_2 = 0$, $T_1 = 0$ in terms of their first derivatives and geometrically-intuitive quantities. The solution of that problem requires the introduction of two further families of orthogonal trajectories that are, at the same time, orthogonal trajectories to the lines $T_2 = 0$ or $T_1 = 0$.

We shall let φ denote the angle between the lines $T_2 = 0$ and $T_1 = 0$, such that:

$$\sum (dx)_{T_1} (dx)_{T_2} = \cos \varphi.$$

We then take:

$$T_1' = \frac{T_2 + T_1 \cos\varphi}{\sin\varphi}, \qquad T_2' = \frac{T_1 + T_2 \cos\varphi}{\sin\varphi}.$$

We will then have:

(3) $$(d\mathfrak{F})_{T_1'} = \frac{-\cos\varphi (d\mathfrak{F})_{T_1} + (d\mathfrak{F})_{T_2}}{\sin\varphi}, \qquad (d\mathfrak{F})_{T_2'} = \frac{(d\mathfrak{F})_{T_1} - \cos\varphi (d\mathfrak{F})_{T_2}}{\sin\varphi},$$

and

$$\sum (dx)_{T_1'}^2 = \sum (dx)_{T_2'}^2 = 1, \qquad \sum (dx)_{T_1} (dx)_{T_1'} = 0, \qquad \sum (dx)_{T_2} (dx)_{T_2'} = 0,$$

$$\sum (dx)_{T_1} (dx)_{T_2'} = \sum (dx)_{T_2} (dx)_{T_1'} = \sin \varphi, \qquad \sum (dx)_{T_1} (dx)_{T_2'} = -\cos \varphi.$$

That shows that the curves $T_2' = 0$, $T_1' = 0$ are orthogonal trajectories of the given family of curves, and furthermore that the curves $T_2' = 0$ are orthogonal trajectories to the curves $T_2 = 0$, and the lines $T_1' = 0$ are likewise orthogonal trajectories to the lines $T_1 = 0$. Finally, $(d\mathfrak{F})_{T_1'}$ or $(d\mathfrak{F})_{T_2'}$ is the derivative of $\mathfrak{F}$ with respect to arc-length of the curves $T_2' = 0$ or $T_1' = 0$, resp.

We shall direct our attention to certain curvatures of an orthogonal trajectory to a given family of curves and apply their defining equations to the families considered, namely, $T_2 = 0$, $T_1 = 0$, $T_2' = 0$ or $T_1' = 0$.

We define the *normal curvature* by the formula:

$$\frac{1}{h} = -\frac{\sum \delta x \, \delta\xi}{\delta s^2} = \frac{\sum \xi \, \delta^2 x}{\delta s^2}.$$

If we denote the normal curvatures to the curves $T_2 = 0$, $T_1 = 0$ by $1/h_{T_1}$, $1/h_{T_2}$, resp., then it will follow that:

$$\frac{1}{h_{T_1}} = \sum \xi \, (dx)_{T_1^2}, \qquad \frac{1}{h_{T_2}} = \sum \xi \, (dx)_{T_2^2}.$$

The curvature axis of an orthogonal trajectory cuts the normal plane of the curve (p = const., q = const.) at a point that shall be called the center of its *geodetic curvature* $1 / R$. If ξ', η', ζ' are the direction cosines of the line that is perpendicular to the direction (δx, δy, δz), as well as the tangent ξ, η, ζ, then one will have:

$$\frac{1}{R} = \frac{-\sum \delta x \, \delta \xi'}{\delta s^2} = \frac{\sum \xi' \delta^2 x}{\delta s^2}.$$

If we denote the geodetic curvatures of the curves $T_2 = 0$, $T_1 = 0$, $T_2' = 0$, $T_1' = 0$ by:

$$\frac{1}{R_{T_1}}, \quad \frac{1}{R_{T_2}}, \quad \frac{1}{R_{T_1'}}, \quad \frac{1}{R_{T_2'}},$$

respectively, then it will follow that:

$$\frac{1}{R_{T_1}} = \sum (dx)_{T_1'} (dx)_{T_1^2}, \qquad \frac{1}{R_{T_2}} = \sum (dx)_{T_2'} (dx)_{T_2^2},$$

$$\frac{1}{R_{T_1'}} = -\sum (dx)_{T_1'} (dx)_{T_1 T_1'} = \frac{\cot \varphi}{R_{T_1}} - \frac{1}{\sin \varphi} \sum (dx)_{T_1'} (dx)_{T_1 T_2},$$

$$\frac{1}{R_{T_2'}} = -\sum (dx)_{T_2'} (dx)_{T_1 T_2'} = \frac{\cot \varphi}{R_{T_2}} - \frac{1}{\sin \varphi} \sum (dx)_{T_2'} (dx)_{T_2 T_1}.$$

In addition to the normal and geodetic curvature for an arbitrary curve, it is recommended that we should consider a third one that we call the *curvature of the curve relative to a normal surface*; i.e., relative to a rectilinear surface whose generators are normals to the curve, and which we shall define as limiting values as follows: Let the coordinates x, y, z of the points of a curve be functions of the variable t, and the normal surface will be determined when we are given the direction cosines a, b, c of its generators. The lines that are perpendicular to the generators and the associated curve tangents might possess the direction cosines a', b', c'. One considers a regular curve segment PP' whose initial point and endpoint correspond to the values t and $t + \Delta t$, resp. A half-line goes through P that defines angles with the positive halves of the coordinate axes whose cosines are a, b, c. One associates the point P_α on that half-line with the point P_α' on the curve segment PP' in such a way that any line segment PP_α will be equal to the arc-length PP_α', and draws a straight line L_α through the point P_α that is parallel to the line (a', b', c') and goes through the point P_α'. The perpendicular projection of a line L_α onto the normal plane to the curve that belongs to P cuts the line (a', b', c') that belongs to P at the point Q_α. The limiting case $P_\alpha = P$ corresponds to the point $Q_\alpha = Q$. Q shall then be regarded as the center of curvature of the curve that belongs to P relative to the normal surface that it is based upon. If one takes the abscissa of Q relative to P to be equal to l and denotes the line element of the curve by ds, as usual, then one will have:

$$\frac{1}{l} = -\sum a \frac{da'}{ds} = \sum a' \frac{da}{ds},$$

and the coordinates of Q will be:

$$x' = x + l\,a', \quad y' = y + l\,b', \quad z' = z + l\,c'.$$

One must not forget that the choice of half-line (a, b, c) will have an effect. If one uses the half-line ($-a$, $-b$, $-c$) then one will get the center of curvature as the point that corresponds symmetrically to the point (x', y', z') and has the coordinates:

$$x'' = x - l\,a', \quad y'' = y - l\,b', \quad z'' = z - l\,c'.$$

The curvature $1 / l$ vanishes when the normal surface (a, b, c) is developable, in which case, the normal surface (a', b', c') will also be developable. If one lets the generators of the normal surface coincide with the binormals to a curved line then $1 / l$ will be the second curvature of the line.

The concept of the curvature of a curve relative to a normal surface that was developed can be made fruitful for the study of the curvature of families of curves in two ways when one focuses on the tangents to one and the same family of orthogonal trajectories along either an orthogonal trajectory to the tangent (ξ, η, ζ) or a curve p = const., q = const. In that way, one will get the following values for the curvature of the curve $T_2 = 0$, $T_1 = 0$, $T_2' = 0$, $T_1' = 0$, relative to the normal surface (ξ, η, ζ):

$$\frac{1}{l_{T_1}} = \sum (dx)_{T_1'} (d\xi)_{T_1} = \frac{1}{\sin\varphi}\left\{\frac{\cos\varphi}{h_{T_1}} + \sum (dx)_{T_2} (d\xi)_{T_1}\right\},$$

$$\frac{1}{l_{T_1}} = \sum (dx)_{T_1'} (d\xi)_{T_1} = \frac{1}{\sin\varphi}\left\{\frac{\cos\varphi}{h_{T_1}} + \sum (dx)_{T_2} (d\xi)_{T_1}\right\},$$

$$\frac{1}{l_{T_2}} = \sum (dx)_{T_2'} (d\xi)_{T_2} = \frac{1}{\sin\varphi}\left\{\sum (dx)_{T_1} (d\xi)_{T_2} + \frac{\cos\varphi}{h_{T_2}}\right\},$$

$$\frac{1}{l_{T_1'}} = \sum (dx)_{T_1} (d\xi)_{T_1'} = \frac{1}{\sin\varphi}\left\{\frac{\cos\varphi}{h_{T_1}} + \sum (dx)_{T_1} (d\xi)_{T_2}\right\},$$

$$\frac{1}{l_{T_2'}} = \sum (dx)_{T_2} (d\xi)_{T_2'} = \frac{1}{\sin\varphi}\left\{\sum (dx)_{T_2} (d\xi)_{T_1} + \frac{\cos\varphi}{h_{T_2}}\right\},$$

such that:

$$\frac{1}{l_{T_1}} + \frac{1}{l_{T_2}} = \frac{1}{l_{T_1'}} + \frac{1}{l_{T_2'}}.$$

One gets the following expressions for the curvature $1/L_{T_1}$ ($1/L_{T_2}$, resp.) of a curve p = const., q = const. relative to the normal surface $((dx)_{T_1}, (dy)_{T_1}, (dz)_{T_1})$ [$((dx)_{T_1}, (dy)_{T_1}$, $(dz)_{T_1})$, resp.]:

$$\frac{1}{L_{T_1}} = \sum (dx)_{T_1'} (dx)_{T_1 T_0} = \frac{1}{\sin\varphi} \sum (dx)_{T_2} (dx)_{T_1 T_0} ,$$

$$\frac{1}{L_{T_2}} = \sum (dx)_{T_2'} (dx)_{T_2 T_0} = \frac{1}{\sin\varphi} \sum (dx)_{T_1} (dx)_{T_2 T_0} ,$$

such that:

(4)
$$\frac{1}{L_{T_1}} + \frac{1}{L_{T_2}} = -(d\varphi)_{T_0} .$$

That equation corresponds to a general property of the curvature that we speak of, namely, that the sum of the curvatures of a curve relative to two normal surfaces plus the derivative of the angle between the two surfaces with respect to the arc-length of the curve will yield the value zero.

Finally, we direct our attention to two more points along the tangents to the curves $T_2 = 0$, $T_1 = 0$ that go through the point (x, y, z), namely, the intersection points of those tangents with the curvature axis of the curve p = const., q = const. that belongs to the point (x, y, z). We denote the radius of the first curvature of that curve by ρ, the direction cosines of its principal normal by a_1, b_1, c_1, and those of its binormal by a_2, b_2, c_2. One will then have:

(5)
$$a_1 = \rho\,(d\xi)_{T_0} , \qquad a_2 = \rho[\eta(d\zeta)_{T_0} - \zeta(d\eta)_{T_0}] .$$

The intersection point of the curvature axis with the tangent to the curve $T_2 = 0$ possesses the abscissa P_{T_1} relative to the point (x, y, z). For the coordinates of that intersection point, one has on the one hand, the expressions:

$$x + (dx)_{T_1} P_{T_1} , \quad y + (dy)_{T_1} P_{T_1} , \quad z + (dz)_{T_1} P_{T_1} ,$$

but on the other hand, when the intersection point has a distance of ρ' from the center of the first curvature of the curve p = const., q = const., it will also have the expressions:

$$x + \rho^2 (d\xi)_{T_0} + \rho\rho' [\eta(d\zeta)_{T_0} - \zeta(d\eta)_{T_0}], \text{ etc.}$$

One will then have:

$$P_{T_1} = \sum (d\xi)_{T_0} (dx)_{T_1} .$$

That will imply the corresponding abscissas for the intersection points of the tangents to the curves $T_1 = 0$, $T_2' = 0$, $T_1' = 0$, with that axis of curvature relative to the point (x, y, z):

$$\frac{1}{P_{T_2}} = \sum (d\xi)_{T_0} (dx)_{T_2} , \qquad \frac{1}{P_{T_1'}} = \sum (d\xi)_{T_0} (dx)_{T_1'} , \qquad \frac{1}{P_{T_2'}} = \sum (d\xi)_{T_0} (dx)_{T_2'} .$$

The formulas that were developed put us into a position of exhibiting the desired expressions for the second derivatives of the coordinates with respect to the arc-lengths of the curves p = const., q = const.; $T_2 = 0$, $T_1 = 0$. One gets the following expressions for the coordinates x :

$$
(6)\quad
\begin{cases}
(dx)_{T_1^2} = \dfrac{(dx)_{T_1'}}{R_{T_1}} + \dfrac{\xi}{h_{T_1}}, \\[2ex]
(dx)_{T_1 T_2} = \left(\dfrac{\cos\varphi}{R_{T_1}} - \dfrac{\sin\varphi}{R_{T_1'}}\right)(dx)_{T_1'} + \left(\dfrac{\cos\varphi}{h_{T_2}} - \dfrac{\sin\varphi}{h_{T_2}}\right)\xi, \\[2ex]
(dx)_{T_1 T_0} = \dfrac{(dx)_{T_1'}}{L_{T_1}} - \dfrac{\xi}{P_{T_1'}}, \\[2ex]
(dx)_{T_2 T_1} = \left(\dfrac{\cos\varphi}{R_{T_2}} - \dfrac{\sin\varphi}{R_{T_2'}}\right)(dx)_{T_2'} + \left(\dfrac{\cos\varphi}{h_{T_1}} - \dfrac{\sin\varphi}{l_{T_1}}\right)\xi, \\[2ex]
(dx)_{T_2^2} = \dfrac{(dx)_{T_2'}}{R_{T_2}} + \dfrac{\xi}{h_{T_2}}, \qquad (dx)_{T_2 T_0} = \dfrac{(dx)_{T_2'}}{L_{T_2}} - \dfrac{\xi}{P_{T_2}}, \\[2ex]
(d\xi)_{T_1} = \dfrac{(dx)_{T_1}}{h_{T_1}} + \dfrac{(dx)_{T_1'}}{l_{T_1}}, \quad (dx)_{T_2} = -\dfrac{(dx)_{T_2}}{h_{T_2}} + \dfrac{(dx)_{T_2'}}{l_{T_2}}, \\[2ex]
\qquad\qquad\qquad\qquad (dx)_{T_0} = \dfrac{(dx)_{T_1}}{P_{T_1}} + \dfrac{(dx)_{T_1'}}{P_{T_1'}}.
\end{cases}
$$

The corresponding equations for the coordinates y and z emerge from these by simultaneously switching x, ξ with y, η or z, ζ, resp.

The vanishing of a coefficient that appears in those equations can be regarded as a salient geometric property of the curve family considered. If $1/l_{T_1}$ or $1/l_{T_2}$ vanishes then the curves $T_2 = 0$ or $T_1 = 0$ will be lines of curvature of the second kind.

An orthogonal trajectory whose normal curvature vanishes everywhere shall be called an *asymptotic line* of the curve family. If such a thing is not straight then its binormals will coincide with the tangents (ξ, η, ζ). In the event that $1/h_{T_1}$ or $1/h_{T_2}$ is zero, the curves $T_2 = 0$ or $T_1 = 0$, resp., will be asymptotic lines.

An orthogonal trajectory whose geodetic curvature vanishes everywhere shall be called a *geodetic line* of the family of curves. If such a thing is not straight then its principal normals will coincide with the tangents (ξ, η, ζ). In the event that $1/R_{T_1}$ or $1/R_{T_2}$ is zero, the curves $T_2 = 0$ or $T_1 = 0$, resp., will be geodetic lines.

An orthogonal trajectory whose tangents are, at the same time, principal normals or binormals of the curves p = const., q = const., shall be called a *principal normal line* or *binormal line* to the family of curves. If $1/P_{T_1}$ or $1/P_{T_2}$ vanishes then the curves $T_2 = 0$ or $T_1 = 0$ will be binormal lines, but when $1/P_{T_1'}$ or $1/P_{T_2'}$ equals zero, the curves $T_2 = 0$ or $T_1 = 0$, resp., will be principal normal lines.

If $1/L_{T_1}$ or $1/L_{T_2}$ vanishes then the tangents to the curves $T_2 = 0$ or $T_1 = 0$ along the curves p = const., q = const., resp., define a developable surface.

It remains for us to consider the coefficients of $(dx)_{T_1T_2}$ and $(dx)_{T_2T_1}$. If we set:

$$(dx)_{T_1T_2} = \beta_{12}(dx)_{T_1'} + \Theta_{12}\xi, \qquad (dx)_{T_2T_1} = \beta_{21}(dx)_{T_2'} + \Theta_{21}\xi,$$

to abbreviate, then for finite values of R_{T_1} and $R_{T_1'}$, the equation $\beta_{12} = 0$ will say that the tangents to the curves $T_1 = 0$ will be perpendicular to the connecting line of the center of geodetic curvature of the curves $T_2 = 0$ and $T_2' = 0$. Likewise, for finite values of R_{T_2} and $R_{T_2'}$, the equation $\beta_{12} = 0$ means that the tangents to the curves $T_2 = 0$ are perpendicular to the connecting lines of the centers of geodetic curvature for the curves $T_1 = 0$ and $T_1' = 0$.

A family (A) of orthogonal trajectories shall be called *adjoint* to a family (B) of such things when the tangent to a curve (A) at any regular point is perpendicular to that tangent ($\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$) that neighbors the tangent (ξ, η, ζ) along the curve (B), or in other words, when the tangents to the curve (A) possess the directions of the lines of intersection of the two neighboring normal planes to the curves p = const., q = const. along the curve (B).

Now, since:

$$\Theta_{12} = -\sum (dx)_{T_1}(d\xi)_{T_2}, \qquad \Theta_{21} = -\sum (dx)_{T_2}(d\xi)_{T_1},$$

the equation $\Theta_{12} = 0$ or $\Theta_{21} = 0$ means that the curves $T_2 = 0$ or $T_1 = 0$ are adjoint to the curves $T_1 = 0$ or $T_2 = 0$, resp.

A remark in regard to the lines of curvature of the first kind might find a place here. A family of orthogonal trajectories can also be defined by setting the ratio $T_1 : T_2$ equal to a function of p, q, r. One then finds the following expression for the normal curvature of the trajectories in question:

$$\frac{1}{h} = \frac{\Theta_{11}T_1^2 + (\Theta_{12} + \Theta_{21})T_1T_2 + \Theta_{22}T_2^2}{T_1^2 + 2\cos\varphi\, T_1T_2 + T_2^2},$$

in which one sets:

$$\frac{1}{h_{T_1}} = \Theta_{11}, \qquad \frac{1}{h_{T_2}} = \Theta_{22},$$

to abbreviate, and the equation of the lines of curvature of the first type will be:

$$T_1^2\left(\frac{\Theta_{12} + \Theta_{21}}{2} - \Theta_{11}\cos\varphi\right) + (\Theta_{22} - \Theta_{11})\,T_1\,T_2 + [\Theta_{22}\cos\varphi - \tfrac{1}{2}(\Theta_{12} + \Theta_{21})]T_2^2 = 0.$$

In order for the lines of curvature in question to be defined by the curves $T_2 = 0$, $T_1 = 0$, one must have:

$$\tfrac{1}{2}(\Theta_{12} + \Theta_{21}) - \Theta_{11}\cos\varphi = 0,$$

$$\tfrac{1}{2}(\Theta_{12} + \Theta_{21}) - \Theta_{22}\cos\varphi = 0.$$

Should the determinant $\Theta_{11} - \Theta_{22}$ vanish here, then we would be dealing with an isotropic family of curves. Thus, the conditions would read:

$$\cos\varphi = 0, \qquad \Theta_{12} + \Theta_{21} = 0$$

or

(7) $$\cos\varphi = 0, \qquad \frac{1}{l_{T_1}} + \frac{1}{l_{T_2}} = 0.$$

The lines of curvature of the first kind then define a system of two mutually-perpendicular families of orthogonal trajectories for which the sum of the curvatures relative to the normal surface (ξ, η, ζ) vanishes at every point.

§ 7. – Effect of transposing two successive derivatives with respect to different arc-lengths. Lines of curvature of the first kind as coordinate lines. Fundamental equations.

Just as the difference $(d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_0T_1}$ was expressed in terms of the derivatives $(d\mathfrak{F})_{T_1}$ and $(d\mathfrak{F})_{T_0}$, and some geometrically-intuitive quantities in the first two paragraphs, the three differences $(d\mathfrak{F})_{T_1T_2} - (d\mathfrak{F})_{T_2T_1}$, $(d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_0T_1}$, $(d\mathfrak{F})_{T_2T_0} - (d\mathfrak{F})_{T_0T_2}$ will now be represented in terms of the derivatives $(d\mathfrak{F})_{T_1}$, $(d\mathfrak{F})_{T_2}$, $(d\mathfrak{F})_{T_0}$, and some geometrically-intuitive quantities. That is equivalent to finding the integrability conditions for a linear form in T_1, T_2, T_3 .

We set:

$$\alpha_1 = \frac{a_{13}}{a_{33}}, \qquad \alpha_2 = \frac{a_{23}}{a_{33}},$$

to abbreviate. We will then have:

$$\mathfrak{F}_p = \frac{\partial\mathfrak{F}}{\partial p} - \alpha_1\frac{\partial\mathfrak{F}}{\partial r}, \qquad \mathfrak{F}_q = \frac{\partial\mathfrak{F}}{\partial q} - \alpha_2\frac{\partial\mathfrak{F}}{\partial r}.$$

That will yield:

$$(\mathfrak{F}_p)_q - (\mathfrak{F}_q)_p = ((\alpha_2)_p - (\alpha_1)_q)\frac{\partial\mathfrak{F}}{\partial r},$$

$$\frac{\partial\mathfrak{F}_p}{\partial r} - \left(\frac{\partial\mathfrak{F}}{\partial r}\right)_p = -\frac{\partial\alpha_1}{\partial r}\frac{\partial\mathfrak{F}}{\partial r}, \qquad \frac{\partial\mathfrak{F}_q}{\partial r} - \left(\frac{\partial\mathfrak{F}}{\partial r}\right)_q = -\frac{\partial\alpha_2}{\partial r}\frac{\partial\mathfrak{F}}{\partial r}.$$

On the other hand, one has:

$$\mathfrak{F}_p = \alpha_{11}(d\mathfrak{F})_{T_1} + \alpha_{21}(d\mathfrak{F})_{T_2}\,, \quad \mathfrak{F}_q = \alpha_{12}(d\mathfrak{F})_{T_1} + \alpha_{22}(d\mathfrak{F})_{T_2}\,, \quad \frac{\partial \mathfrak{F}}{\partial r} = \sqrt{a_{33}}\,(d\mathfrak{F})_{T_0}\,.$$

The determinant $\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21}$ will be set equal to D. It will then follow that:

$$(1) \qquad \left\{ \begin{aligned} & EG - F^2 = D^2 \sin^2\varphi, \\ & f - f' = D(\Theta_{12} - \Theta_{21}). \end{aligned} \right.$$

It further follows that:

$$\begin{aligned} (\mathfrak{F}_p)_q = (\alpha_{11})_q(d\mathfrak{F})_{T_1} + (\alpha_{21})(d\mathfrak{F})_{T_2} + \alpha_{11}[\alpha_{12}(d\mathfrak{F})_{T_1^2} + \alpha_{22}(d\mathfrak{F})_{T_1 T_2}] \\ + \alpha_{21}[\alpha_{12}(d\mathfrak{F})_{T_2\,T_1} + \alpha_{22}(d\mathfrak{F})_{T_2^2}], \end{aligned}$$

$$\begin{aligned} (\mathfrak{F}_q)_p = (\alpha_{12})_q(d\mathfrak{F})_{T_1} + (\alpha_{22})(d\mathfrak{F})_{T_2} + \alpha_{12}[\alpha_{11}(d\mathfrak{F})_{T_1^2} + \alpha_{21}(d\mathfrak{F})_{T_1 T_2}] \\ + \alpha_{22}[\alpha_{11}(d\mathfrak{F})_{T_2\,T_1} + \alpha_{21}(d\mathfrak{F})_{T_2^2}]. \end{aligned}$$

A second expression for the difference $(\mathfrak{F}_p)_q - (\mathfrak{F}_q)_p$ will emerge from this. If one sets it equal to the expression that was found above then that will yield the desired relation in the form:

$$(2) \qquad \left\{ \begin{aligned} (d\mathfrak{F})_{T_1 T_2} - (d\mathfrak{F})_{T_2 T_1} = \frac{(\alpha_{12})_p - (\alpha_{11})_q}{D}(d\mathfrak{F})_{T_1} + \frac{(\alpha_{22})_p - (\alpha_{21})_q}{D}(d\mathfrak{F})_{T_2} \\ + \frac{((\alpha_2)_p - (\alpha_1)_q)\sqrt{a_{33}}}{D}(d\mathfrak{F})_{T_0}. \end{aligned} \right.$$

If one proceeds correspondingly with the differences $\dfrac{\partial \mathfrak{F}_p}{\partial r} - \left(\dfrac{\partial \mathfrak{F}}{\partial r}\right)_p$ and $\dfrac{\partial \mathfrak{F}_q}{\partial r} - \left(\dfrac{\partial \mathfrak{F}}{\partial r}\right)_q$ then it will further follow that:

$$(3) \qquad \left\{ \begin{aligned} (d\mathfrak{F})_{T_1 T_0} - (d\mathfrak{F})_{T_0 T_1} = \frac{\alpha_{21}\dfrac{\partial \alpha_{12}}{\partial r} - \alpha_{22}\dfrac{\partial \alpha_{11}}{\partial r}}{D\sqrt{a_{33}}}(d\mathfrak{F})_{T_1} + \frac{-\alpha_{22}\dfrac{\partial \alpha_{21}}{\partial r} + \alpha_{21}\dfrac{\partial \alpha_{22}}{\partial r}}{D\sqrt{a_{33}}}(d\mathfrak{F})_{T_2} \\ + \frac{1}{D}\left\{\alpha_{22}\left[\tfrac{1}{2}(\log a_{33})_p - \frac{\partial \alpha_1}{\partial r}\right] - \alpha_{21}\left[\tfrac{1}{2}(\log a_{33})_q - \frac{\partial \alpha_2}{\partial r}\right]\right\}(d\mathfrak{F})_{T_0}, \end{aligned} \right.$$

$$(4)\quad \begin{cases} (d\mathfrak{F})_{T_2T_0} - (d\mathfrak{F})_{T_0T_2} = \dfrac{\alpha_{12}\dfrac{\partial\alpha_{11}}{\partial r} - \alpha_{11}\dfrac{\partial\alpha_{12}}{\partial r}}{D\sqrt{a_{33}}}(d\mathfrak{F})_{T_1} + \dfrac{\alpha_{12}\dfrac{\partial\alpha_{21}}{\partial r} - \alpha_{11}\dfrac{\partial\alpha_{22}}{\partial r}}{D\sqrt{a_{33}}}(d\mathfrak{F})_{T_2} \\ \quad + \dfrac{1}{D}\left\{\alpha_{11}\left[\tfrac{1}{2}(\log a_{33})_q - \dfrac{\partial\alpha_1}{\partial r}\right] - \alpha_{12}\left[\tfrac{1}{2}(\log a_{33})_p - \dfrac{\partial\alpha_1}{\partial r}\right]\right\}(d\mathfrak{F})_{T_0}. \end{cases}$$

We introduce some abbreviations for the coefficients on the right-hand sides of (2), (3), (4), and set:

$$(5)\quad \begin{cases} (d\mathfrak{F})_{T_1T_2} - (d\mathfrak{F})_{T_2T_1} = c_{11}(d\mathfrak{F})_{T_1} + c_{12}(d\mathfrak{F})_{T_2} + c_{10}(d\mathfrak{F})_{T_0}, \\ (d\mathfrak{F})_{T_1T_0} - (d\mathfrak{F})_{T_0T_1} = c_{21}(d\mathfrak{F})_{T_1} + c_{22}(d\mathfrak{F})_{T_2} + c_{20}(d\mathfrak{F})_{T_0}, \\ (d\mathfrak{F})_{T_2T_0} - (d\mathfrak{F})_{T_0T_2} = c_{31}(d\mathfrak{F})_{T_1} + c_{32}(d\mathfrak{F})_{T_2} + c_{30}(d\mathfrak{F})_{T_0}. \end{cases}$$

If one takes x, y, z in place of $\mathfrak{F}$ here in succession and compares the relations that thus arise with the corresponding ones for the system (6) in the previous paragraph then that will yield new expressions for the quantities $c_{\mu\nu}$ in a geometrically-intuitive form, namely:

$$(6)\quad \begin{cases} c_{11} = -\dfrac{\cos^2\varphi}{\sin\varphi\, R_{T_1}} + \dfrac{\cos\varphi}{R_{T_1'}} - \dfrac{\cos\varphi}{\sin\varphi\, R_{T_2}} + \dfrac{1}{R_{T_2'}}, \\ c_{12} = \dfrac{\cos^2\varphi}{\sin\varphi\, R_{T_2}} - \dfrac{\cos\varphi}{R_{T_2'}} + \dfrac{\cos\varphi}{\sin\varphi\, R_{T_1}} - \dfrac{1}{R_{T_1'}}, \\ c_{10} = \cos\varphi\left(\dfrac{1}{h_{T_2}} - \dfrac{1}{h_{T_1}}\right) - \sin\varphi\left(\dfrac{1}{l_{T_2}} - \dfrac{1}{l_{T_1}}\right), \\ c_{21} = -\cot\varphi\left(\dfrac{1}{L_{T_2}} - \dfrac{1}{L_{T_1}}\right) + \dfrac{1}{h_{T_1}}, \quad c_{22} = \dfrac{1}{\sin\varphi}\left(\dfrac{1}{L_{T_2}} - \dfrac{1}{L_{T_1}}\right), \quad c_{20} = -\dfrac{1}{R_{T_1}}, \\ c_{31} = \dfrac{1}{\sin\varphi}\left(\dfrac{1}{L_{T_2}} - \dfrac{1}{L_{T_1}}\right), \quad c_{32} = -\cot\varphi\left(\dfrac{1}{L_{T_2}} - \dfrac{1}{L_{T_1}}\right) + \dfrac{1}{h_{T_2}}, \quad c_{30} = -\dfrac{1}{P_{T_1}}. \end{cases}$$

One can now take $\mathfrak{F}$ in (5) to be the nine direction cosines ξ, $(dx)_{T_1}$, $(dx)_{T_2}$, etc., and one would then get twelve differential equations in the various curvatures. There is little reason to pursue that tedious path. First of all, one would give preference to the rectangular coordinates in the choice of curvilinear coordinates, and one would then avoid the introduction of arbitrary functions (m_1, n_1, m_2, n_2) and employ systems of orthogonal trajectories that are determined by only the given family of curves. We have learned of two such systems, namely, the lines of curvature of the first kind and the system of principal normals and binormals. However, the latter will be undetermined when the given family of curves consists of nothing but straight lines. We then take the lines of curvature of the first kind to be coordinate lines and introduce some special notations under that assumption. We shall write $\mathfrak{S}_1$ or $\mathfrak{S}_2$ in place of T_1 or T_2, resp. The first family of lines of curvature in question ($\mathfrak{S}_2 = 0$) shall be the one whose tangents possess the direction cosines $\kappa_1, \lambda_1, \mu_1$, while the tangents to the second family ($\mathfrak{S}_1 = 0$) possess the direction cosines $\kappa_2, \lambda_2, \mu_2$. If we take:

$$\alpha_{11} = \sigma_1, \qquad \alpha_{12} = \sigma_2, \qquad \alpha_{21} = \sigma_3, \qquad \alpha_{22} = \sigma_4$$

here then it will follow that:

$$(7) \qquad \begin{cases} \mathfrak{S}_1 = \sigma_1\, dp + \sigma_2\, dq, & \mathfrak{S}_2 = \sigma_3\, dp + \sigma_4\, dq, \\ \sigma_1 = \dfrac{E + F\, t_1}{V_1} = \dfrac{V_1\, t_2}{t_2 - t_1}, & \sigma_2 = \dfrac{F + G\, t_1}{V_1} = -\dfrac{V_1\, t_2}{t_2 - t_1}, \\ \sigma_3 = \dfrac{E + F\, t_3}{V_2} = \dfrac{-V_2\, t_1}{t_2 - t_1}, & \sigma_4 = \dfrac{F + G\, t_2}{V_2} = \dfrac{V_2}{t_2 - t_1}, \end{cases}$$

$$\delta x = \kappa_1\, \mathfrak{S}_1 + \kappa_2\, \mathfrak{S}_2 .$$

We couple the last of those equations with the following remark (Cf., **Darboux**, *Leçons sur la théorie générale des surfaces*, t. II, pp. 3): Consider the tangents to four orthogonal trajectories that go through the regular point (x, y, z). Each of the latter is an individual curve of a family that is determined by the fact that the ratio dq / dp is set equal to a function of p, q, r. The four functions in question shall be denoted by $\tau_1, \tau_2, \tau_3, \tau_4$, and the corresponding values of $\dfrac{\mathfrak{S}_2}{\mathfrak{S}_1}$ by $\dfrac{\mathfrak{S}_{21}}{\mathfrak{S}_{11}}, \dfrac{\mathfrak{S}_{22}}{\mathfrak{S}_{12}}, \dfrac{\mathfrak{S}_{23}}{\mathfrak{S}_{13}}, \dfrac{\mathfrak{S}_{24}}{\mathfrak{S}_{14}}$, resp. The double ratio of the tangents that we speak of is:

$$\frac{\left(\dfrac{\mathfrak{S}_{21}}{\mathfrak{S}_{11}} - \dfrac{\mathfrak{S}_{23}}{\mathfrak{S}_{13}}\right)\left(\dfrac{\mathfrak{S}_{22}}{\mathfrak{S}_{12}} - \dfrac{\mathfrak{S}_{24}}{\mathfrak{S}_{14}}\right)}{\left(\dfrac{\mathfrak{S}_{21}}{\mathfrak{S}_{11}} - \dfrac{\mathfrak{S}_{24}}{\mathfrak{S}_{14}}\right)\left(\dfrac{\mathfrak{S}_{22}}{\mathfrak{S}_{12}} - \dfrac{\mathfrak{S}_{23}}{\mathfrak{S}_{13}}\right)} = \frac{(\tau_1 - \tau_3)(\tau_2 - \tau_4)}{(\tau_1 - \tau_4)(\tau_2 - \tau_3)}.$$

The equation $\mathfrak{F}(p, q) =$ const. determines a surface that is formed from nothing but curves of the given family. One now lays four such surfaces through an individual curve of the family and calculates the quantities τ_ν with the help of their equations $\mathfrak{F}_\nu(p, q) = c_\nu$. In that way, the quantities τ_ν will be functions of p and q alone, and therefore the double ratio considered, as well. However, that is equal to the double ratio of the tangent planes to the four surfaces at a point of the curve, and we will see that it does not change along the curve.

We further denote the values $\frac{1}{R_{T_1}}$, $\frac{1}{R_{T_2}}$, $\frac{1}{P_{T_1}}$, $\frac{1}{P_{T_2}}$ by $\frac{1}{R_1}$, $\frac{1}{R_2}$, $\frac{1}{P_1}$, $\frac{1}{P_2}$, resp., under the prevailing assumption that the curves $T_2 = 0$, $T_1 = 0$ are lines of curvature of the first kind. ε is set to $1/l_{T_1}$, in such a way that from (7), § **6**, $1/l_{T_2}$ must be replaced with $-\varepsilon$. When ϑ is written for $1/L_{T_1}$, from (4), § **6**, $1/L_{T_2}$ will go to $-\vartheta$. Finally, the derivatives $(d\mathfrak{F})_{T_1}$, $(d\mathfrak{F})_{T_2}$, $(d\mathfrak{F})_{T_0}$ shall be denoted by $g_1(\mathfrak{F})$, $g_2(\mathfrak{F})$, $g_3(\mathfrak{F})$, while the second derivative $(d\mathfrak{F})_{T_\alpha T_\beta}$ will be replaced with $g_{\alpha\beta}(\mathfrak{F})$.

When one takes:

$$\sigma_1\,\sigma_4 - \sigma_2\,\sigma_3 = \sigma,$$

the determinations of the quantities $c_{\mu\nu}$ that are contained in equations (2), (3), (4), on the one hand, and (6) on the other, will now imply the relations:

$$(8)\qquad \left\{\begin{aligned}
c_{11} &= \frac{(\sigma_2)_p - (\sigma_1)_q}{\sigma} = \frac{1}{R_1},\\
c_{12} &= \frac{(\sigma_4)_p - (\sigma_2)_q}{\sigma} = -\frac{1}{R_2},\\
c_{10} &= \frac{(\alpha_2)_p - (\alpha_1)_q}{\sigma}\sqrt{a_{23}} = 2\varepsilon,\\
c_{21} &= \frac{\sigma_3\,g_0(\sigma_2) - \sigma_4\,g_0(\sigma_1)}{\sigma} = \frac{1}{h_1},\\
c_{22} &= \frac{\sigma_3\,g_0(\sigma_4) - \sigma_4\,g_0(\sigma_3)}{\sigma} = \vartheta - \varepsilon,\\
c_{20} &= \frac{1}{\sigma}\left\{\sigma_4\left[\tfrac{1}{2}(\log a_{33})_p - \frac{\partial a_1}{\partial r}\right] - \sigma_3\left[\tfrac{1}{2}(\log a_{33})_q - \frac{\partial a_3}{\partial r}\right]\right\} = -\frac{1}{P_1},\\
c_{31} &= \frac{\sigma_2\,g_0(\sigma_1) - \sigma_1\,g_0(\sigma_2)}{\sigma} = \varepsilon - \vartheta,\\
c_{32} &= \frac{\sigma_2\,g_0(\sigma_3) - \sigma_1\,g_0(\sigma_4)}{\sigma} = \frac{1}{h_2},\\
c_{30} &= \frac{1}{\sigma}\left\{\sigma_1\left[\tfrac{1}{2}(\log a_{33})_q - \frac{\partial a_2}{\partial r}\right] - \sigma_2\left[\tfrac{1}{2}(\log a_{33})_p - \frac{\partial a_1}{\partial r}\right]\right\} = -\frac{1}{P_2}.
\end{aligned}\right.$$

We shall now infer an important consequence from the fifth and seventh of these equations. From (7), one has:

$$t_1 = -\frac{\sigma_3}{\sigma_4}, \qquad\qquad t_2 = -\frac{\sigma_1}{\sigma_2},$$

and as a result, one will have:

$$\vartheta - \varepsilon = \frac{\sigma_4^2}{\sigma} g_0(t_1) = \frac{\sigma_2^2}{\sigma} g_0(t_2) .$$

If $\vartheta - \varepsilon$ vanishes then $\frac{\sigma_3}{\sigma_4}$, as well as $\frac{\sigma_1}{\sigma_4}$, will be independent of r.

Equations (2), (3), and (4) will now assume the form:

$$(9) \quad \begin{cases} g_{12}(\mathfrak{F}) - g_{21}(\mathfrak{F}) = \dfrac{1}{R_1} g_1(\mathfrak{F}) - \dfrac{1}{R_2} g_2(\mathfrak{F}) + 2\varepsilon\, g_0(\mathfrak{F}), \\ g_{10}(\mathfrak{F}) - g_{01}(\mathfrak{F}) = \dfrac{1}{h_1} g_1(\mathfrak{F}) + (\vartheta - \varepsilon) g_2(\mathfrak{F}) - \dfrac{1}{P_1} g_0(\mathfrak{F}), \\ g_{20}(\mathfrak{F}) - g_{02}(\mathfrak{F}) = (\varepsilon - \vartheta) g_1(\mathfrak{F}) + \dfrac{1}{h_2} g_2(\mathfrak{F}) - \dfrac{1}{P_2} g_0(\mathfrak{F}). \end{cases}$$

It follows from equations (6), § **6** that:

$$(10) \quad \begin{cases} d\kappa_1 = \left(\dfrac{\kappa_2}{R_1} + \dfrac{\xi}{h_1}\right)\mathfrak{S}_1 + \left(-\dfrac{\kappa_2}{R_2} + \varepsilon\,\xi\right)\mathfrak{S}_2 + \left(\vartheta\,\kappa_2 - \dfrac{\xi}{P_1}\right)T_0, \\ d\kappa_2 = \left(-\dfrac{\kappa_1}{R_1} - \varepsilon\,\xi\right)\mathfrak{S}_1 + \left(\dfrac{\kappa_1}{R_2} + \dfrac{\xi}{h_2}\right)\mathfrak{S}_2 + \left(-\vartheta\,\kappa_1 - \dfrac{\xi}{P_2}\right)T_0, \\ d\xi = \left(-\dfrac{\kappa_1}{h_1} + \xi\,\kappa_2\right)\mathfrak{S}_1 + \left(-\varepsilon\,\kappa_1 - \dfrac{\kappa_2}{h_2}\right)\mathfrak{S}_2 + \left(\dfrac{\kappa_1}{P_1} + \dfrac{\kappa_2}{P_1}\right)T_0. \end{cases}$$

An application of equations (9) to the representations (10) will yield the differential equations:

$$(11) \quad \begin{cases} g_1\left(\dfrac{1}{h_1}\right) - g_1(\varepsilon) = \dfrac{1}{R_1}\left(\dfrac{1}{h_1} - \dfrac{1}{h_2}\right) - \dfrac{2\varepsilon}{R_1}, \\ g_1\left(\dfrac{1}{h_2}\right) + g_2(\varepsilon) = \dfrac{1}{R_2}\left(\dfrac{1}{h_2} - \dfrac{1}{h_1}\right) + \dfrac{2\varepsilon}{P_2}, \\ g_0\left(\dfrac{1}{h_1}\right) + g_1\left(\dfrac{1}{P_1}\right) = \dfrac{1}{P_1^2} + \dfrac{1}{h_1^2} + \dfrac{1}{P_2 R_1} - \varepsilon^2, \\ g_0(\varepsilon) - g_1\left(\dfrac{1}{P_2}\right) = \dfrac{1}{P_1}\left(\dfrac{1}{R_1} - \dfrac{1}{P_2}\right) + \varepsilon\left(\dfrac{1}{h_1} + \dfrac{1}{h_2}\right) + \vartheta\left(\dfrac{1}{h_1} - \dfrac{1}{h_2}\right), \\ g_2\left(\dfrac{1}{R_1}\right) + g_1\left(\dfrac{1}{R_2}\right) = \dfrac{1}{h_1 h_2} + \dfrac{1}{R_1^2} + \dfrac{1}{R_1^2} + \varepsilon^2 + 2\varepsilon\vartheta, \end{cases}$$

$$
(11\text{ cont.})\qquad \left\{\begin{aligned}
g_0\left(\frac{1}{h_2}\right)+g_2\left(\frac{1}{P_2}\right)&=\frac{1}{h_1^2}+\frac{1}{P_1^{\,2}}+\frac{1}{R_2\,P_1}-\varepsilon^2,\\
g_0\left(\frac{1}{R_2}\right)-g_1(\vartheta)&=\frac{1}{h_1}\left(\frac{1}{R_1}-\frac{1}{P_2}\right)-\frac{\varepsilon+\vartheta}{P_1}+\frac{\varepsilon-\vartheta}{R_2},\\
g_0\left(\frac{1}{R_2}\right)+g_2(\vartheta)&=\frac{1}{h_2}\left(\frac{1}{R_2}-\frac{1}{P_1}\right)+\frac{\varepsilon+\vartheta}{P_2}-\frac{\varepsilon-\vartheta}{R_1},\\
g_2\left(\frac{1}{P_2}\right)+g_0(\varepsilon)&=\frac{1}{P_2}\left(\frac{1}{P_1}-\frac{1}{R_2}\right)+\varepsilon\left(\frac{1}{h_1}+\frac{1}{h_2}\right)-\vartheta\left(\frac{1}{h_1}-\frac{1}{h_2}\right).
\end{aligned}\right.
$$

Those equations play the same role in the theory of families of curves that the so-called fundamental equations do in the theory of surfaces.

If one represents the family of curves considered with the help of a new variable r_1 and two new parameters p_1 and q_1, as was done in § **4**, then the derivatives $g_1(\mathfrak{F})$, $g_2(\mathfrak{F})$, $g_0(\mathfrak{F})$ that are defined with the new independent variables will be equal to the corresponding derivatives that are defined with the old independent variables. We can then call those derivatives *invariant operations*. Similarly, the quantities $\frac{1}{R_1}$, $\frac{1}{R_2}$, $\frac{1}{P_1}$, $\frac{1}{P_2}$, ε, and ϑ will not change their values when they are calculated with the help of the new independent variables. On that basis, we shall endow the quantities in question, and likewise $\frac{1}{h_1}$ and $\frac{1}{h_2}$, with the common name of *geometric invariants.*

§ 8. – Ray systems. Families of planar curves. Orthogonal trajectories of a family of surfaces that belong to a triply-orthogonal system of surfaces.

Having concluded the necessary theoretical discussions, we now turn to the practical questions that should next relate to the curves of a given family. When are they straight lines? When are they curved, planar lines?

In § **6**, (5), we found the following expressions for the direction cosines of the principal normals and binormals of the curves in question:

$$a_1 = \rho\, g_0(\xi),\quad a_2 = \rho\,[\eta\, g_0(\xi) - \zeta\, g_0(\eta)],$$

in which ρ means the radius of the first curvature. If we establish, once and for all, that the signs of the cosines κ_1, κ_2, etc., are chosen such that:

$$\xi = \lambda_1\,\mu_2 - \mu_1\,\lambda_2,\qquad \eta = \mu_1\,\kappa_2 - \kappa_1\,\mu_2,\qquad \zeta = \kappa_1\,\lambda_2 - \lambda_1\,\kappa_2$$

then it will now follow that:

(1) $$a_1 = \rho\left(\frac{\kappa_1}{P_1}+\frac{\kappa_2}{P_2}\right), \qquad a_2 = \rho\left(-\frac{\kappa_1}{P_2}+\frac{\kappa_2}{P_1}\right),$$

(2) $$\frac{1}{\rho} = \sqrt{\frac{1}{P_1^2}+\frac{1}{P_2^2}}\,.$$

Therefore, the curves of the family well be straight lines when $1 / P_1$ vanishes, as well as $1 / P_2$. From (8), § **7**, one will then have:

$$\tfrac{1}{2}(\log a_{33})_p - \frac{\partial \frac{a_{13}}{a_{33}}}{\partial r} = 0 \qquad \text{and} \qquad \tfrac{1}{2}(\log a_{33})_q - \frac{\partial \frac{a_{23}}{a_{33}}}{\partial r} = 0.$$

Knowing the three quantities a_{13}, a_{23}, a_{33} will then suffice to decide whether one is dealing with a ray system or not.

We obtain the following equation for the second curvature:

$$\frac{1}{\rho'} = \sum a_1\, g_0(a_2) = \rho^2 \sum\left(\frac{\kappa_1}{P_1}+\frac{\kappa_2}{P_2}\right) g_0\left(-\frac{\kappa_1}{P_2}+\frac{\kappa_2}{P_1}\right);$$

i.e.:

(3) $$\frac{1}{\rho'} = -\,\vartheta + g_0\left(\arctan\frac{P_2}{P_1}\right).$$

We will then be dealing with a family of planar, curved lines when P_1 and P_2 possess finite values, in general, and when:

(4) $$\vartheta = g_0\left(\arctan\frac{P_2}{P_1}\right).$$

When is the family of curves considered a normal family?

From (1), § **7**, one has:

(5) $$\frac{f - f'}{2\sqrt{EG-F^2}} = \varepsilon.$$

If equation (4) is true then the planes of the individual curves of the families are, at the same time, contact planes of a surface whose coordinates x_0 , y_0 , z_0 depend upon p and q. Here, one sets:

$$E_0 = \sum\left(\frac{\partial x_0}{\partial p}\right)^2, \qquad G_0 = \sum\left(\frac{\partial x_0}{\partial q}\right)^2, \qquad F_0 = \sum\frac{\partial x_0}{\partial p}\frac{\partial x_0}{\partial q},$$

and if one understands u and v to mean functions of p, q, r then the coordinates of the family of curves can be brought into the form:

$$x = x_0 + \frac{\frac{\partial x_0}{\partial p}}{\sqrt{E_0}} u + \frac{\frac{\partial x_0}{\partial q}}{\sqrt{G_0}} v, \qquad y = y_0 + \frac{\frac{\partial y_0}{\partial p}}{\sqrt{E_0}} u + \frac{\frac{\partial y_0}{\partial q}}{\sqrt{G_0}} v, \qquad z = z_0 + \frac{\frac{\partial z_0}{\partial p}}{\sqrt{E_0}} u + \frac{\frac{\partial z_0}{\partial q}}{\sqrt{G_0}} v.$$

The expression $f - f'$ now depends upon only E_0, F_0, G_0, in addition to u and v.

One now fixes u and v while directing one's attention to the bending surfaces of the surface (x_0, y_0, z_0), instead of that surface. In that way, any curve p = const., q = const. of the first family will be associated with one of the second, whereby only the position, but not the form, will be different from the original, and the quantity $f - f'$ will remain the same. If the family of curves is a normal family then that property will not be lost when the arrangement of its individual curves is changed by a bending of the surface (x_0, y_0, z_0). **Ribaucour** found that theorem by a different method [J. de Math. **7** (1891), pp. 251].

We now return from that digression to equation (5)! Since $\varepsilon = 1/l_{\mathfrak{S}_1}$, for $\varepsilon = 0$, the curve tangents (ξ, η, ζ) along the lines of curvature of the first kind will define a developable surface. Therefore, the lines of curvature of the first kind will be, at the same time, the lines of curvature of the family of surfaces whose orthogonal trajectories define the family of curves.

In order for the family of surfaces in question to belong to a triply-orthogonal system of surfaces, from **Dupin**'s theorem, the tangents to the lines of curvature along every curve p = const., q = const. must define a developable surface. One further condition $\vartheta = 0$ is necessary for that to be true. It shall now be shown that the two conditions $\varepsilon = 0$, $\vartheta = 0$ are also sufficient. If ε vanishes then, from § **3**, the differential form T_0 will possess an integrating factor, and one can set:

$$T_0 = n\, dw.$$

From a remark that was made in the previous paragraph, the quotients σ_1 / σ_3 and σ_3 / σ_4 will be independent of r when $\varepsilon - \vartheta = 0$, and the differential forms $\mathfrak{S}_1 / \sigma_1$ and $\mathfrak{S}_2 / \sigma_4$ will then possess integrating factors that depend upon only p and q. There will then be equations of the form:

$$\mathfrak{S}_1 = l\, du, \qquad \mathfrak{S}_2 = m\, dv.$$

The square of the line element in space will generally have the expression:

$$ds^2 = dx^2 + dy^2 + dz^2 = \mathfrak{S}_1^2 + \mathfrak{S}_2^2 + T_0^2.$$

In the case $\varepsilon = \vartheta = 0$, we will then get:

$$ds^2 = l\, du^2 + m\, dv^2 + n\, dw^2.$$

The family of surfaces u = const., v = const., w = const. then define a triply-orthogonal system, and the given family of curves will then consist of the lines of intersection of the surfaces u = const., v = const.

Let the general expression for the square of the line element in terms of the differentials dp, dq, dr be:

$$a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2 + 2a_{13}\,dp\,dr + 2a_{23}\,dq\,dr + a_{33}\,dr^2.$$

In order to establish whether ε does or does not vanish, we need to know only the coefficients a_{13} , a_{23} , a_{33} . We will now show that knowing all of the coefficients $a_{\mu\nu}$ will suffice for us to decide whether ϑ does or does not also vanish when $\varepsilon = 0$. To that end, we shall derive a new expression for ϑ under the assumption that $\varepsilon = 0$.

The system (10) of the previous paragraph implies that:

$$\vartheta = \sum \kappa_2\, g_0(\kappa_1) = - \sum \kappa_1\, g_0(\kappa_2),$$

such that:

$$2\,\vartheta = \sum \kappa_2\, g_0(\kappa_1) - \sum \kappa_1\, g_0(\kappa_2).$$

In order to convert that equation, we employ the expression:

$$\kappa_1 = \frac{x_p + x_q\, t_1}{V_1}, \qquad \kappa_2 = \frac{x_p + x_q\, t_2}{V_2},$$

and first assume that t_1 and t_2 have finite values. We will then find that:

$$(6)\begin{cases} \sum \kappa_2\, g_0(\kappa_1) - \sum \kappa_2\, g_0(\kappa_1) \\ = \dfrac{1}{V_1 V_2}\left\{(t_1 - t_2)\left[\sum x_p\, g_0(x_q) - \sum x_q\, g_0(x_p)\right] + F\, g_0(t_1 - t_2) + G[t_2\, g_0(t_1) - t_1\, g_0(t_2)]\right\}. \end{cases}$$

From (3), § **4**, one has:

$$\sum x_p\, g_0(x_q) - \sum x_q\, g_0(x_p) = f' - f.$$

The sum on the left will then vanish with ε. The quantities t_1 and t_2 are the roots of equation (7), § **4**:

$$(7) \qquad [F\,g_0\,(G) - G\,g_0\,(F)]\,t^2 + [E\,g_0\,(G) - G\,g_0\,(E)]\,t + E\,g_0\,(F) - F\,g_0\,(E) = 0,$$

so:

$$[F\,g_{00}\,(G) - G\,g_{00}\,(F)]\,t^2 + [E\,g_{00}\,(G) - G\,g_{00}\,(E)]\,t + E\,g_{00}\,(F) - F\,g_{00}\,(E)$$

$$+ g_0\,(t)\{2t\,[F\,g_0\,(G) - G\,g_0\,(F)] + E\,g_0\,(F) - F\,g_0\,(E)\} = 0 .$$

One writes that in the form:

$$M + N\,g_0\,(t) = 0,$$

to abbreviate. If one replaces the quantity t^2 in M with its expression that one infers from equation (7) then it will follow that:

$$M = -\frac{\mathsf{A}(G\,t + F)}{F\,g_0(G) - G\,g_0(F)},$$

in which:

$$\mathsf{A} = \begin{vmatrix} E & F & G \\ g_0(E) & g_0(F) & g_0(G) \\ g_{00}(E) & g_{00}(F) & g_{00}(G) \end{vmatrix}.$$

Furthermore, one has:

$$N = [F\,g_0\,(G) - G\,g_0\,(F)]\,(2t - t_1 - t_2).$$

One then has:

$$g_0\,(t_1) = \frac{\mathsf{A}}{[F\,g_0(G) - G\,g_0(F)]^2}\frac{G\,t_1 + F}{t_1 - t_2}, \qquad g_0\,(t_2) = \frac{\mathsf{A}}{[F\,g_0(G) - G\,g_0(F)]^2}\frac{G\,t_2 + F}{t_2 - t_1},$$

and for $\varepsilon = 0$, one will get:

$$\vartheta = \frac{\mathsf{A}(EG - F^2)}{V_1 V_2\,[F\,g_0(G) - G\,g_0(F)]^2 (t_2 - t_1)},$$

in place of (6). When $F\,g_0\,(G) - G\,g_0\,(F)$ vanishes, a root of (7) – e.g., t_1 – will be infinitely large. Now, since:

$$F + G\,t_2 = 0,$$

one will have:

$$\kappa_1 = \frac{x_q}{\sqrt{G}}, \qquad \kappa_2 = \frac{G\,x_p - F\,x_q}{\sqrt{G}\,\sqrt{EG - F^2}}.$$

That shows that for $\varepsilon = 0$, the difference:

$$\sum \kappa_2\,g_0(\kappa_1) - \sum \kappa_1\,g_0(\kappa_2)$$

will also vanish, and therefore ϑ. The determinant A will become:

$$\frac{1}{G}[G\,g_0\,(E) - E\,g_0\,(G)]\,[G\,g_{00}\,(F) - F\,g_{00}\,(G)],$$

and will then vanish in any event. As a result, the equation $\mathsf{A} = 0$ will be the necessary and sufficient condition for the vanishing of ϑ when $\varepsilon = 0$. That will lead to the value of A in the general case.

One has:

$$E = \sigma_1^2 + \sigma_3^2, \qquad F = \sigma_1\,\sigma_2 + \sigma_3\,\sigma_4, \qquad G = \sigma_2^2 + \sigma_4^2.$$

If follows from the system (8), § **7** that:

$$g_0(\sigma_1) = -\frac{\sigma_1}{h_1} - \sigma_3(\varepsilon - \vartheta), \qquad g_0(\sigma_2) = -\frac{\sigma_2}{h_1} - \sigma_4(\varepsilon - \vartheta),$$

$$g_0(\sigma_3) = -\frac{\sigma_3}{h_2} - \sigma_1(\varepsilon - \vartheta), \qquad g_0(\sigma_4) = -\frac{\sigma_4}{h_2} - \sigma_4(\varepsilon - \vartheta).$$

Hence:

$$g_0(E) = -2\left(\frac{\sigma_1^2}{h_1} + \frac{\sigma_3^2}{h_2}\right), \quad g_0(F) = -2\left(\frac{\sigma_1\,\sigma_2}{h_1} + \frac{\sigma_3\,\sigma_4}{h_2}\right), \quad g_0(G) = -2\left(\frac{\sigma_2^2}{h_1} + \frac{\sigma_4^2}{h_2}\right),$$

$$F\,g_0(G) - G\,g_0(F) = 2\sigma\,\sigma_2\,\sigma_4\left(\frac{1}{h_1} - \frac{1}{h_2}\right),$$

$$G\,g_0(E) - E\,g_0(G) = -2\sigma(\sigma_1\,\sigma_4 + \sigma_2\,\sigma_3)\left(\frac{1}{h_1} - \frac{1}{h_2}\right),$$

$$E\,g_0(F) - F\,g_0(E) = 2\sigma\,\sigma_1\,\sigma_3\left(\frac{1}{h_1} - \frac{1}{h_2}\right),$$

$$\mathsf{A} = 4\sigma^3\left(\frac{1}{h_1} - \frac{1}{h_2}\right)^2 (\vartheta - \varepsilon).$$

If A vanishes then ε will be equal to ϑ.

One further has:

$$\mathsf{A} = \frac{1}{\left(\sqrt{a_{33}}\right)^3}\begin{vmatrix} E & F & G \\ \dfrac{\partial E}{\partial r} & \dfrac{\partial F}{\partial r} & \dfrac{\partial G}{\partial r} \\ \dfrac{\partial^2 E}{\partial r^2} & \dfrac{\partial^2 F}{\partial r^2} & \dfrac{\partial^2 G}{\partial r^2} \end{vmatrix}$$

and

$$E = a_{11} - \frac{a_{13}^2}{a_{33}}, \qquad F = a_{12} - \frac{a_{13}\,a_{23}}{a_{33}}, \qquad G = a_{22} - \frac{a_{23}^2}{a_{33}}.$$

The connection between the determinant A and the coefficients of the square of the line element is made clear with that.

If we think of the conditions $\varepsilon = \vartheta = 0$ as being fulfilled and set:

$$\mathfrak{S}_1 = l\,du, \qquad \mathfrak{S}_2 = m\,dv, \qquad T_0 = n\,dw,$$

as above, then for an arbitrary function $\mathfrak{F}$ of p, q, r, we will have:

$$g_1(\mathfrak{F}) = \frac{1}{l}\frac{\partial \mathfrak{F}}{\partial u}, \qquad g_2(\mathfrak{F}) = \frac{1}{m}\frac{\partial \mathfrak{F}}{\partial v}, \qquad g_0(\mathfrak{F}) = \frac{1}{n}\frac{\partial \mathfrak{F}}{\partial w}.$$

The differential equations (11), § **7** are then equivalent to the differential equations that **Lamé** presented as (14) and (15) in *Leçons sur les coordonées curvilignes et leurs diverses applications*, pp. 80, 81. **Lamé** denoted the quantities h_1, h_2, R_1, R_2, P_1, P_2 by r', r'', r_2', r_1', r_1, r_2, respectively.

§ 9. – Cyclic families of curves.

One calls a family of curves that is composed of nothing but circles *cyclic*, following **Ribaucour**'s analysis. (Cf., **Ribaucour**, C. R. Acad. Sci. Paris, t. 76, pp. 478, 830, as well as the presentation in **Bianchi**'s *Lezioni di geometria differenziale* and **Darboux**'s *Leçons*, t., II.)

The most important theorem that **Ribaucour** presented in regard to those families of curves says that when such a family consists of the orthogonal trajectories to a family of surfaces, the latter will belong to a triply-orthogonal system of surfaces. I will give two proofs of that theorem, the first of which, which was published before (Math. Ann., Bd. 44, pp. 456), starts with a well-defined form for the analytical representation of the family of curves, while the second one employs the relations that were developed in § **7**.

We can establish a doubly-infinite family of circles by the equations:

$$\begin{aligned} x &= x_0 + R(\alpha_1 \cos r + \beta_1 \sin r), \\ y &= y_0 + R(\alpha_2 \cos r + \beta_2 \sin r), \\ z &= z_0 + R(\alpha_3 \cos r + \beta_3 \sin r). \end{aligned}$$

In this, x_0, y_0, z_0 (viz., the coordinates of the surface that is described by the centers of the circles), as well as R; $\alpha_1, \alpha_2, \alpha_3$; $\beta_1, \beta_2, \beta_3$, are functions of only p and q. $\alpha_1, \alpha_2, \alpha_3$, like $\beta_1, \beta_2, \beta_3$, are the direction cosines of two mutually-perpendicular lines. We denote the direction cosines of the lines that are perpendicular to them by $\gamma_1, \gamma_2, \gamma_3$. If we set:

$$c_{11} = \frac{\partial R}{\partial p} + \sin r \sum \beta_1 \frac{\partial x_0}{\partial p} + \cos r \sum \alpha_1 \frac{\partial x_0}{\partial p},$$

$$c_{21} = \frac{\partial R}{\partial q} + \sin r \sum \beta_1 \frac{\partial x_0}{\partial q} + \cos r \sum \alpha_1 \frac{\partial x_0}{\partial q},$$

$$c_{12} = \sum \gamma_1 \frac{\partial x_0}{\partial p} + R\cos r \sum \gamma_1 \frac{\partial \alpha_1}{\partial p} + R \sin r \sum \gamma_1 \frac{\partial \beta_0}{\partial p},$$

$$c_{22} = \sum \gamma_1 \frac{\partial x_0}{\partial q} + R\cos r \sum \gamma_1 \frac{\partial \alpha_1}{\partial q} + R \sin r \sum \gamma_1 \frac{\partial \beta_0}{\partial q},$$

then we will have:

$$x_p = (\alpha_1 \cos r + \beta_1 \sin r)\, c_{11} + \gamma_1\, c_{12}\,, \qquad x_q = (\alpha_1 \cos r + \beta_1 \sin r)\, c_{21} + \gamma_1\, c_{22}\,,$$

$$\xi_p = \frac{1}{R}\left[(\alpha_1 \cos r + \beta_1 \sin r)\frac{\partial c_{11}}{\partial r} + \gamma_1 \frac{\partial c_{12}}{\partial r}\right],$$

$$\xi_q = \frac{1}{R}\left[(\alpha_1 \cos r + \beta_1 \sin r)\frac{\partial c_{21}}{\partial r} + \gamma_1 \frac{\partial c_{22}}{\partial r}\right].$$

In order for the circles of the family to be orthogonal trajectories of a family of surfaces, $f - f'$ must vanish; i.e.:

(1) $$c_{11}\frac{\partial c_{21}}{\partial r} + c_{12}\frac{\partial c_{22}}{\partial r} = c_{21}\frac{\partial c_{11}}{\partial r} + c_{22}\frac{\partial c_{12}}{\partial r}.$$

As a calculation will show, that condition will assume the form:

$$A + B \cos r + C \sin r = 0,$$

in which A, B, C are independent of r. It can then be fulfilled identically only when $A = B = C = 0$; i.e.:

(2) $$\begin{cases} R^2\left(\sum \gamma_1 \frac{\partial \alpha_1}{\partial p} \sum \gamma_1 \frac{\partial \beta_1}{\partial q} - \sum \gamma_1 \frac{\partial \alpha_1}{\partial q} \sum \gamma_1 \frac{\partial \beta_1}{\partial p}\right) + \sum \beta_1 \frac{\partial x_0}{\partial q} \sum \alpha_1 \frac{\partial x_0}{\partial p} - \sum \alpha_1 \frac{\partial x_0}{\partial q} \sum \beta_1 \frac{\partial x_0}{\partial p} = 0, \\ R\left(\sum \gamma_1 \frac{\partial x_0}{\partial q} \sum \gamma_1 \frac{\partial \beta_1}{\partial p} - \sum \gamma_1 \frac{\partial x_0}{\partial p} \sum \gamma_1 \frac{\partial \beta_1}{\partial q}\right) + \frac{\partial R}{\partial q} \sum \beta_1 \frac{\partial x_0}{\partial p} - \frac{\partial R}{\partial p} \sum \beta_1 \frac{\partial x_0}{\partial q} = 0, \\ R\left(\sum \gamma_1 \frac{\partial x_0}{\partial p} \sum \gamma_1 \frac{\partial \alpha_1}{\partial q} - \sum \gamma_1 \frac{\partial x_0}{\partial q} \sum \gamma_1 \frac{\partial \alpha_1}{\partial p}\right) + \frac{\partial R}{\partial p} \sum \alpha_1 \frac{\partial x_0}{\partial q} - \frac{\partial R}{\partial q} \sum \alpha_1 \frac{\partial x_0}{\partial p} = 0. \end{cases}$$

One can also give the first of those equations the form:

(3) $$\frac{\partial^2 c_{11}}{\partial r^2}\frac{\partial c_{21}}{\partial r} + \frac{\partial^2 c_{12}}{\partial r^2}\frac{\partial c_{22}}{\partial r} - \frac{\partial^2 c_{21}}{\partial r^2}\frac{\partial c_{11}}{\partial r} - \frac{\partial^2 c_{22}}{\partial r^2}\frac{\partial c_{12}}{\partial r} = 0.$$

It must now be shown that the determinant:

$$J = \begin{vmatrix} E & F & G \\ \dfrac{\partial E}{\partial r} & \dfrac{\partial F}{\partial r} & \dfrac{\partial G}{\partial r} \\ \dfrac{\partial^2 E}{\partial r^2} & \dfrac{\partial^2 F}{\partial r^2} & \dfrac{\partial^2 G}{\partial r^2} \end{vmatrix}$$

vanishes. When one recalls (1), one will have:

$$E = c_{11}^2 + c_{12}^2, \qquad F = c_{11}\,c_{21} + c_{12}\,c_{22}, \qquad G = c_{21}^2 + c_{22}^2,$$

$$\frac{\partial E}{\partial r} = 2c_{11}\frac{\partial c_{11}}{\partial r} + 2c_{12}\frac{\partial c_{12}}{\partial r}, \qquad \frac{\partial F}{\partial r} = 2\left(c_{21}\frac{\partial c_{11}}{\partial r} + c_{22}\frac{\partial c_{12}}{\partial r}\right) = 2\left(c_{11}\frac{\partial c_{11}}{\partial r} + c_{12}\frac{\partial c_{22}}{\partial r}\right),$$

$$\frac{\partial G}{\partial r} = 2\left(c_{21}\frac{\partial c_{21}}{\partial r} + c_{22}\frac{\partial c_{22}}{\partial r}\right).$$

As a result, one will have:

$$F\frac{\partial G}{\partial r} - G\frac{\partial F}{\partial r} = 2c\left(c_{22}\frac{\partial c_{21}}{\partial r} - c_{21}\frac{\partial c_{22}}{\partial r}\right),$$

$$G\frac{\partial E}{\partial r} - E\frac{\partial G}{\partial r} = 2c\left(c_{11}\frac{\partial c_{22}}{\partial r} + c_{21}\frac{\partial c_{12}}{\partial r} - c_{22}\frac{\partial c_{11}}{\partial r} - c_{12}\frac{\partial c_{21}}{\partial r}\right),$$

$$E\frac{\partial F}{\partial r} - F\frac{\partial E}{\partial r} = 2c\left(c_{12}\frac{\partial c_{11}}{\partial r} - c_{11}\frac{\partial c_{12}}{\partial r}\right),$$

in which one has set $c = c_{12}\,c_{21} - c_{11}\,c_{22}$.

If one replaces the second derivative $\dfrac{\partial^2 F}{\partial r^2}$ in J with:

$$2\left(c_{21}\frac{\partial^2 c_{22}}{\partial r^2} + c_{22}\frac{\partial^2 c_{12}}{\partial r^2} + \frac{\partial c_{12}}{\partial r}\frac{\partial c_{11}}{\partial r} + \frac{\partial c_{22}}{\partial r}\frac{\partial c_{12}}{\partial r}\right)$$

then all of the terms in J that include only first derivatives of the quantities $c_{\alpha\beta}$ will cancel, and it will become:

$$J = 4c\left\{c\left(-\frac{\partial^2 c_{11}}{\partial r^2}\frac{\partial c_{21}}{\partial r} - \frac{\partial^2 c_{11}}{\partial r^2}\frac{\partial c_{21}}{\partial r} + \frac{\partial^2 c_{21}}{\partial r^2}\frac{\partial c_{11}}{\partial r} + \frac{\partial^2 c_{22}}{\partial r^2}\frac{\partial c_{12}}{\partial r}\right)\right.$$
$$\left. + \left(c_{21}\frac{\partial c_{11}}{\partial r} - c_{22}\frac{\partial c_{11}}{\partial r}\right)\left(c_{21}\frac{\partial^2 c_{11}}{\partial r^2} + c_{22}\frac{\partial^2 c_{12}}{\partial r^2} - c_{11}\frac{\partial^2 c_{21}}{\partial r^2} - c_{12}\frac{\partial^2 c_{22}}{\partial r^2}\right)\right\}.$$

However, the right-hand side of that equation will vanish as a consequence of (1) and (3).

In order to achieve a second proof of the theorem that we speak of, we start from the remark that the first curvature of a curve of a cylic family is independent of r, and its second curvature will vanish. That is expressed as a result of equations (2) and (3) of § **8** in the form of:

(4) $$\frac{\vartheta}{P_1} = -g_0\left(\frac{1}{P_2}\right), \qquad \frac{\vartheta}{P_2} = g_0\left(\frac{1}{P_1}\right).$$

One now takes F to be the quantity $1 / P_2$ in the second equation of the system (9), § **7** and takes F to be the quantity $1 / P_1$ in the third, and considers ε to be zero. When one recalls (4), it will then follow that:

$$g_{10}\left(\frac{1}{P_2}\right) + \frac{1}{P_1}g_1(\vartheta) + \vartheta\, g_1\left(\frac{1}{P_1}\right) = \frac{1}{h_1}g_1\left(\frac{1}{P_2}\right) + \vartheta\left[g_2\left(\frac{1}{P_2}\right) + \frac{1}{P_1^2}\right],$$

$$g_{20}\left(\frac{1}{P_1}\right) - \frac{1}{P_2}g_2(\vartheta) - \vartheta\, g_2\left(\frac{1}{P_2}\right) = \frac{1}{h_2}g_2\left(\frac{1}{P_1}\right) - \vartheta\left[g_1\left(\frac{1}{P_1}\right) + \frac{1}{P_2^2}\right].$$

If one subtracts the second of those equations from the first then one will have:

(5) $$g_{10}\left(\frac{1}{P_2}\right) - g_{20}\left(\frac{1}{P_1}\right) + \frac{1}{P_1}g_1(\vartheta) + \frac{1}{P_2}g_2(\vartheta) - \frac{1}{h_1}g_1\left(\frac{1}{P_2}\right) + \frac{1}{h_2}g_2\left(\frac{1}{P_1}\right) - \vartheta\left(\frac{1}{P_1^2} + \frac{1}{P_2^2}\right) = 0.$$

In order to simplify this relation, we employ the system (11), § **7**. The fourth and ninth equations imply that:

$$g_{10}\left(\frac{1}{P_2}\right) - g_{20}\left(\frac{1}{P_1}\right) = \vartheta\left(\frac{1}{P_2 R_1} + \frac{1}{P_1 R_2}\right) - \frac{1}{P_1}g_0\left(\frac{1}{R_1}\right) + \frac{1}{P_2}g_0\left(\frac{1}{R_2}\right),$$

the seventh and eighth imply that:

$$-\frac{1}{P_1}\left[g_0\left(\frac{1}{R_1}\right) - g_1(\vartheta)\right] + \frac{1}{P_2}\left[g_0\left(\frac{1}{R_2}\right) + g_2(\vartheta)\right]$$

$$= -\frac{1}{h_1 P_1}\left(\frac{1}{R_1}-\frac{1}{P_2}\right)+\frac{1}{h_2 P_2}\left(\frac{1}{R_2}-\frac{1}{P_1}\right)+\vartheta\left(\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_1 R_2}+\frac{1}{P_2 R_1}\right),$$

and the fourth and ninth imply that:

$$-\frac{1}{h_1}g_1\left(\frac{1}{P_2}\right)+\frac{1}{h_2}g_2\left(\frac{1}{P_1}\right) = \frac{1}{h_1 P_1}\left(\frac{1}{R_1}-\frac{1}{P_2}\right)+\frac{1}{h_2 P_2}\left(\frac{1}{P_1}-\frac{1}{R_2}\right)+\vartheta\left(\frac{1}{h_1}-\frac{1}{h_2}\right)^2.$$

As a result of that, the relation (5) will be equivalent to the following one:

$$\vartheta\left(\frac{1}{h_1}-\frac{1}{h_2}\right)^2 = 0\;;$$

however: ϑ vanishes when ε is equal to zero.

We prove the further theorem: If a doubly-infinite family of circles with constant radius $1 / c$ consists of the orthogonal trajectories to a family of surfaces then the individual surfaces of the latter will possess constant Gaussian curvature $-c^2$.

Here, in addition to the assumption that $\varepsilon = \vartheta = 0$, one also has the assumption that:

$$c^2 = \frac{1}{P_1^2}+\frac{1}{P_2^2},$$

such that:

$$\frac{1}{P_1}g_0\left(\frac{1}{P_1}\right)+\frac{1}{P_2}g_0\left(\frac{1}{P_2}\right) = 0.$$

It then follows from the third and fourth equation in (11), § **7** that:

$$g_0\left(\frac{1}{h_1}\right) = c^2+\frac{1}{h_1^2},$$

and it follows from the seventh and last equation that:

$$g_0\left(\frac{1}{h_2}\right) = c^2+\frac{1}{h_2^2}.$$

Furthermore, from (11), § **7**, one has:

$$g_2\left(\frac{1}{h_1}\right) = \frac{1}{R_1}\left(\frac{1}{h_1}-\frac{1}{h_2}\right), \qquad g_1\left(\frac{1}{h_2}\right) = \frac{1}{R_2}\left(\frac{1}{h_2}-\frac{1}{h_1}\right),$$

$$g_0\left(\frac{1}{R_1}\right) = \frac{1}{h_1}\left(\frac{1}{R_1} - \frac{1}{P_2}\right), \qquad g_0\left(\frac{1}{R_2}\right) = \frac{1}{h_2}\left(\frac{1}{R_2} - \frac{1}{P_1}\right).$$

For $\mathfrak{F} = 1 / h_1$, the third equation in (9), § **7** will become:

$$g_{20}\left(\frac{1}{h_1}\right) - g_{02}\left(\frac{1}{h_1}\right) = \frac{1}{h_2}\,g_2\left(\frac{1}{h_1}\right) - \frac{1}{P_2}\,g_0\left(\frac{1}{h_1}\right).$$

However:

$$g_{20}\left(\frac{1}{h_1}\right) - g_{02}\left(\frac{1}{h_1}\right) = \left(\frac{1}{h_1} - \frac{1}{h_2}\right)\left(\frac{1}{R_1\,h_2} - \frac{1}{h_1 P_2}\right),$$

so:

$$\frac{1}{h_1\,h_2\,P_2} = -\,\frac{c^2}{P_2}.$$

If one forms the second equation in (9), § **7** for $\mathfrak{F} = 1 / h_2$ then one will recognize that one correspondingly has:

$$\frac{1}{h_1\,h_2\,P_1} = -\,\frac{c^2}{P_1}.$$

However, the two quantities $1 / P_1$ and $1 / P_2$ cannot vanish simultaneously without the circles degenerating into straight lines. We then get:

$$\frac{1}{h_1\,h_2} = -\,c^2,$$

as stated.

A ray system is given at the same time as a cyclic family of curves. One will obtain it when one lays a perpendicular to the plane of the circle through its center. The centers then define a surface with coordinates:

$$x_0 = x + \frac{(P_2\,\kappa_1 + P_1\,\kappa_2)P_1\,P_2}{P_1^{\,2} + P_2^{\,2}}, \quad y_0 = y + \frac{(P_2\,\lambda_1 + P_1\,\lambda_2)P_1\,P_2}{P_1^{\,2} + P_2^{\,2}}, \quad z_0 = z + \frac{(P_2\,\mu_1 + P_1\,\mu_2)P_1\,P_2}{P_1^{\,2} + P_2^{\,2}},$$

and the normals to the planes of the circles possess the direction cosines:

$$\xi' = \frac{P_2\,\kappa_2 - P_1\,\kappa_1}{\sqrt{P_1^{\,2} + P_2^{\,2}}}, \qquad \eta' = \frac{P_2\,\lambda_2 - P_1\,\lambda_1}{\sqrt{P_1^{\,2} + P_2^{\,2}}}, \qquad \zeta' = \frac{P_2\,\mu_2 - P_1\,\mu_1}{\sqrt{P_1^{\,2} + P_2^{\,2}}}.$$

We will come back to this in § **12**.

§ 10. – Families of orthogonal trajectories, referred to the lines of curvature of the first kind. Normal curvature. Geodetic curvature.

One will get a family of orthogonal trajectories of curves p = const., q = const. in such a way that one represents p, q, r by functions of one variable t and two parameters a, b that satisfy:

$$a_{13}\frac{\partial p}{\partial t}+a_{23}\frac{\partial q}{\partial t}+a_{33}\frac{\partial r}{\partial t} = 0.$$

Along such a trajectory, one will have, on the one hand:

$$dx = \frac{\partial x}{\partial t}dt = \left(x_p\frac{\partial p}{\partial t}+x_q\frac{\partial q}{\partial t}\right)dt,$$

and on the other hand:

$$dx = \kappa_1\,\mathfrak{S}_1 + \kappa_2\,\mathfrak{S}_2,$$

such that:

$$\mathfrak{S}_1 : \mathfrak{S}_2 = \sum\kappa_1\frac{\partial x}{\partial t}:\sum\kappa_1\frac{\partial x}{\partial t}.$$

If we set:

$$\frac{\sum\kappa_1\frac{\partial x}{\partial t}}{\sqrt{\sum\left(\frac{\partial x}{\partial t}\right)^2}} = \alpha_1, \qquad \frac{\sum\kappa_2\frac{\partial x}{\partial t}}{\sqrt{\sum\left(\frac{\partial x}{\partial t}\right)^2}} = \alpha_2$$

then α_1 and α_2 will be cosines of the angles that the tangent to the trajectory that goes through the point (p, q, r) will make with the tangents to the lines of curvature of the first kind.

Conversely, if one thinks of α_1 and α_2 as being functions of p, q, and r and seeks to determine the corresponding family of orthogonal trajectories then one should consider that:

$$x_p\frac{\partial p}{\partial t}+x_q\frac{\partial q}{\partial t} = \lambda\,(\kappa_1\,\alpha_1 + \kappa_2\,\alpha_2),$$

in which λ means a proportionality factor. It then follows from this that:

$$\sigma_1\frac{\partial p}{\partial t}+\sigma_2\frac{\partial q}{\partial t} = \alpha_1\,\lambda, \qquad \sigma_3\frac{\partial p}{\partial t}+\sigma_4\frac{\partial q}{\partial t} = \alpha_2\,\lambda;$$

i.e.:

$$\frac{\partial p}{\partial t} = \lambda\frac{\alpha_1\sigma_4-\alpha_2\sigma_2}{\sigma_1\sigma_4-\sigma_2\sigma_3}, \qquad \frac{\partial q}{\partial t} = \lambda\frac{-\sigma_3\alpha_1+\sigma_1\alpha_2}{\sigma_1\sigma_4-\sigma_2\sigma_3}.$$

One then gets the finite equations of the family of trajectories by integrating the simultaneous systems:

$$dp : dq : dr = \alpha_1\,\sigma_4 - \alpha_2\,\sigma_2 \ : - \sigma_3\,\alpha_1 + \sigma_1\,\alpha_2 : \frac{a_{13}(\alpha_2\,\sigma_3 - \alpha_1\,\sigma_4) + a_{23}(\sigma_3\,\alpha_1 - \sigma_1\,\alpha_2)}{a_{33}}.$$

Along with a family of orthogonal trajectories, one is also given the family of those orthogonal trajectories that intersect the first one perpendicularly.

If we set:

$$T_1 = \alpha_1\,\mathfrak{S}_1 + \alpha_2\,\mathfrak{S}_2, \qquad T_2 = -\alpha_2\,\mathfrak{S}_1 + \alpha_1\,\mathfrak{S}_2$$

then the first family will be established by the differential equation $T_2 = 0$, while the second one will be established by the differential equation $T_1 = 0$. Now, for an arbitrary function $\mathfrak{F}$ of p, q, r, one has the relations:

$$(d\mathfrak{F})_{T_1} = \alpha_1\,g_1(\mathfrak{F}) + \alpha_2\,g_2(\mathfrak{F}), \qquad (d\mathfrak{F})_{T_2} = -\alpha_2\,g_1(\mathfrak{F}) + \alpha_1\,g_2(\mathfrak{F}),$$

such that, in particular:

$$(d\xi)_{T_1} = -\kappa_1\left(\frac{\alpha_1}{h_1} + \varepsilon\,\alpha_2\right) + \kappa_2\left(\varepsilon\,\alpha_2 - \frac{\alpha_2}{h_2}\right),$$

$$(d\xi)_{T_2} = \kappa_1\left(\frac{\alpha_2}{h_1} - \varepsilon\,\alpha_1\right) - \kappa_2\left(\varepsilon\,\alpha_2 + \frac{\alpha_1}{h_2}\right).$$

For the normal curvatures of the families in question, we get:

$$\frac{1}{h_{T_1}} = -\sum (d\xi)_{T_1}(dx)_{T_1} = \frac{\alpha_1^2}{h_1} + \frac{\alpha_2^2}{h_2},$$

$$\frac{1}{h_{T_2}} = -\sum (d\xi)_{T_2}(dx)_{T_2} = \frac{\alpha_2^2}{h_1} + \frac{\alpha_1^2}{h_2},$$

$$\frac{1}{h_{T_1}} + \frac{1}{h_{T_2}} = \frac{1}{h_1} + \frac{1}{h_2}.$$

Those relations say the same thing for families of curves that **Euler**'s theorem on the radii of curvature of normal sections says in the theory of surfaces, and implies the theorem on the sum of the curvatures of two mutually-perpendicular normal sections.

The equation of a family of asymptotic lines will be:

$$\frac{\alpha_1^2}{h_1} + \frac{\alpha_2^2}{h_2} = 0.$$

The asymptotic lines then define two distinct families, in general. They will be imaginary when $1 / h_1\,h_2$ is positive. If we assume that $1 / h_1$ vanishes then, from the first equation in (11), § **7**, we will have:

$$g_1(\varepsilon) = \frac{1}{R_1 h_1} + \frac{2\varepsilon}{P_1}.$$

However, R_1 is the radius of the first curvature of the asymptotic lines, which coincide with the first family of lines of curvature of the first kind here. We will then be dealing with straight asymptotic lines when:

$$g_1(\varepsilon) = \frac{2\varepsilon}{P_1}.$$

Coincident asymptotic lines in a normal family will always be straight then.

If $1 / h_1 h_2$ is less than zero, so $1 / h_1$ is positive and $1 / h_2$ is negative, then the asymptotic lines will define two real families. The direction cosines of the tangents to the one are:

$$\frac{\kappa_1\sqrt{h_1} + \kappa_2\sqrt{-h_2}}{\sqrt{h_1 - h_2}}, \text{ etc.},$$

and for the other, they are:

$$\frac{\kappa_1\sqrt{h_1} - \kappa_2\sqrt{-h_2}}{\sqrt{h_1 - h_2}}, \text{ etc.}$$

It emerges from this that the lines of curvature of the first kind bisect the angles that are formed by the asymptotic lines. Those angles will be right angles when $h_1 + h_2 = 0$.

For the geodetic curvature of the families considered, we get:

$$-\frac{1}{R_{T_1}} = \sum (dx)_{T_1} (dx)_{T_2 T_1} = -\alpha_1 (d\alpha_2)_{T_1} + \alpha_2 (d\alpha_1)_{T_1} + \sum \kappa_1 (d\kappa_2)_{T_1}.$$

However:

$$\alpha_1 (d\alpha_2)_{T_1} - \alpha_2 (d\alpha_1)_{T_1} = g_1(\alpha_2) - g_2(\alpha_1)$$

and

$$\sum \kappa_1 (d\kappa_2)_{T_1} = -\frac{\alpha_1}{R_1} + \frac{\alpha_2}{R_2},$$

so:

$$\frac{1}{R_{T_1}} = g_1(\alpha_2) - g_2(\alpha_1) + \frac{\alpha_1}{R_1} - \frac{\alpha_2}{R_2},$$

and correspondingly:

$$\frac{1}{R_{T_2}} = -g_1(\alpha_1) - g_2(\alpha_2) + \frac{\alpha_1}{R_1} + \frac{\alpha_2}{R_2}.$$

The equation:

$$g_2(\alpha_1) - g_1(\alpha_2) - \frac{\alpha_1}{R_1} + \frac{\alpha_2}{R_2} = 0$$

is consequently the differential equation for the geodetic lines of the family of curves, while:

$$g_1(\alpha_1) + g_2(\alpha_2) - \frac{\alpha_1}{R_1} - \frac{\alpha_2}{R_2} = 0$$

is the differential equation for those orthogonal trajectories whose perpendicular penetration curves (*Durchdringungscurven*) consist of geodetic lines of the given family of curves, as long they are simultaneously penetration curves of that family.

We add some remarks about that. Should the curves $T_2 = 0$, $T_1 = 0$ be, at the same time, geodetic lines then we will have:

$$\frac{1}{R_1} = g_1\left(\arctan\frac{\alpha_1}{\alpha_2}\right), \qquad \frac{1}{R_2} = -g_2\left(\arctan\frac{\alpha_1}{\alpha_2}\right),$$

and therefore:

$$g_2\left(\frac{1}{R_1}\right) + g_1\left(\frac{1}{R_2}\right) = g_{12}\left(\arctan\frac{\alpha_1}{\alpha_2}\right) - g_{21}\left(\arctan\frac{\alpha_1}{\alpha_2}\right).$$

If we now consider the first equation in (9), § **7**, as well as the fifth one in (11), then it will follow that:

$$\frac{1}{\rho_1\,\rho_2} = 2\varepsilon\left[g_0\left(\arctan\frac{\alpha_1}{\alpha_2}\right) - \vartheta\right].$$

If ε vanishes then the family of curves must be special in order for it to possess two systems of mutually-perpendicular, orthogonal trajectories that consist of geodetic lines. If ε is non-zero then it will have the property in question only when the differential form:

$$\frac{\mathfrak{S}_1}{R_1} - \frac{\mathfrak{S}_2}{R_2} + \left(\frac{1}{2\rho_1\,\rho_2\,\varepsilon} + \vartheta\right)T_0$$

is a complete differential.

From § **8**, (1), for the family of principal normal lines, one has:

$$\alpha_1 = \frac{\rho}{P_1}, \qquad \alpha_2 = \frac{\rho}{P_2},$$

and from § **8**, (3), the second curvature of the curves p = const., q = const. is determined by the equation:

$$\frac{1}{\rho'} = -\vartheta + g_0\left(\arctan\frac{P_2}{P_1}\right).$$

When the principal normals, as well as the binormals, are geodetic lines, one will then have:

$$\frac{1}{\rho_1 \rho_2} - \frac{2\varepsilon}{\rho'} = 0.$$

We shall now consider the curvature of the curves $T_2 = 0$, $T_1 = 0$ relative to the normal surfaces (ξ, η, ζ). We get:

$$\frac{1}{l_{T_1}} = \sum (dx)_{T_2} (d\xi)_{T_1} = \alpha_1\, \alpha_2 \left(\frac{1}{h_1} - \frac{1}{h_2}\right) + \varepsilon,$$

$$\frac{1}{l_{T_2}} = \sum (dx)_{T_1} (d\xi)_{T_2} = \alpha_1\, \alpha_2 \left(\frac{1}{h_1} - \frac{1}{h_2}\right) - \varepsilon.$$

The curves $T_2 = 0$ will be lines of curvature of the second kind when:

$$\alpha_1\, \alpha_2 \left(\frac{1}{h_1} - \frac{1}{h_2}\right) + \varepsilon = 0$$

or

$$\alpha_1^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\varepsilon^2}{\left(\frac{1}{h_1} - \frac{1}{h_2}\right)^2}}.$$

If follows further from (16), § **4** that:

$$\alpha_1^2 = \frac{1}{2} \pm \frac{1}{2} \frac{\frac{1}{\rho_1} - \frac{1}{\rho_2}}{\frac{1}{h_1} - \frac{1}{h_2}}.$$

If the quantities $1 / \rho_1$ and $1 / \rho_2$ are real and distinct then the lines of curvature of the second kind will define two separate families. The direction cosines of the tangents to the one family are:

$$\frac{\kappa_1 \sqrt{\frac{1}{h_1} - \frac{1}{\rho_3}} + \kappa_2 \sqrt{\frac{1}{h_1} - \frac{1}{\rho_1}}}{\sqrt{\frac{1}{h_1} - \frac{1}{h_2}}}, \text{ etc.},$$

and those of the other are:

$$\frac{\kappa_1 \sqrt{\frac{1}{h_1} - \frac{1}{\rho_1}} + \kappa_2 \sqrt{\frac{1}{h_1} - \frac{1}{\rho_2}}}{\sqrt{\frac{1}{h_1} - \frac{1}{h_2}}}, \text{ etc.}$$

The roots that appear here in the numerators must be determined such that:

$$\sqrt{\frac{1}{h_1}-\frac{1}{\rho_1}}+\sqrt{\frac{1}{h_1}-\frac{1}{\rho_2}}+\varepsilon=0.$$

For the angle φ between the tangents in question, one has:

$$\cos\varphi=\frac{-2\varepsilon}{\frac{1}{h_1}-\frac{1}{h_2}},\qquad \sin\varphi=\frac{\frac{1}{\rho_1}-\frac{1}{\rho_2}}{\frac{1}{h_1}-\frac{1}{h_2}}.$$

The lines of curvature of the second kind possess the same angle bisectors as the lines of curvature of the first kind and will be mutually-perpendicular only when ε vanishes.

The curvature of the one family of asymptotic lines relative to the normal surfaces (ζ, η, ζ) will be:

$$\sqrt{-\frac{1}{h_1 h_2}}+\varepsilon,$$

and that of the other will be:

$$\varepsilon-\sqrt{-\frac{1}{h_1 h_2}}.$$

Since the curvature in question is equal to the second curvature of those lines for non-planar asymptotic lines, for $\varepsilon = 0$, one will have **Enneper**'s theorem, according to which the square of the second curvature of an asymptotic line on a surface is equal to the absolute value of the **Gaussian** curvature of the surface. (Göttinger Nachrichten from the year 1870, pp. 499)

We finally consider the family of curves that is adjoint to the family of curves $T_2 = 0$. We put the direction cosines of its tangents into the form:

$$\kappa_1\beta_1+\kappa_2\beta_2,\qquad \lambda_1\beta_1+\lambda_2\beta_2,\qquad \mu_1\beta_1+\mu_2\beta_2.$$

From § **6**, the equation:

$$\sum(\kappa_1\beta_1+\kappa_2\beta_2)(d\xi)_{T_1}=0$$

will serve to determine β_1, β_2, from which it will follow that:

$$\frac{\beta_1}{\beta_2}=\frac{\varepsilon\alpha_1-\frac{\alpha_2}{h_2}}{\frac{\alpha_1}{h_1}+\varepsilon\alpha_2}.$$

One now focuses on a regular point P of the family of curves. The last equation will then associate every tangent (α_1, α_2) that goes through P with a tangent (β_1, β_2) that goes through P and which one calls its *adjoint*. That association is projective. All of the tangents (α_1, α_2) will be associated with only a single tangent (β_1, β_2) when:

$$\varepsilon^2 + \frac{1}{h_1\,h_2} = \frac{1}{\rho_1\,\rho_2} = 0 ;$$

i.e., when the basic family of curves is special.

The elements of the projectivity that correspond to themselves will be determined by the equation:

$$\frac{\alpha_1^2}{h_1} + \frac{\alpha_2^2}{h_2} = 0 ;$$

i.e., they will coincide with the tangents to the asymptotic lines that go through P.

The projectivity will be an involution when ε vanishes. The concept of "adjoint tangent" will then be equivalent to the concept of "conjugate tangent" in surface theory.

We would like to call the tangent that is adjoint to the tangent (β_1, β_2) the *second adjoint* of the tangent (α_1, α_2), the one that is adjoint to the second tangent will be the *third adjoint* to the tangent (α_1, α_2), etc. The condition for the μ^{th} adjoint to coincide with the tangent (α_1, α_2) is that:

$$\rho_1\,\rho_2\,\varepsilon^2 = \cos^2\frac{\kappa}{\mu} .$$

(Cf., **Serret**, *Handbuch der höheren Algebra*. Second edition, Bd. 2, pp. 203)

A. Voss began the consideration of the projectivity that was spoken of in his papers on families of curves. (Math. Ann., Bd. 16, pp. 556 and Bd. 23, pp. 45)

We conclude this paragraph with the easily-proved equations:

$$\frac{1}{P_{T_1}} = \sum (d\xi)_{T_0}\,(dx)_{T_1} = \frac{\alpha_1}{P_1} + \frac{\alpha_2}{P_2} ,$$

$$\frac{1}{P_{T_2}} = \sum (d\xi)_{T_0}\,(dx)_{T_2} = \frac{\alpha_2}{P_1} - \frac{\alpha_1}{P_2} ,$$

$$\frac{1}{L_{T_1}} = \sum (dx)_{T_0}\,(dx)_{T_1 T_0} = -\,\alpha_2\,g_0\,(\alpha_1) + \alpha_1\,g_0\,(\alpha_2) + \vartheta,$$

$$\frac{1}{L_{T_2}} = -\frac{1}{L_{T_1}} .$$

§ 11. – Family of curves that relates to a second one. Locus of centers of geodetic curvature of the lines of curvature of a ray system.

We say that a family of curves (C') is *related* to a family of curves (C) when every point (x, y, z) of the latter is associated with a point (x', y', z') of the former, and every tangent (ξ, η, ζ) to the latter is associated with a tangent (ξ', η', ζ') to the former. That can be expressed by the equations:

$$x' = x + a_1\,\kappa_1 + a_2\,\kappa_2 + a_0\,\xi\,,$$

$$y' = y + a_1\,\lambda_1 + a_2\,\lambda_2 + a_0\,\eta\,,$$

$$z' = z + a_1\,\mu_1 + a_2\,\mu_2 + a_0\,\zeta\,,$$

$$\xi' = \xi + \alpha_1\kappa_1 + \alpha_2\kappa_2 + \alpha_0\xi\,,\quad \eta' = \eta + \alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_0\eta\,,\quad \zeta' = \zeta + \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_0\zeta\,.$$

Here, a_1, a_2, a_0 are arbitrarily-chosen functions of p, q, r, while α_1, α_2, α_0 are ones that satisfy the relation:

$$\alpha_1^2 + \alpha_2^2 + \alpha_0^2 = 1.$$

We shall characterize the quantities κ_1, κ_2, R_1, R_2, etc., that pertain to the family of curves (C') by a prime. The problem of calculating those quantities with the help of the corresponding quantities that are true for the family (C) can be solved completely on the grounds of the developments that were given in §§ **6** and **7**. It represents the analogue of the problem in the theory of families of curves that **Ribaucour** called "geometry around a reference surface" in the theory of surfaces. [J. de Math. **7** (1891)].

We would like to explain the procedure, which is generally quite complicated, with two examples, and we shall choose the first one to be the case in which the family (C') consists of orthogonal trajectories to the family (C). Here, we must set $a_1 = a_2 = a_0 = \alpha_0 = 0$, such that when we preserve the notations of the previous paragraph:

$$\xi' = (dx)_{T_1}\,,\qquad \eta' = (dy)_{T_1}\,,\qquad \zeta' = (dz)_{T_1}\,.$$

Since:

$$\sum (dx)_{T_1}\,\kappa_1' = \sum (dx)_{T_1}\,\kappa_2' = 0,$$

here, we will have:

$$\kappa_1' = \xi \cos\beta + (dx)_{T_2}\sin\beta,\qquad \kappa_2' = -\,\xi\sin\beta + (dx)_{T_2}\cos\beta.$$

The following two representations are true for the differential dx :

$$(dx)_{T_1}\,T_1 + (dx)_{T_2}\,T_2 + x\,T_0 = \kappa_1'\,\mathfrak{S}_1' + \kappa_2'\,\mathfrak{S}_2' + (dx)_{T_1}\,T_0'\,.$$

One will then get:

$$\mathfrak{S}_1' = \sin\beta\, T_2 + \cos\beta\, T_0\,, \qquad \mathfrak{S}_1' = \cos\beta\, T_2 - \sin\beta\, T_0\,, \qquad T_0' = T_1\,.$$

For an arbitrary function $\mathfrak{F}$, that will then yield:

$$(d\mathfrak{F})_{T_1} = g_0'(\mathfrak{F})\,, \qquad (d\mathfrak{F})_{T_2} = g_1'(\mathfrak{F})\sin\beta + g_2'(\mathfrak{F})\cos\beta\,,$$
$$(d\mathfrak{F})_{T_0} = g_1'(\mathfrak{F})\cos\beta - g_2'(\mathfrak{F})\sin\beta\,.$$

If we apply that to $\xi' = (dx)_{T_1}$ then it will follow that:

$$\frac{(dx)_{T_2}}{R_{T_1}} + \frac{\xi}{h_{T_1}} = \frac{\kappa_1'}{P_1'} + \frac{\kappa_2'}{P_2'}\,,$$

$$\frac{(dx)_{T_2}}{R_{T_2}} + \frac{\xi}{l_{T_2}} = \left(\frac{\kappa_1'}{h_1'} - \varepsilon'\,\kappa_2'\right)\sin\beta + \left(\varepsilon'\,\kappa_1' + \frac{\kappa_2'}{h_2'}\right)\cos\beta\,,$$

$$\frac{(dx)_{T_2}}{L_{T_2}} - \frac{\xi}{P_{T_1}} = \left(-\frac{\kappa_1'}{h_1'} + \varepsilon'\,\kappa_2'\right)\cos\beta + \left(\varepsilon'\,\kappa_1' + \frac{\kappa_2'}{h_2'}\right)\sin\beta\,,$$

so:

$$\frac{1}{P_1'} = \frac{\sin\beta}{R_{T_1}} + \frac{\cos\beta}{h_{T_1}}, \quad \frac{1}{P_2'} = \frac{\cos\beta}{R_{T_1}} - \frac{\sin\beta}{h_{T_1}}\,,$$

$$\frac{\sin\beta}{h_1'} + \varepsilon'\cos\beta = \frac{\sin\beta}{R_{T_2}} + \frac{\cos\beta}{h_{T_2}}, \qquad \frac{\cos\beta}{h_2'} - \varepsilon'\sin\beta = \frac{\cos\beta}{R_{T_2}} - \frac{\sin\beta}{l_{T_2}}\,,$$

$$-\frac{\cos\beta}{h_1'} + \varepsilon'\sin\beta = \frac{\sin\beta}{L_{T_1}} - \frac{\cos\beta}{P_{T_1}}, \qquad \frac{\sin\beta}{h_2'} + \varepsilon'\cos\beta = \frac{\cos\beta}{L_{T_1}} + \frac{\sin\beta}{P_{T_1}}\,.$$

Those equations determine the six quantities P_1', P_2', h_1', h_2', ε', and β. We emphasize the special relations:

$$2\varepsilon' = \frac{1}{l_{T_2}} + \frac{1}{L_{T_1}}, \qquad \sin 2\beta\left(\frac{1}{R_{T_2}} - \frac{1}{P_{T_1}}\right) + \cos 2\beta\left(\frac{1}{l_{T_2}} - \frac{1}{L_{T_1}}\right) = 0.$$

The family $T_2 = 0$ is then a normal family when $\frac{1}{l_{T_2}} + \frac{1}{L_{T_1}} = 0$. Its lines of curvature of the first kind will be the curves p = const., q = const., and the curves $T_1 = 0$ when:

$$\frac{1}{l_{T_2}} = \frac{1}{L_{T_1}}\,.$$

If one considers the orthogonal family $T_1 = 0$, instead of the family $T_2 = 0$, then one can switch T_1 with T_2 and T_2 with T_1 everywhere, such that:

$$2\varepsilon' = \frac{1}{l_{T_1}} + \frac{1}{L_{T_2}}.$$

In the event that both families $T_1 = 0$ and $T_2 = 0$ are normal families, one will have, as a consequence of (4), § **6** :

$$\frac{1}{l_{T_1}} + \frac{1}{L_{T_2}} = 0.$$

From (7), § **6**, that fact shows that one is dealing with lines of curvature of the first kind. Conversely, the two families of lines of curvature of the first kind are or are not always simultaneously normal families, since for the one family one has:

$$2\varepsilon' = -\varepsilon + \vartheta,$$

and for the other:

$$2\varepsilon' = \varepsilon - \vartheta.$$

If the curves $T_2 = 0$ define a normal family then one can easily convince oneself that there is an integrating factor for the differential form T_1. Namely, from (6), § **7**, one has $2\varepsilon' = -c_{31}$, and from (4), § **7**, the vanishing of c_{31} emerges from the equation:

$$\alpha_{12}\frac{\partial\alpha_{11}}{\partial r} - \alpha_{11}\frac{\partial\alpha_{12}}{\partial r} = 0.$$

However, that represents the condition for the existence of an integrating factor for the differential form:

$$T_1 = \alpha_{11}\,dp + \alpha_{12}\,dq\,.$$

It still remains for us to determine the quantities R_1', R_2', and ϑ'. We have:

$$\frac{1}{R_1'} = \sum \kappa_2'\, g_1'(\kappa_1'), \qquad \frac{1}{R_2'} = \sum \kappa_1'\, g_2'(\kappa_2'), \qquad \vartheta' = \sum \kappa_2'\, g_0'(\kappa_1').$$

Now:

$$\begin{aligned} g_1'(\mathfrak{F}) &= \cos\beta\,(d\mathfrak{F})_{T_0} + \sin\beta\,(d\mathfrak{F})_{T_2}, \\ g_2'(\mathfrak{F}) &= -\sin\beta\,(d\mathfrak{F})_{T_0} + \cos\beta\,(d\mathfrak{F})_{T_2}, \\ g_0'(\mathfrak{F}) &= (d\mathfrak{F})_{T_1}, \end{aligned}$$

which then implies that:

$$\frac{1}{R_1'} = \cos\beta\left[\frac{1}{P_{T_0}} + (d\beta)_{T_0}\right] - \sin\beta\left[\frac{1}{h_{T_2}} - (d\beta)_{T_2}\right],$$

$$\frac{1}{R_2'} = \sin\beta\left[\frac{1}{P_{T_0}} + (d\beta)_{T_0}\right] + \cos\beta\left[\frac{1}{h_{T_2}} - (d\beta)_{T_2}\right],$$

$$\vartheta' = \frac{1}{l_{T_1}} + (d\beta)_{T_1}.$$

Let the following be remarked in regard to the condition above that:

$$\alpha_{12}\, g_0\,(\alpha_{11}) - \alpha_{11}\, g_0\,(\alpha_{12}) = 0.$$

Since:

$$T_1 = \alpha_1\, \mathfrak{S}_1 + \alpha_2\, \mathfrak{S}_2\,,$$

one will have:

$$\alpha_{11} = \alpha_1\, \sigma_1 + \alpha_2\, \sigma_3\,, \qquad \alpha_{12} = \alpha_1\, \sigma_2 + \alpha_2\, \sigma_4\,,$$

and as a result of equations (8), § **7**, the condition in question will go to:

$$g_0\left(\arctan\frac{\alpha_1}{\alpha_2}\right) + \varepsilon - \vartheta - \alpha_1\,\alpha_2\left(\frac{1}{h_1} - \frac{1}{h_2}\right) = 0.$$

One next assumes that the family of curves consists of the normals to a family of parallel surfaces. ε, $1 / P_1$, and $1 / P_2$ will then vanish, but also ϑ, since one family of parallel surfaces always belongs to a triply-orthogonal system of surfaces, which will also emerge directly from the last equation in (11), § **7** directly.

Moreover, if:

$$\alpha_1 = \frac{1}{R_1\sqrt{\dfrac{1}{R_1^2} + \dfrac{1}{R_2^2}}}, \qquad \alpha_2 = \frac{-1}{R_2\sqrt{\dfrac{1}{R_1^2} + \dfrac{1}{R_2^2}}}$$

then the tangents to the curves $T_2 = 0$ will be parallel to the connecting lines of the centers of geodetic curvature of the lines of curvature.

The seventh and eighth equations in (11), § **7** possess the following forms here:

$$g_0\left(\frac{1}{R_1}\right) = \frac{1}{h_1\, R_1}, \qquad g_0\left(\frac{1}{R_2}\right) = \frac{1}{R_2\, h_2},$$

and the quantity ε' vanishes. As a result, we obtain the theorem:

If one lays a line through any point of a surface that is parallel to the connecting line from the point to the center of geodetic curvature of the lines of curvature that belong to the point and one performs the same construction at all points of the surfaces that are parallel to the surface considered then the lines that one speaks of will define the tangents to a normal family of curves.

As a second example, we consider the surface of centers of curvature of a family of surfaces. We must then take ε equal to zero from the outset, and we will have:

$$x' = x + h_1\,\xi, \quad y' = y + h_1\,\eta, \quad z' = z + h_1\,\zeta,$$

$$\xi' = \kappa_1\,, \qquad \eta' = \lambda_1\,, \qquad \zeta' = \mu_1$$

for a family of center surfaces, moreover.

One gets the two expressions for dx':

$$\xi\,g_1(h_1)\,\mathfrak{S}_1 + \left[\xi\,g_2(h_1) + \left(1 - \frac{h_1}{h_2}\right)\kappa_2\right]\mathfrak{S}_1 + \left[\xi\,(1 + g_0(h_1)) + \frac{h_1\,\kappa_1}{P_1} + \frac{h_1\,\kappa_2}{P_2}\right]T_0$$

$$= \kappa_1'\,\mathfrak{S}_1' + \kappa_2'\,\mathfrak{S}_2' + \kappa_1\,T_0'\,.$$

Now, let:

$$\kappa_1' = \xi\cos\psi + \kappa_2\sin\psi, \qquad \kappa_2' = \xi\sin\psi - \kappa_2\cos\psi\,.$$

It will then follow that:

$$\frac{h_1}{P_1}T_0 = T_0'\,,$$

$$\left(1 - \frac{h_1}{h_2}\right)\mathfrak{S}_2 + \frac{h_1}{P_2}T_0 = \sin\psi\,\mathfrak{S}_1' - \cos\psi\,\mathfrak{S}_2'\,,$$

$$g_1(h_1)\,\mathfrak{S}_1 + g_2(h_1)\,\mathfrak{S}_2 + [1 + g_0(h_1)]\,T_0 = \cos\psi\,\mathfrak{S}_1' + \sin\psi\,\mathfrak{S}_2'\,.$$

The first of these equations shows that the family of curves considered is a normal family, since if μ is an integrating factor of T_0 then $\mu\,P_1 / h_1$ will be one for T_0'. In the event that $1 / P_1$ and $1 / P_2$ vanish, from the third equation in (11), § **7**, one will have:

$$1 + g_0(h_1) = 0,$$

and therefore:

$$g_0(x') = 0, \qquad g_0(y') = 0, \qquad g_0(z') = 0\,.$$

When expressed in words, that means that only one center of curvature surface belongs to a family of parallel surfaces.

When one considers the first and third equation of the system (11), § **7**, one finds that:

$$T_0 = \frac{P_1}{h_1} T_0',$$

$$\mathfrak{S}_2 = \frac{h_2}{h_2 - h_1} \left\{ \sin\psi\, \mathfrak{S}_1' - \cos\psi\, \mathfrak{S}_2' - \frac{P_1}{P_2} T_0' \right\},$$

$$\mathfrak{S}_1 = \frac{1}{g_1(h_1)} \left\{ (\cos\psi + \frac{h_1}{R_2}\sin\psi)\, \mathfrak{S}_1' + (\sin\psi - \frac{h_1}{R_1}\cos\psi)\, \mathfrak{S}_2' + h_1 P_1 \left[\frac{1}{P_1^{\,2}} - g_1\left(\frac{1}{P_2}\right) \right] T_0' \right\},$$

and it follows from this that for an arbitrary function $\mathfrak{F}$ of p, q, r :

$$g_1'(\mathfrak{F}) = \frac{g_1(\mathfrak{F})}{g_1(h_1)} \left(\cos\psi + \frac{h_1}{R_1}\sin\psi \right) + \frac{g_2(\mathfrak{F})\, h_2}{h_2 - h_1} \sin\psi,$$

$$g_2'(\mathfrak{F}) = \frac{g_1(\mathfrak{F})}{g_1(h_1)} \left(\sin\psi - \frac{h_1}{R_1}\cos\psi \right) - \frac{g_2(\mathfrak{F})\, h_2}{h_2 - h_1} \cos\psi,$$

$$g_0'(\mathfrak{F}) = \frac{g_1(\mathfrak{F})\, h_1 P_1}{g_1(h_1)} \left[\frac{1}{P_1^{\,2}} - g_1\left(\frac{1}{P_1}\right) \right] - \frac{g_2(\mathfrak{F})\, h_2 P_1}{(h_2 - h_1) P_2} + \frac{g_0(\mathfrak{F})\, P_1}{h_1}.$$

The quantities that come into question for the family of curves considered can be easily calculated with the help of those formulas.

The angle ψ will be given by the vanishing of ε' or $\sum \kappa_2'\, g_1'(\kappa_1)$; i.e., by the equation:

$$\frac{h_1}{g_1(h_1)} \cdot \left(\frac{\sin\psi}{h_1} - \frac{\cos\psi}{R_1} \right) \left(\frac{\sin\psi}{R_1} + \frac{\cos\psi}{h_1} \right) + \frac{h_2 \sin\psi \cos\psi}{R_2 (h_2 - h_1)} = 0.$$

It further follows that:

$$\frac{1}{h_1'} = \frac{h_2 \sin^2\psi}{R_2 (h_2 - h_1)} - \frac{h_1}{g_1(h_1)} \left(\frac{\sin\psi}{R_1} + \frac{\cos\psi}{h_1} \right)^2,$$

$$\frac{1}{h_2'} = \frac{h_2 \cos^2\psi}{R_2 (h_2 - h_1)} - \frac{h_1}{g_1(h_1)} \left(\frac{\cos\psi}{R_1} - \frac{\sin\psi}{h_1} \right)^2,$$

$$\frac{1}{R_1'} = \frac{\sin\psi}{h_2 - h_1} - g_1'(\psi),$$

$$\frac{1}{R_2'} = \frac{\cos\psi}{h_2 - h_1} + g_2'(\psi),$$

$$\frac{1}{P_1'} = \frac{h_1 P_1}{g_1(h_1)}\left[\frac{1}{P_1^2} - g_1\left(\frac{1}{P_1}\right)\right]\left(\frac{\sin\psi}{R_1} + \frac{\cos\psi}{h_1}\right) + \frac{\sin\psi \cdot h_2 P_1}{R_2(h_2 - h_1)P_2} + \left(\vartheta\sin\psi - \frac{\cos\psi}{P_1}\right)\frac{P_1}{h_1},$$

$$\frac{1}{P_2'} = \frac{h_1 P_1}{g_1(h_1)}\left[\frac{1}{P_1^2} - g_1\left(\frac{1}{P_1}\right)\right]\left(\frac{\sin\psi}{h_1} - \frac{\cos\psi}{R_1}\right) - \frac{\cos\psi \cdot h_2 P_1}{R_2 P_2(h_2 - h_1)} - \left(\frac{\sin\psi}{P_1} + \vartheta\cos\psi\right)\frac{P_1}{h_1},$$

$$\vartheta' = \frac{h_2 P_1}{h_1 P_2(h_2 - h_1)} - g_0'(\psi).$$

The second family of center of curvature surfaces, which are represented by the equations:

$$x'' = x + h_2\,\xi, \qquad y'' = y + h_2\,\eta, \qquad z'' = z + h_2\,\zeta,$$

$$\xi'' = \kappa_2, \qquad \lambda'' = \lambda_2, \qquad \zeta'' = \mu_2$$

can be treated in the same way.

One can correspondingly examine the families of curves that are determined by the centers of geodetic curvature of the lines of curvature of a given family of curves. For ray systems, those families will either be likewise ray systems or they will consist of hyperbolas. The proof of that assertion might define the conclusion of this section.

We saw in § **8** that the quantities $1 / P_1$ and $1 / P_2$ vanish for a ray system. As a result, from (10), § **7**, we have:

$$d\xi = \kappa_1\left(-\frac{\mathfrak{S}_1}{h_1} - \varepsilon\,\mathfrak{S}_2\right) + \kappa_2\left(\varepsilon\,\mathfrak{S}_1 - \frac{\mathfrak{S}_2}{h_2}\right).$$

If we take:

$$H_1 = -\frac{\mathfrak{S}_1}{h_1} - \varepsilon\,\mathfrak{S}_2, \qquad H_2 = \varepsilon\,\mathfrak{S}_1 - \frac{\mathfrak{S}_2}{h_2}$$

then H_1 and H_2 will be linear differential forms in dp and dq whose coefficients depend upon only p and q. For an arbitrary function $\mathfrak{F}$, one gets:

$$g_1(\mathfrak{F}) = -\frac{1}{h_1}(d\mathfrak{F})_{H_1} + \varepsilon\,(d\mathfrak{F})_{H_2}, \qquad g_2(\mathfrak{F}) = -\,\varepsilon\,(d\mathfrak{F})_{H_1} - \frac{1}{h_2}(d\mathfrak{F})_{H_2}.$$

We then get:

$$\frac{1}{R_1} = \sum \kappa_2\,g_1(\kappa_1) = -\frac{1}{h_1}\sum \kappa_2 (d\kappa_1)_{H_1} + \varepsilon \sum \kappa_2 (d\kappa_1)_{H_2},$$

$$\frac{1}{R_2} = \sum \kappa_1\, g_2(\kappa_2) = -\,\varepsilon \sum \kappa_1 (d\kappa_2)_{H_1} - \frac{1}{h_2}\sum \kappa_1 (d\kappa_2)_{H_2}\,.$$

The differential equations:

$$H_1 = 0, \qquad H_2 = 0$$

determine two families of curves on the unit sphere (ξ, η, ζ) whose tangents are parallel to the tangents to the lines of curvature of the first kind of the ray system. If we denote their geodetic curvatures by $1 / K_1$ and $1 / K_1$ then, from § **2**, we will have:

$$\frac{1}{K_1} = -\sum \kappa_1 (d\kappa_2)_{H_1}\,, \qquad \frac{1}{K_2} = -\sum \kappa_1 (d\kappa_1)_{H_2}\,.$$

As a result, we will get:

$$\frac{1}{R_1} = -\frac{1}{h_1 K_1} - \frac{\varepsilon}{K_2}\,, \qquad \frac{1}{R_2} = \frac{\varepsilon}{K_1} - \frac{1}{h_2 K_2}\,.$$

In order to find the geometric locus of the centers of geodetic curvature of the lines of curvature of the first kind along a ray, we must address the type of dependency of the quantities R_1 and R_2 on r in the event that the ray system is represented by the equations:

$$x = x_0 + r\,\xi, \quad y = y_0 + r\,\eta, \quad z = z_0 + r\,\zeta,$$

in which x_0, y_0, z_0 are functions of p and q alone.

In regard to the quantities h_1 and h_2, we find from (12) and (14), § **4** that:

$$\frac{1}{h_1} = \frac{\mathfrak{r}_2}{\rho_1\,\rho_2}\,, \qquad \frac{1}{h_2} = \frac{\mathfrak{r}_1}{\rho_1\,\rho_2}\,.$$

The values of $\mathfrak{r}_1$, $\mathfrak{r}_2$, ρ_1, ρ_2 for $r = 0$ shall be denoted by $\mathfrak{r}_{10}$, $\mathfrak{r}_{20}$, ρ_{10}, ρ_{20}, resp. We will then have:

$$\mathfrak{r}_1 = \mathfrak{r}_{10} - r, \qquad \mathfrak{r}_2 = \mathfrak{r}_{20} - r,$$

$$\rho_1 = \rho_{10} - r, \qquad \rho_2 = \rho_{20} - r,$$

$$\frac{1}{h_1} = \frac{\mathfrak{r}_{20} - r}{(\rho_{10} - r)(\rho_{20} - r)}\,, \qquad \frac{1}{h_2} = \frac{\mathfrak{r}_{10} - r}{(\rho_{10} - r)(\rho_{20} - r)}\,.$$

The quantity ε was defined by the equation (see pp. 34, line 5 from bottom):

$$\varepsilon^2 = \frac{(f - f')^2}{4(EG - F^2)}\,.$$

It follows from the formulas on (pp. 39, line 4 from bottom) that r means the same thing here as l did in *loc. cit.*, namely:

$$f - f' = f_0 - f_0',$$

$$EG - F^2 = (H\Psi - \Phi^2)\,(\rho_{10} - r)^2\,(\rho_{20} - r)^2.$$

If we then set:

$$\varepsilon' = \frac{f_0 - f_0'}{2\sqrt{H\Psi - \Phi^2}}$$

then ε will be independent of r, and it will follow that:

$$\varepsilon = \frac{\varepsilon'}{(\rho_{10} - r)(\rho_{20} - r)}.$$

In the case considered, one has:

$$a_{33} = 1, \qquad g_0\,(\mathfrak{F}) = \frac{\partial \mathfrak{F}}{\partial r},$$

and expressions that were exhibited for $1 / h_1$, $1 / h_2$, and ε will satisfy the third, sixth, and last differential equation, resp., of the system (11), § **7**.

We now get:

$$\frac{1}{R_1} = \frac{\dfrac{r - \mathfrak{r}_{20}}{K_1} - \dfrac{\varepsilon'}{K_2}}{(\rho_{10} - r)(\rho_{20} - r)}, \qquad \frac{1}{R_2} = \frac{\dfrac{\varepsilon'}{K_1} + \dfrac{r - \mathfrak{r}_{10}}{K_2}}{(\rho_{10} - r)(\rho_{20} - r)}.$$

Along one and the same ray, the centers of geodetic curvature for the lines of curvature of the first kind will generally lie along two hyperbolas then. Their equations will then possess the common discriminant:

$$\frac{\varepsilon'}{4}\left[\frac{\mathfrak{r}_{10} - \mathfrak{r}_{20}}{K_1\,K_2} - \varepsilon'\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right)\right].$$

The hyperbolas will then be lines when $\varepsilon' = 0$; i.e., when the ray system is a normal system or when:

$$1 = \frac{\varepsilon'}{\mathfrak{r}_{10} - \mathfrak{r}_{20}}\left(\frac{K_1}{K_2} + \frac{K_2}{K_1}\right).$$

That relation can be easily put into a geometrically intuitive form. In § **10**, we found that the angle φ between the two focal planes had the equation:

$$\cos\varphi = \frac{-2\varepsilon}{\dfrac{1}{h_1} - \dfrac{1}{h_2}}.$$

When we employ the expressions that were given above for $1 / h_1$, $1 / h_2$, and ε, it will take the form:

$$\cos \varphi = \frac{2\varepsilon'}{\mathfrak{r}_{10} - \mathfrak{r}_{20}}.$$

If we take:

$$\frac{K_1}{K_2} = \tan \psi$$

then ψ will mean the angle that the connecting line of the centers of geodetic curvature of the spherical curves $H_1 = 0$ and $H_2 = 0$ makes with the spherical tangent (κ_1, λ_1, μ_1), and the relation in question will be equivalent to the following one:

$$\cos \varphi = \sin 2\psi.$$

In order to find the locus of the centers of geodetic curvature of the lines of curvature of the second kind along a ray, one notes that equation (11), § **5** must be written in the form:

$$\delta x_0 = - \rho_{10}\ \kappa_3\ S_1 - \rho_{20}\ \kappa_4\ S_2$$

here, such that:

$$\delta x = \delta x_0 + r\ d\xi = (r - \rho_{10})\ \kappa_3\ S_1 + (r - \rho_{20})\ \kappa_4\ S_2\,.$$

If one then takes:

$$T_1 = (r - \rho_{10})\ S_1\,, \qquad T_2 = (r - \rho_{20})\ S_2$$

then:

$$(d\mathfrak{F})_{T_1} = \frac{(d\mathfrak{F})_{S_1}}{r - \rho_{10}}, \qquad (d\mathfrak{F})_{T_2} = \frac{(d\mathfrak{F})_{S_2}}{r - \rho_{20}}$$

will be the derivatives of the function $\mathfrak{F}$ with respect to arc-length of the lines of curvature of the second kind.

We denote the geodetic curvatures of the curves $T_2 = 0$, $T_1 = 0$, $T_2' = 0$, $T_1' = 0$, by $1/R_2$, $1 / R_4$, $1/R_2'$, $1/R_4'$, respectively, and likewise those of the spherical curves $S_2 = 0$, $S_1 = 0$, $S_2' = 0$, $S_1' = 0$ will be denoted by $1 / K_2$, $1 / K_4$, $1/K_2'$, $1/K_4'$, resp. The equation that was found in § **10**:

$$\sin \varphi = \frac{\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}}{\dfrac{1}{h_1} - \dfrac{1}{h_2}},$$

in conjunction with the one that is true for $\cos \varphi$ above, will imply that:

$$\cot \varphi = - \frac{2\varepsilon}{\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}} = \frac{2\varepsilon'}{\rho_{10} - \rho_{20}},$$

and will yield the following expressions for A and B:

$$A = \frac{1}{K_3'} - \frac{2\varepsilon'}{(\rho_{10} - \rho_{20})K_3}, \qquad B = \frac{1}{K_4'} - \frac{2\varepsilon'}{(\rho_{10} - \rho_{20})K_4}.$$

One now gets:

$$\frac{1}{R_3} = \sum \kappa_3' (d\kappa_3)_{T_1} = \frac{1}{(r - \rho_{10})K_3},$$

$$\frac{1}{R_4} = \sum \kappa_4' (d\kappa_4)_{T_2} = \frac{1}{(r - \rho_{20})K_4}.$$

The locus of centers of geodetic curvature of the lines of curvature of the second kind along a ray will then be defined by two lines that go through the focal points.

It further arises that:

$$\frac{1}{R_3'} = -\sum \kappa_3' (d\kappa_3)_{T_1'} = \frac{\cos\varphi \sum \kappa_3'(d\kappa_3)_{T_1} - \sum \kappa_2'(d\kappa_3)_{T_2}}{\sin\varphi} = \frac{\cot\varphi}{(r - \rho_{10})K_3} + \frac{A}{r - \rho_{20}},$$

$$\frac{1}{R_4'} = -\sum \kappa_4' (d\kappa_4)_{T_2'} = \frac{-\sum \kappa_4'(d\kappa_4)_{T_1} + \cos\varphi \sum \kappa_4'(d\kappa_4)_{T_2}}{\sin\varphi} = \frac{B}{r - \rho_{10}} + \frac{\cot\varphi}{(r - \rho_{20})K_4}.$$

Along a ray, the locus of centers of geodetic curvature of those orthogonal trajectories of the ray system that are, at the same time, perpendicular penetrating curves of the lines of curvature of the second kind will then consist of two hyperbolas.

The first of them, which corresponds to the curves $T_2' = 0$, will degenerate into a line for $A = 0$, while the second one will degenerate into a line for $B = 0$.

If one gives the latter equations the form:

$$\frac{A}{r - \rho_{20}} = \frac{1}{R_3'} - \frac{\cot\varphi}{R_3}, \qquad \frac{B}{r - \rho_{10}} = \frac{1}{R_4'} - \frac{\cot\varphi}{R_4}$$

then that will show that for $A = B = 0$, the lines of curvature of the second kind, like their spherical images (§ **5**), will possess the property that the tangents to the curves $T_2 = 0$ or $T_1 = 0$ are perpendicular to the connecting lines of the centers of geodetic curvature of the curves $T_1 = 0$ and $T_1' = 0$ or $T_2 = 0$ and $T_2' = 0$.

§ 12. – Transformations that relate to a family of curves.

Just as we related a family of curves to a second one in the previous paragraphs, we can also relate a single function $\mathfrak{F}$ of p, q, r or x, y, z, and a system of such functions to a family of curves. The problem then arises of expressing the derivatives of $\mathfrak{F}$ with respect to p, q, r or x, y, z in terms of the invariant operations $g_1(\mathfrak{F})$, $g_2(\mathfrak{F})$, $g_0(\mathfrak{F})$.

One gets the first of those transformations in the following way:

One has:

$$d\,\mathfrak{F} = g_1(\mathfrak{F})\,\mathfrak{S}_1 + g_2(\mathfrak{F})\,\mathfrak{S}_2 + g_0(\mathfrak{F})\,T_0\,.$$

Since, from § **6**:

$$T_0 = (a_{12}\,dp + a_{23}\,dq + a_{33}\,dr),$$

and from (7), § **7**:

$$\mathfrak{S}_1 = \sigma_1\,dp + \sigma_2\,dq, \qquad \mathfrak{S}_2 = \sigma_3\,dp + \sigma_4\,dq,$$

that will make:

$$(1)\qquad \left\{\begin{aligned} \frac{\partial\mathfrak{F}}{\partial p} &= \sigma_1\,g_1(\mathfrak{F}) + \sigma_3\,g_2(\mathfrak{F}) + \frac{a_{13}}{\sqrt{a_{33}}}\,g_0(\mathfrak{F}), \\ \frac{\partial\mathfrak{F}}{\partial q} &= \sigma_2\,g_1(\mathfrak{F}) + \sigma_4\,g_2(\mathfrak{F}) + \frac{a_{23}}{\sqrt{a_{33}}}\,g_0(\mathfrak{F}), \\ \frac{\partial\mathfrak{F}}{\partial r} &= \sqrt{a_{33}}\,g_0(\mathfrak{F}). \end{aligned}\right.$$

In order to give an application of that transformation, we consider the surface (x_0, y_0, z_0) that was derived at the conclusion of § **9** from a cyclic family of curves under the assumption that the family of circles was a normal family, and the radii to the circles kept the constant value $1 / c$ (*). We will then have:

$$x_0 = x + \frac{1}{c^2}\left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2}\right), \qquad y_0 = y + \frac{1}{c^2}\left(\frac{\lambda_1}{P_1} + \frac{\lambda_2}{P_2}\right), \qquad z_0 = z + \frac{1}{c^2}\left(\frac{\mu_1}{P_1} + \frac{\mu_2}{P_2}\right).$$

We next determine the direction cosines of the normals to our surface. Since:

$$g_0(x_0) = g_0(y_0) = g_0(z_0) = 0,$$

we will have:

$$\begin{vmatrix} \dfrac{\partial y_0}{\partial p} & \dfrac{\partial y_0}{\partial q} \\ \dfrac{\partial z_0}{\partial p} & \dfrac{\partial z_0}{\partial q} \end{vmatrix} = \sigma \begin{vmatrix} g_1(y_0) & g_2(y_0) \\ g_1(z_0) & g_2(z_0) \end{vmatrix}.$$

We found in § **9** that:

(*) Cf., **Bianchi**, *Lezioni di Geometria differenziale*, pp. 322, no. 186.

$$g_0\left(\frac{1}{h_1}\right) = c^2 + \frac{1}{h_1}, \qquad g_0\left(\frac{1}{h_2}\right) = c^2 + \frac{1}{h_2}.$$

As a result of this, one will get from the third, fourth, ninth, and sixth equations in the system (11), § **7** that:

$$g_1\left(\frac{1}{P_1}\right) = -\frac{1}{P_2}\left(\frac{1}{P_2} - \frac{1}{R_1}\right), \qquad g_1\left(\frac{1}{P_2}\right) = \frac{1}{P_1}\left(\frac{1}{P_2} - \frac{1}{R_1}\right),$$

$$g_2\left(\frac{1}{P_1}\right) = \frac{1}{P_2}\left(\frac{1}{P_1} - \frac{1}{R_2}\right), \qquad g_2\left(\frac{1}{P_2}\right) = -\frac{1}{P_1}\left(\frac{1}{P_1} - \frac{1}{R_2}\right).$$

One then has:

$$g_1(x_0) = \frac{1}{c^2 P_1}\left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2} + \frac{\xi}{h_1}\right), \qquad g_2(x_0) = \frac{1}{c^2 P_2}\left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2} + \frac{\xi}{h_2}\right),$$

and

$$\begin{vmatrix} g_1(y_0) & g_2(y_0) \\ g_1(z_0) & g_2(z_0) \end{vmatrix} = \left(\frac{1}{h_1} - \frac{1}{h_2}\right)\left(\frac{\kappa_2}{P_1} - \frac{\kappa_1}{P_1}\right).$$

The normal to the surface at the point (x_0, y_0, z_0) is then, at the same time, perpendicular to the plane of the circle whose center has the coordinates x_0, y_0, z_0. Its direction cosines then coincide with ξ', η', ζ' (§ **9**). In regard to the latter, one has:

$$g_1(\xi') = -\frac{1}{cP_2}\left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2} + \frac{\xi}{h_1}\right), \qquad g_2(\xi') = \frac{1}{cP_1}\left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2} + \frac{\xi}{h_2}\right).$$

It follows from the proportionalities:

$$g_1(x_0) : g_1(y_0) : g_1(z_0) = g_1(\xi') : g_1(\eta') : g_1(\zeta'),$$

$$g_2(x_0) : g_2(y_0) : g_2(z_0) = g_2(\xi') : g_2(\eta') : g_2(\zeta')$$

that the operations g_1 () and g_2 () imply differentiations in the directions of the lines of curvature of the surface (x_0, y_0, z_0). If one then defines the two operations $g_1'(\mathfrak{F})$ and $g_2'(\mathfrak{F})$ by the equations:

$$g_1'(\mathfrak{F}) = \frac{c^2 P_1}{\sqrt{c^2 + \dfrac{1}{h_1^2}}}\, g_1(\mathfrak{F}), \qquad g_2'(\mathfrak{F}) = \frac{c^2 P_2}{\sqrt{c^2 + \dfrac{1}{h_2^2}}}\, g_2(\mathfrak{F}),$$

then $g_1'(x_0)$, $g_1'(y_0)$, $g_1'(z_0)$ will be the direction cosines of the tangents to those lines of curvature whose arc-length is $\sqrt{c^2+\frac{1}{h_1^2}}\cdot\frac{\mathfrak{S}_1}{c^2P_1}$, and $g_2'(x_0)$, $g_2'(y_0)$, $g_2'(z_0)$ will be the direction cosines of the tangents to those lines of curvature whose arc-length is $\sqrt{c^2+\frac{1}{h_2^2}}\cdot\frac{\mathfrak{S}_2}{c^2P_2}$.

That implies that the radius of curvature ρ_1 of the first line of curvature satisfies:

$$\frac{1}{\rho_1} = -\sum g_1'(x_0)\,g_1'(\xi') = \frac{c\,P_1}{P_2},$$

and the radius of curvature ρ_2 of the second satisfies:

$$\frac{1}{\rho_2} = -\sum g_2'(x_0)\,g_2'(\xi') = -\frac{c\,P_2}{P_1}.$$

Therefore, the surface (x_0, y_0, z_0) possesses a mean curvature of $c\left(\frac{P_1}{P_2}-\frac{P_2}{P_1}\right)$, but constant, negative Gaussian curvature $-c^2$.

The radii of geodetic curvature of the lines of curvature, which we would like to denote by R_1' and R_2', are also easy to calculate. In order to do that, it is simplest to employ the relations:

$$g_2'\left(\frac{1}{\rho_1}\right) = \frac{1}{R_1'}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right), \qquad g_1'\left(\frac{1}{\rho_2}\right) = \frac{-1}{R_2'}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right),$$

which are true because of the first two equations in (11), § **7**, and get:

$$\frac{1}{R_1'} = \frac{c^2P_1}{\sqrt{c^2+\frac{1}{h_2^2}}}\left(\frac{1}{R_2}-\frac{1}{P_1}\right), \quad \frac{1}{R_2'} = \frac{c^2P_2}{\sqrt{c^2+\frac{1}{h_1^2}}}\left(\frac{1}{R_1}-\frac{1}{P_2}\right).$$

In order to express the partial derivatives of a function $\mathfrak{F}$ with respect to x, y, z in terms of the operations $g_\alpha(\mathfrak{F})$, we apply the conversions (1) to the system:

$$\frac{\partial\mathfrak{F}}{\partial p} = \frac{\partial\mathfrak{F}}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial\mathfrak{F}}{\partial y}\frac{\partial y}{\partial p}+\frac{\partial\mathfrak{F}}{\partial z}\frac{\partial z}{\partial p},$$

$$\frac{\partial\mathfrak{F}}{\partial q} = \frac{\partial\mathfrak{F}}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial\mathfrak{F}}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial\mathfrak{F}}{\partial z}\frac{\partial z}{\partial q},$$

$$\frac{\partial\mathfrak{F}}{\partial r} = \frac{\partial\mathfrak{F}}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\mathfrak{F}}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial\mathfrak{F}}{\partial z}\frac{\partial z}{\partial r},$$

and obtain:

$$\sigma_1\left[g_1(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\kappa_1\right] + \sigma_2\left[g_2(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\kappa_2\right] + \frac{a_{13}}{\sqrt{a_{33}}}\left[g_0(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\xi\right] = 0,$$

$$\sigma_2\left[g_1(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\kappa_1\right] + \sigma_4\left[g_2(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\kappa_2\right] + \frac{a_{23}}{\sqrt{a_{33}}}\left[g_0(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\xi\right] = 0,$$

$$g_0(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\xi = 0.$$

The first two of these equations give:

$$g_1(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\kappa_1 = 0,$$

$$g_2(\mathfrak{F}) - \sum\frac{\partial\mathfrak{F}}{\partial x}\kappa_2 = 0,$$

as a consequence of the last one.

One will then have the transformation equations:

$$(2) \qquad \left\{\begin{aligned} \frac{\partial\mathfrak{F}}{\partial x} &= \kappa_1\, g_1(\mathfrak{F}) + \kappa_2\, g_2(\mathfrak{F}) + \xi\, g_0(\mathfrak{F}), \\ \frac{\partial\mathfrak{F}}{\partial y} &= \lambda_1\, g_1(\mathfrak{F}) + \lambda_2\, g_2(\mathfrak{F}) + \eta\, g_0(\mathfrak{F}), \\ \frac{\partial\mathfrak{F}}{\partial z} &= \mu_1\, g_1(\mathfrak{F}) + \mu_2\, g_2(\mathfrak{F}) + \zeta\, g_0(\mathfrak{F}). \end{aligned}\right.$$

Every n^{th} derivative of $\mathfrak{F}$ will be an n-fold linear form of the nine direction cosines κ_1, κ_2, ξ, …

We apply the system (2) in order to transform the **Lamé** differential parameters. For the first differential parameter, we get:

$$(3) \qquad \Delta_1^2(\mathfrak{F}) = \left(\frac{\partial\mathfrak{F}}{\partial x}\right)^2 + \left(\frac{\partial\mathfrak{F}}{\partial y}\right)^2 + \left(\frac{\partial\mathfrak{F}}{\partial z}\right)^2 = g_1(\mathfrak{F})^2 + g_2(\mathfrak{F})^2 + g_0(\mathfrak{F})^2,$$

and for the second one:

$$(4) \quad \begin{cases} \Delta_2(\mathfrak{F}) = \dfrac{\partial^2 \mathfrak{F}}{\partial x^2} + \dfrac{\partial^2 \mathfrak{F}}{\partial y^2} + \dfrac{\partial^2 \mathfrak{F}}{\partial z^2} \\ = g_{11}(\mathfrak{F}) + g_{22}(\mathfrak{F}) + g_{00}(\mathfrak{F}) - g_1(\mathfrak{F})\left(\dfrac{1}{R_2} + \dfrac{1}{P_1}\right) - g_2(\mathfrak{F})\left(\dfrac{1}{R_1} + \dfrac{1}{P_2}\right) - g_0(\mathfrak{F})\left(\dfrac{1}{h_2} + \dfrac{1}{h_1}\right). \end{cases}$$

If one would like to make the partial derivatives of $\mathfrak{F}$ with respect to p, q, r more evident in these conversions then it would simplest for one to appeal to the abbreviations that were introduced in § **3** :

$$\mathfrak{F}_p = \frac{\partial \mathfrak{F}}{\partial p} - \frac{a_{13}}{a_{33}} \frac{\partial \mathfrak{F}}{\partial r}, \qquad \mathfrak{F}_q = \frac{\partial \mathfrak{F}}{\partial q} - \frac{a_{23}}{a_{33}} \frac{\partial \mathfrak{F}}{\partial r},$$

with which, one will have:

$$g_1(\mathfrak{F}) = \frac{\sigma_4 \mathfrak{F}_p - \sigma_2 \mathfrak{F}_q}{\sigma_1 \sigma_4 - \sigma_2 \sigma_3}, \qquad g_2(\mathfrak{F}) = \frac{-\sigma_2 \mathfrak{F}_p + \sigma_1 \mathfrak{F}_q}{\sigma_1 \sigma_4 - \sigma_2 \sigma_3}.$$

One then has the following representation for the first differential parameter:

$$(5) \qquad \Delta_1^2(\mathfrak{F}) = \frac{G \mathfrak{F}_p^2 - 2F \mathfrak{F}_p \mathfrak{F}_q + E \mathfrak{F}_q^2}{EG - F^2} + \frac{1}{a_{33}} \left(\frac{\partial \mathfrak{F}}{\partial r}\right)^2.$$

In order to arrive at a corresponding statement for the second differential parameter, we set, for the moment:

$$A = \frac{G \mathfrak{F}_p - F \mathfrak{F}_q}{\sigma}, \qquad B = \frac{E \mathfrak{F}_q - F \mathfrak{F}_p}{\sigma},$$

so:

$$A = \sigma_4\, g_1(\mathfrak{F}) - \sigma_2\, g_2(\mathfrak{F}), \qquad B = \sigma_1\, g_2(\mathfrak{F}) - \sigma_3\, g_1(\mathfrak{F}),$$

and

$$g_1(\mathfrak{F}) = \frac{\sigma_1}{\sigma} A + \frac{\sigma_2}{\sigma} B, \qquad g_2(\mathfrak{F}) = \frac{\sigma_3}{\sigma} A + \frac{\sigma_4}{\sigma} B.$$

It now follows that:

$$g_{11}(\mathfrak{F}) + g_{22}(\mathfrak{F}) = A\left\{g_1\left(\frac{\sigma_1}{\sigma}\right) + g_2\left(\frac{\sigma_3}{\sigma}\right)\right\} + B\left\{g_1\left(\frac{\sigma_2}{\sigma}\right) + g_2\left(\frac{\sigma_4}{\sigma}\right)\right\} + \frac{1}{\sigma}(A_p + B_q).$$

However [cf., (8), § **7**]:

$$g_1\left(\frac{\sigma_1}{\sigma}\right) + g_2\left(\frac{\sigma_3}{\sigma}\right) = \frac{1}{\sigma^2}\{\sigma_1(\sigma_{3q} - \sigma_{4p}) + \sigma_3(\sigma_{2p} - \sigma_{1q})\} = \frac{1}{\sigma}\left(\frac{\sigma_1}{R_2} + \frac{\sigma_3}{R_1}\right),$$

$$g_1\left(\frac{\sigma_2}{\sigma}\right)+g_2\left(\frac{\sigma_4}{\sigma}\right) = \frac{1}{\sigma^2}\{\sigma_3(\sigma_{3q}-\sigma_{4p})+\sigma_4(\sigma_{2p}-\sigma_{1q})\} = \frac{1}{\sigma}\left(\frac{\sigma_2}{R_2}+\frac{\sigma_4}{R_1}\right).$$

The desired result then reads:

$$(6)\qquad \left\{\begin{aligned} \Delta_2(\mathfrak{F}) = \frac{1}{\sqrt{EG-F^2}}&\left\{\left(\frac{G\,\mathfrak{F}_p - F\,\mathfrak{F}_q}{\sqrt{EG-F^2}}\right)_p - \left(\frac{E\,\mathfrak{F}_q - F\,\mathfrak{F}_p}{\sqrt{EG-F^2}}\right)_q\right\} \\ &- \frac{g_1(\mathfrak{F})}{P_1} - \frac{g_2(\mathfrak{F})}{P_2} - g_0(\mathfrak{F})\left(\frac{1}{h_2}+\frac{1}{h_1}\right) + g_{00}(\mathfrak{F}). \end{aligned}\right.$$

In order to apply equation (3), we shall use the method that was developed to prove a theorem that goes back to **Weingarten** on the condition for a family of surfaces to belong to a triply-orthogonal system of surfaces. (J. f. reine angew. Math., Bd. 83, pp. 4) Let ε = 0, and let μ denote an integrating factor for the differential form:

$$a_{13}\,dp + a_{23}\,dq + a_{33}\,dr\,,$$

such that:

$$\mu\,(a_{13}\,dp + a_{23}\,dq + a_{33}\,dr) = dt,$$

$$t = f\,(p, q, r).$$

Now, if $\mathfrak{F}$ is a function of p, q, r and one imagines replacing r with its expression in terms of p, q, t, then the complete derivative of $\mathfrak{F}$ with respect to p (q, resp.) will be represented in terms of $\mathfrak{F}_p$ ($\mathfrak{F}_q$, resp.), since:

$$dr = \frac{dt}{\mu\,a_{33}} - \frac{a_{13}}{a_{33}}\,dp - \frac{a_{23}}{a_{33}}\,dq\,,$$

and the derivative of $\mathfrak{F}$ with respect t will become $\dfrac{1}{\mu\,a_{33}}\dfrac{\partial\mathfrak{F}}{\partial r}$.

Since the complete derivatives of t with respect to p and q vanish, g_1 (t) and g_2 (t) will also be zero, and the equation for the first **Lamé** differential parameter of t will become:

$$\sqrt{\left(\frac{\partial t}{\partial x}\right)^2+\left(\frac{\partial t}{\partial y}\right)^2+\left(\frac{\partial t}{\partial z}\right)^2} = g_0\,(t) = \mu\sqrt{a_{33}}\,.$$

If we set:

$$\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2 = a'_{33}$$

then that will imply that:

$$\sqrt{a'_{33}} = \frac{1}{\mu\sqrt{a_{33}}}.$$

It follows from the differential equations that are true for μ :

$$\frac{\partial\mu\, a_{13}}{\partial q} = \frac{\partial\mu\, a_{23}}{\partial p}, \qquad \frac{\partial\mu\, a_{13}}{\partial r} = \frac{\partial\mu\, a_{33}}{\partial p}, \qquad \frac{\partial\mu\, a_{33}}{\partial r} = \frac{\partial\mu\, a_{33}}{\partial q}$$

that

$$(\log \mu)_p = \frac{\dfrac{\partial a_{13}}{\partial r} - \dfrac{\partial a_{33}}{\partial p}}{a_{33}}, \qquad (\log \mu)_q = \frac{\dfrac{\partial a_{23}}{\partial r} - \dfrac{\partial a_{33}}{\partial q}}{a_{33}},$$

and if one recalls (8), § **7**:

(7) $$g_1(\log \mu\sqrt{a_{33}}) = \frac{1}{P_1}, \qquad g_2(\log \mu\sqrt{a_{33}}) = \frac{1}{P_2},$$

such that one further has:

$$g_1(\sqrt{a'_{33}}) = -\frac{\sqrt{a'_{33}}}{P_1}, \qquad g_2(\sqrt{a'_{33}}) = -\frac{\sqrt{a'_{33}}}{P_2}.$$

Weingarten's theorem says that the expression:

$$\frac{\partial\sqrt{a'_{33}}}{\partial x}d\xi + \frac{\partial\sqrt{a'_{33}}}{\partial y}d\eta + \frac{\partial\sqrt{a'_{33}}}{\partial z}d\zeta$$

is a complete differential along any surface t = const., so the first equation in (9), § **7** will be true when the family of surfaces t = const. belongs to a triply-orthogonal system.

If φ is a function of x, y, z then under an application of our transformation formulas, the differential form:

$$\frac{\partial\varphi}{\partial x}d\xi + \frac{\partial\varphi}{\partial y}d\eta + \frac{\partial\varphi}{\partial z}d\zeta$$

will go to:

$$-\frac{g_1(\varphi)\mathfrak{S}_1}{h_1} - \frac{g_2(\varphi)\mathfrak{S}_2}{h_2}.$$

Should the first of equations (9), § **7** be true for the latter differential form, then one would need to have:

$$-\frac{g_{12}(\varphi)}{h_1} + \frac{g_{21}(\varphi)}{h_2} = -\frac{g_1(\varphi)}{R_1 h_2} + \frac{g_2(\varphi)}{R_2 h_1}$$

or:

$$g_{12}(\varphi) + \frac{g_2(\varphi)}{R_2} = 0.$$

When $\varphi = \sqrt{a'_{33}}$, the last equation will be replaced with the following one:

$$-g_2\left(\frac{1}{P_1}\right) + \frac{1}{P_1 P_2} - \frac{1}{R_2 P_2} = 0\ ;$$

however, because of the last equation of the system (11), § **7** :

$$\vartheta = 0,$$

which proves **Weingarten**'s theorem.

One obtains a known theorem on parallel surfaces from equations (7). If $\mathfrak{F}(x, y, z) = t$ is the equation of a family of surfaces then $\Delta_1 t$ will depend upon only t when the complete derivatives $(\Delta_1 t)_p$ and $(\Delta_1 t)_q$ vanish. However, that comes from the vanishing of $g_1(\Delta_1 t)$ and $g_2(\Delta_1 t)$; i.e., from (7), the quantities $1 / P_1$ and $1 / P_2$ are zero. As a result of that, the orthogonal trajectories of the family of surfaces define a ray system, and the family itself will consist of parallel surfaces.

An application of equation (4) will then yield the solution to the following problem: Discover the conditions under which the family of curves will consist of the orthogonal trajectories of an isothermal family of surfaces when $\varepsilon = 0$. In the stated case:

$$A = \frac{\Delta_2(t)}{\Delta_1^2(t)}$$

depends upon only t, so the complete derivatives A_p and A_q will vanish, or what amounts to the same thing, $g_1(A)$ and $g_2(A)$ will.

Now, from (4), one has:

$$\Delta_2(t) = g_0\left(\mu\sqrt{a_{33}}\right) - \mu\sqrt{a_{33}}\left(\frac{1}{h_1} + \frac{1}{h_2}\right),$$

so:

$$A = \frac{1}{\mu\sqrt{a_{33}}}\left[g_0\left(\log\mu\sqrt{a_{33}}\right) - \frac{1}{h_1} - \frac{1}{h_2}\right].$$

The equations:

$$g_1(A) = 0, \qquad g_2(A) = 0$$

assume the form:

$$-\ g_1\left(\log\mu\sqrt{a_{33}}\right)\left[g_0\left(\log\mu\sqrt{a_{33}}\right) - \frac{1}{h_1} - \frac{1}{h_2}\right] + g_{01}\left(\log\mu\sqrt{a_{33}}\right) - g_1\left(\frac{1}{h_1}\right) - g_1\left(\frac{1}{h_2}\right) = 0,$$

$$-\ g_2\left(\log\mu\sqrt{a_{33}}\right)\left[g_0\left(\log\mu\sqrt{a_{33}}\right)-\frac{1}{h_1}-\frac{1}{h_2}\right]+g_{02}\left(\log\mu\sqrt{a_{33}}\right)-g_2\left(\frac{1}{h_1}\right)-g_2\left(\frac{1}{h_2}\right)=0.$$

If one considers (9) and (11), § **7** then the last two equations will go to:

$$g_0\left(\frac{1}{P_1}\right)-g_1\left(\frac{1}{h_1}\right)=-\frac{1}{h_2\,P_1}+\frac{1}{R_2}\left(\frac{1}{h_2}-\frac{1}{h_2}\right)+\frac{\vartheta}{P_2},$$

$$g_0\left(\frac{1}{P_2}\right)-g_1\left(\frac{1}{h_2}\right)=-\frac{1}{h_1\,P_2}+\frac{1}{R_1}\left(\frac{1}{h_1}-\frac{1}{h_2}\right)-\frac{\vartheta}{P_1}.$$

The essence of that result consists of the fact that the conditions that were found for the existence of the integrating factor did not contain μ anywhere, while it would not be possible to construct A if one did not know such a factor. One gets the following expression for the second curvature of the orthogonal trajectories of an isothermal family of surfaces:

$$\frac{1}{P_1}g_2\left(\frac{1}{h_1}+\frac{1}{h_2}\right)-\frac{1}{P_2}g_1\left(\frac{1}{h_1}+\frac{1}{h_2}\right)+\left(\frac{1}{h_1}+\frac{1}{h_2}\right)\frac{1}{P_1\,P_2}.$$

If the family consists of nothing but minimal surfaces then the trajectories in question will be plane curves.

PART THREE

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Doubly-infinite families of curves defined by differential equations.

§ 13. – Normal family. Special family. Orthogonal trajectories and the most distinguished types of them.

We now turn to the case in which a family of curves is defined by differential equations of the form:

$$dx : dy : dz = \xi : \eta : \zeta ,$$

in which:

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

as before, while ξ, η, ζ mean functions of x, y, z.

The family of curves considered will be a *normal family* when it is possible to convert the expression:

$$\xi\, dx + \eta\, dy + \zeta\, dz$$

into the differential of a function of x, y, z by multiplying by a suitable factor. As is known, the necessary condition for that reads:

$$\text{(1)} \qquad \xi\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)+\eta\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)+\zeta\left(\frac{\partial\eta}{\partial x}-\frac{\partial\zeta}{\partial y}\right)= 0.$$

The left-hand side of that equation must be proportional to the quantity ε. In order to determine the proportionality factor, one notes that, from (2), § **12** and (10), § **7** :

$$\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}= 2\ \varepsilon\ \xi+\frac{\kappa_2}{P_1}-\frac{\kappa_1}{P_2}\,.$$

The desired proportionality factor is then equal to 2.

When one employs the notations:

$$\frac{\partial\xi}{\partial x}= \xi_1\,, \qquad \frac{\partial\xi}{\partial y}= \xi_2\,, \qquad \frac{\partial\xi}{\partial z}= \xi_3\,, \qquad \frac{\partial\eta}{\partial x}= \xi_1\,, \qquad \text{etc.,}$$

$$\text{(2)} \qquad e_1 = \tfrac{1}{2}(\zeta_2 - \eta_3), \qquad e_2 = \tfrac{1}{2}(\xi_3 - \zeta_1), \qquad e_3 = \tfrac{1}{2}(\eta_1 - \xi_2),$$

it will follow that:

(3) $$\varepsilon = e_1\,\xi + e_2\,\eta + e_3\,\zeta.$$

As in § **6**, we understand a_1, b_1, c_1 or a_2, b_2, c_2 to mean the direction cosines of the principal normals or binormals to the curves of the family and understand ρ and ρ' to mean the radii of their first or second curvature.

One will then get from the first **Frenet** formula that:

(4) $$\begin{cases} \dfrac{a_1}{\rho} = \xi_1\,\xi + \xi_2\,\eta + \xi_3\,\zeta = 2(e_2\,\zeta - e_2\,\eta), \\ \dfrac{b_1}{\rho} = \eta_1\,\xi + \eta_2\,\eta + \eta_3\,\zeta = 2(e_3\,\xi - e_1\,\zeta), \\ \dfrac{c_1}{\rho} = \zeta_1\,\xi + \zeta_2\,\eta + \zeta_3\,\zeta = 2(e_1\,\eta - e_2\,\xi). \end{cases}$$

The first curvature then satisfies the equation:

(5) $$\frac{1}{\rho^2} = 4(e_1^2 + e_2^2 + e_3^2 - \varepsilon^2).$$

One will be dealing with a ray system when:

$$\varepsilon^2 = e_1^2 + e_2^2 + e_3^2 \qquad \text{or} \qquad e_1 : e_2 : e_3 = \xi : \eta : \zeta,$$

and the ray system will consist of normals to a surface when:

$$e_1 = e_2 = e_3 = 0,$$

or in other words, when the expression:

$$\xi\,dx + \eta\,dy + \zeta\,dz$$

is a complete differential.

The theorem that was quoted on pp. 101 regarding parallel surfaces can be put into a form that has many uses with the help of that result. If a family of surfaces is given by the equation:

$$\mathfrak{F}(x, y, z) = t$$

then one will have:

$$\xi = \frac{\dfrac{\partial\mathfrak{F}}{\partial x}}{\sqrt{L}}, \qquad \eta = \frac{\dfrac{\partial\mathfrak{F}}{\partial y}}{\sqrt{L}}, \qquad \zeta = \frac{\dfrac{\partial\mathfrak{F}}{\partial z}}{\sqrt{L}},$$

in which:

$$L = \left(\frac{\partial\mathfrak{F}}{\partial x}\right)^2 + \left(\frac{\partial\mathfrak{F}}{\partial y}\right)^2 + \left(\frac{\partial\mathfrak{F}}{\partial z}\right)^2 = (\Delta t)^2.$$

However:

$$2e_1 = \frac{\partial \mathfrak{F}}{\partial z}\,\partial\frac{\partial \dfrac{1}{\sqrt{L}}}{\partial y} - \frac{\partial \mathfrak{F}}{\partial y}\,\partial\frac{\partial \dfrac{1}{\sqrt{L}}}{\partial z}, \qquad \text{etc.}$$

One will then be dealing with a family of parallel surfaces when L is constant or when:

$$\frac{\partial L}{\partial x} : \frac{\partial L}{\partial y} : \frac{\partial L}{\partial z} = \frac{\partial \mathfrak{F}}{\partial x} : \frac{\partial \mathfrak{F}}{\partial y} : \frac{\partial \mathfrak{F}}{\partial z}.$$

If we take:

$$a_2 = \eta\, c_1 - \zeta\, b_1$$

then we will get:

$$(5) \qquad a_2 = 2\rho\,(-\varepsilon\,\xi + e_1), \quad b_2 = 2\rho\,(-\varepsilon\,\eta + e_2), \quad c_2 = 2\rho\,(-\varepsilon\,\zeta + e_3),$$

and we will get the following equation for the second curvature:

$$(6) \quad \left\{ \begin{aligned} \frac{1}{\rho'} = -2\varepsilon - 4\rho^2 \Big\{ & \xi^2\left(e_2\frac{\partial e_3}{\partial x} - e_3\frac{\partial e_2}{\partial x}\right) + \eta^2\left(e_3\frac{\partial e_1}{\partial y} - e_1\frac{\partial e_3}{\partial y}\right) + \zeta^2\left(e_1\frac{\partial e_2}{\partial z} - e_2\frac{\partial e_1}{\partial z}\right) \\ & + \xi\eta\left(e_2\frac{\partial e_3}{\partial x} - e_3\frac{\partial e_2}{\partial x} + e_3\frac{\partial e_1}{\partial x} - e_1\frac{\partial e_3}{\partial x}\right) \\ & + \eta\zeta\left(e_3\frac{\partial e_1}{\partial z} - e_1\frac{\partial e_3}{\partial z} + e_1\frac{\partial e_2}{\partial y} - e_2\frac{\partial e_1}{\partial y}\right) \\ & + \xi\zeta\left(e_1\frac{\partial e_2}{\partial x} - e_2\frac{\partial e_1}{\partial x} + e_2\frac{\partial e_3}{\partial z} - e_3\frac{\partial e_2}{\partial x}\right)\Big\}. \end{aligned} \right.$$

We now move on to a consideration of the orthogonal trajectories of the family of curves. If u, v, w mean three functions of x, y, z then the differential equations:

$$dx : dy : dz = u : v : w$$

will determine a family of orthogonal trajectories when we have:

$$\xi\, dx + \eta\, dy + \zeta\, dz = 0$$

identically. The differential equations shall be said to be *found in normal form* when:

$$u^2 + v^2 + w^2 = 1.$$

The given family of curves is *special* when a system of functions u, v, w can be determined from the property that when one advances along any corresponding orthogonal trajectory, the direction of the tangent (ξ, η, ζ) does not change (§ **3**). The system u, v, w must then satisfy the relations:

$$\begin{aligned} \xi_1\, u + \xi_2\, v + \xi_3\, w &= 0, \\ \eta_1\, u + \eta_2\, v + \eta_3\, w &= 0, \\ \zeta_1\, u + \zeta_2\, v + \zeta_3\, w &= 0, \\ \xi\;\, u + \xi\;\, v + \xi\;\, w &= 0. \end{aligned}$$

If we take:

$$\Delta = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi \\ \eta_1 & \eta_2 & \eta_3 & \eta \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} \tag{7}$$

then the possibility of the coexistence of the previous notations will emerge from the vanishing of the determinant. Namely, if we let $\Delta_{\mu\nu}$ denote the adjoint of the element of Δ that is in the μ^{th} column and the ν^{th} row then we will have:

$$\xi\,\Delta = \Delta_{14}\,, \qquad \eta\,\Delta = \Delta_{24}\,, \qquad \zeta\,\Delta = \Delta_{34}\,.$$

However, the equations:

$$\Delta_{14} = \Delta_{24} = \Delta_{34} = 0$$

imply the necessary and sufficient condition for the existence of a system u, v, w with the property in question.

The normal planes to the curves of a special family define only a doubly-infinite manifold, and thus envelop a surface. That will follow with the help of a remark by **Voss** (Math. Ann., Bd. 23, pp. 48) about the vanishing of Δ, namely:

If one takes:

$$m = -(\xi x + \eta y + \zeta z)$$

then

$$\alpha = \frac{\xi}{m}, \qquad \beta = \frac{\eta}{m}, \qquad \gamma = \frac{\zeta}{m}$$

will be the **Hessian** coordinates of the normal planes. Moreover, if:

$$m_\nu = -(x\,\xi_\nu + y\,\eta_\nu + z\,\zeta_\nu)$$

then the functional determinant J of the quantities α, β, γ will take the form:

$$\frac{1}{m^6}\begin{vmatrix} \xi_1\, m - \xi\, m_1 + \xi^2 & \xi_2\, m - \xi\, m_2 + \xi\eta & \xi_3\, m - \xi\, m_3 + \xi\,\zeta \\ \eta_1\, m - \eta\, m_1 + \xi\,\eta & \eta_2\, m - \eta\, m_2 + \eta^2 & \eta_3\, m - \eta\, m_3 + \eta\zeta \\ \zeta_1\, m - \zeta\, m_1 + \xi\,\zeta & \zeta_2\, m - \zeta\, m_2 + \eta\,\zeta & \zeta_3\, m - \zeta\, m_3 + \zeta^2 \end{vmatrix}.$$

However, one has:

$$\Delta = \frac{1}{m^2}\begin{vmatrix} \xi_1 m - \xi m_1 + \xi^2 & \xi_2 m - \xi m_2 + \xi\eta & \xi_3 m - \xi m_3 + \xi\zeta & \xi \\ \eta_1 m - \eta m_1 + \xi\eta & \eta_2 m - \eta m_2 + \eta^2 & \eta_3 m - \eta m_3 + \eta\zeta & \eta \\ \zeta_1 m - \zeta m_1 + \xi\zeta & \zeta_2 m - \zeta m_2 + \eta\zeta & \zeta_3 m - \zeta m_3 + \zeta^2 & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix}.$$

If one adds the first row in this, multiplied by x / m, the second one, multiplied by y / m, and the third one, multiplied by z / m, to the fourth row then that will yield:

$$\Delta = - m^4 J.$$

As in § **6**, we now focus on the coordinate lines and derivatives with respect to their arc-lengths. The differential equations:

$$dx : dy : dz = u_1 : v_1 : w_1 ,$$
$$dx : dy : dz = u_2 : v_2 : w_2 ,$$

which are assumed to be in normal form, will determine two families of orthogonal trajectories, which might intersect with an angle of φ, when:

$$\xi u_1 + \eta v_1 + \zeta w_1 = \xi u_2 + \eta v_2 + \zeta w_2 = 0.$$

If we take:

$$u_2' = \frac{u_1 - u_2 \cos\varphi}{\sin\varphi}, \qquad u_1' = \frac{-u_1 \cos\varphi + u_2}{\sin\varphi}, \quad \text{etc.}$$

then the differential equations:

$$dx : dy : dz = u_1' : v_1' : w_1',$$
$$dx : dy : dz = u_2' : v_2' : w_2'$$

will determine those orthogonal trajectories of the family of curves that penetrate the previous two families of curves at right angles. If we now set:

$$dx = u_1 T_1 + u_2 T_2 + \xi T_0$$

then one will have:

$$T_1 = \frac{u_2' dx + v_2' dy + w_2' dz}{\sin\varphi},$$

$$T_2 = \frac{u_1' dx + v_1' dy + w_1' dz}{\sin\varphi},$$

$$T_0 = \xi dx + \eta dy + \zeta dz,$$

and the derivatives of a function $\mathfrak{F}$ with respect to the arc-lengths of the curves $T_2 = 0$, $T_1 = 0$ will take the form:

$$(d\mathfrak{F})_{T_1} = u_1 \frac{\partial \mathfrak{F}}{\partial x} + v_1 \frac{\partial \mathfrak{F}}{\partial y} + w_1 \frac{\partial \mathfrak{F}}{\partial z},$$
$$(d\mathfrak{F})_{T_2} = u_2 \frac{\partial \mathfrak{F}}{\partial x} + v_2 \frac{\partial \mathfrak{F}}{\partial y} + w_2 \frac{\partial \mathfrak{F}}{\partial z}.$$

We now connect this with the determination of the distinguished types of orthogonal trajectories that emerged in § **6**.

From the above, the differential equations of the principal normals and binormals are:

$$dx : dy : dz = e_2\,\zeta - e_3\,\eta : e_3\,\xi - e_1\,\zeta : e_1\,\eta - e_2\,\xi,$$
$$dx : dy : dz = \varepsilon\,\xi - e_1 : \varepsilon\,\eta - e_2 : \varepsilon\,\zeta - e_3.$$

In (7), § **6**, we found the following condition for the lines of curvature of the first kind:

$$\cos\varphi = 0, \qquad \sum (dx)_{T_1} (d\xi)_{T_2} + \sum (dx)_{T_2} (d\xi)_{T_1} = 0.$$

When one makes use of the notations:

$$a_{11} = \xi_1, \qquad a_{22} = \eta_2, \qquad a_{33} = \zeta_3,$$

$$a_{12} = \tfrac{1}{2}(\xi_1 + \eta_1), \qquad a_{13} = \tfrac{1}{2}(\xi_3 + \zeta_1), \qquad a_{23} = \tfrac{1}{2}(\eta_3 + \zeta_2),$$

the second of those conditions will assume the form:

$$u_2 (a_{11}\,u_1 + a_{12}\,v_1 + a_{13}\,w_1) + v_2 (a_{12}\,u_1 + a_{22}\,v_1 + a_{23}\,w_1) + w_2 (a_{13}\,u_1 + a_{23}\,v_1 + a_{33}\,w_1) = 0.$$

Since one has:

$$u_2\,u_1 + v_2\,v_1 + w_2\,w_1 = 0, \qquad u_2\,\xi + v_2\,\eta + w_2\,\xi = 0,$$

in addition, the two systems of values (u, v, w) in question will be defined by the equations:

$$\text{(8)} \qquad \left\{ \begin{aligned} &(a_{11}\,u + a_{12} v + a_{13} w)(\eta\,w - \zeta\,v) + (a_{12}\,u + a_{22} v + a_{23} w)(\zeta\,u - \xi\,w) \\ &\qquad\qquad + (a_{13}\,u + a_{23} v + a_{33} w)(\xi\,v - \eta\,u) = 0, \end{aligned} \right.$$

$$\xi\,u + \eta\,v + \zeta\,w = 0, \quad u^2 + v^2 + w^2 = 1.$$

The lines of curvature of the second kind are then given by the condition:

$$\frac{1}{l_{T_1}} = \sum (dx)_{T_1'} (d\xi)_{T_1} = 0$$

or

$$u_1' (\xi_1\, u_1 + \xi_2\, v_1 + \xi_3\, w_1) + v_1' (\eta_1\, u_1 + \eta_2\, v_1 + \eta_3\, w_1) + w_1' (\zeta_1\, u_1 + \zeta_2\, v_1 + \zeta_3\, w_1) = 0\,.$$

If one appends the relations:

$$u_1' \xi + v_1' \eta + w_1' \zeta = 0, \quad u_1' u_1 + v_1' v_1 + w_1' w_1 = 0$$

to the first of those conditions then the desired defining equation will take the form:

$$\text{(9)} \qquad \left\{ \begin{array}{c} (\xi_1 u + \xi_2 v + \xi_3 w)(\eta w - \zeta v) + (\eta_1 u + \eta_2 v + \eta_3 w)(\zeta u - \xi w) \\ (\zeta_1 u + \zeta_2 v + \zeta_3 w)(\xi v - \eta u) = 0. \end{array} \right.$$

The asymptotic lines will be determined by the condition:

$$\frac{1}{h_{T_1}} = - \sum (dx)_{T_1} (d\xi)_{T_1} = 0.$$

That will yield:

$$\text{(10)} \qquad a_{11}\, u^2 + a_{22}\, v^2 + a_{33}\, w^2 + 2\, a_{12}\, u\, v + 2\, a_{13}\, u\, w + 2\, a_{23}\, v\, w = 0.$$

Finally, the quantity:

$$\frac{1}{R_{T_1}} = \sum (dx)_{T_1'} (dx)_{T_1}{}^2$$

will vanish for geodetic lines. That implies the determining equations for u, v, w :

$$\text{(11)} \qquad \left\{ \begin{array}{c} (\eta w - \zeta v)\left(u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} + w \dfrac{\partial u}{\partial z} \right) + (\zeta u - \xi w)\left(u \dfrac{\partial v}{\partial x} + v \dfrac{\partial v}{\partial y} + w \dfrac{\partial v}{\partial z} \right) \\ + (\xi v - \eta u)\left(u \dfrac{\partial w}{\partial x} + v \dfrac{\partial w}{\partial y} + w \dfrac{\partial w}{\partial z} \right) = 0. \end{array} \right.$$

The family of orthogonal trajectories that is adjoint to the family that is defined by the differential equations:

$$dx : dy : dz = u : v : w$$

is determined by the system:

$$\text{(12)} \qquad \left\{ \begin{array}{r} dx : dy : dz = (\eta\, \zeta_1 - \zeta\, \eta_1)\, u + (\eta\, \zeta_2 - \zeta\, \eta_2)\, v + (\eta\, \zeta_3 - \zeta\, \eta_3)\, w : \\ (\zeta\, \xi_1 - \xi\, \zeta_1)\, u + (\zeta\, \xi_2 - \xi\, \zeta_2)\, v + (\zeta\, \eta_3 - \xi\, \zeta_3)\, w : \\ (\xi\, \eta_1 - \eta\, \xi_1)\, u + (\xi\, \eta_2 - \eta\, \xi_2)\, v + (\xi\, \eta_3 - \eta\, \xi_3)\, w. \end{array} \right.$$

§ 14. – Isotropic families of curves. The quantities h_1, h_2, ρ_1, ρ_2, $\mathfrak{r}_1$, $\mathfrak{r}_2$. A way of determining the lines of curvature of the first kind.

The normal curvature of an orthogonal trajectory possesses the expression:

$$\frac{1}{h} = -(a_{11}\,u^2 + a_{22}\,v^2 + a_{33}\,w^2 + 2\,a_{12}\,u\,v + 2\,a_{13}\,u\,w + 2\,a_{22}\,v\,w)\,. \tag{1}$$

We next seek the condition for that expression to be independent of u, v, w; i.e., the condition for an isotropic family of curves. One obtains it most simply by eliminating one of the quantities u, v, w – say w. If one multiplies $1 / h$ by ζ^2 and divides by $\zeta^2\,(u^2 + v^2 + w^2)$ then that will yield:

$$\frac{1}{h} = -\frac{u^2(a_{11}\,\zeta^2 - 2a_{13}\,\xi\,\zeta + a_{33}\,\xi^2) + 2u\,v\,(a_{12}\,\zeta^2 + a_{23}\,\xi\,\eta - a_{13}\,\eta\,\zeta - a_{33}\,\xi\,\zeta) + v^2(a_{22}\,\zeta^2 - 2a_{23}\,\eta\,\zeta + a_{33}\,\eta^2)}{u^2(\xi^2 + \zeta^2) + 2u\,v\,\xi\,\eta + v^2(\eta^2 + \zeta^2)}\,.$$

If λ denotes a proportionality factor then the case in question will occur when:

$$\begin{aligned}
\xi^2 + \zeta^2 &= \lambda\,(a_{11}\,\zeta^2 + a_{33}\,\xi^2 - 2\,a_{13}\,\xi\,\zeta),\\
\zeta^2 + \eta^2 &= \lambda\,(a_{22}\,\zeta^2 + a_{33}\,\xi^2 - 2\,a_{23}\,\eta\,\zeta),\\
\xi\,\eta &= \lambda\,(a_{12}\,\zeta^2 + a_{33}\,\xi\,\eta - a_{13}\,\eta\,\zeta - a_{23}\,\xi\,\zeta)\,.
\end{aligned}$$

After the last of these equations is multiplied by $2\xi\eta$, and when one considers the previous two, that will imply that:

$$\begin{aligned}
2\xi^2\eta^2 = \lambda\,(2\,a_{13}\,\xi\,\eta\,\zeta^2 + 2\,a_{33}\,\xi^2\,\eta^2) + \eta^2\,(\xi^2 + \zeta^2) - \eta^2\,\lambda\,(a_{11}\,\zeta^2 + a_{33}\,\xi^2)&\\
+ \xi^2\,(\zeta^2 + \eta^2) - \lambda\,\xi^2\,(a_{22}\,\zeta^2 + a_{33}\,\xi^2)&,
\end{aligned}$$

or

$$\eta^2 + \xi^2 = \lambda\,(a_{11}\,\eta^2 + a_{22}\,\xi^2 - 2\,a_{12}\,\xi\,\eta)\,.$$

If now follows by addition that:

$$2 = \lambda\,\{a_{11}\,(\zeta^2 + \eta^2) + a_{22}\,(\xi^2 + \zeta^2) + a_{33}\,(\xi^2 + \eta^2) - 2\,a_{12}\,\xi\,\eta - 2\,a_{13}\,\xi\,\zeta - 2\,a_{23}\,\eta\,\zeta\}.$$

The expression in brackets admits a significant simplification. One has:

$$\left\{\begin{aligned}
a_{11}\,\xi + a_{12}\,\eta + a_{13}\,\zeta &= \tfrac{1}{2}(\xi_1\,\xi + \xi_2\,\eta + \xi_3\,\zeta) = \frac{a_1}{2\rho},\\
a_{12}\,\xi + a_{22}\,\eta + a_{23}\,\zeta &= \tfrac{1}{2}(\eta_1\,\xi + \eta_2\,\eta + \eta_3\,\zeta) = \frac{b_1}{2\rho},\\
a_{13}\,\xi + a_{23}\,\eta + a_{33}\,\zeta &= \tfrac{1}{2}(\zeta_1\,\xi + \zeta_2\,\eta + \zeta_3\,\zeta) = \frac{c_1}{2\rho},
\end{aligned}\right. \tag{2}$$

and therefore:

$$2 = \lambda\,(a_{11} + a_{22} + a_{33}).$$

We then find the conditions for the case that we speak of in the form:

$$(3)\qquad \begin{cases} (\xi^2 - \zeta^2)\,a_{11} + (\xi^2 + \zeta^2)\,a_{22} + (\zeta^2 - \xi^2)\,a_{33} + 4a_{13}\,\xi\,\zeta = 0, \\ (\zeta^2 + \eta^2)\,a_{11} + (\eta^2 - \zeta^2)\,a_{22} + (\zeta^2 - \eta^2)\,a_{33} + 4a_{23}\,\eta\,\zeta = 0, \\ (\xi^2 - \eta^2)\,a_{11} + (\eta^2 - \xi^2)\,a_{22} + (\eta^2 + \xi^2)\,a_{33} + 4a_{12}\,\eta\,\xi = 0. \end{cases}$$

Each of these equations is a consequence of the other two.

If one would like to use those conditions to show that the families of curves whose normals define a line complex of degree one are isotropic then one should consider that here one has:

$$\xi = \frac{a + \gamma y - \beta z}{N}, \quad \eta = \frac{b + \alpha z - \gamma x}{N}, \quad \zeta = \frac{c + \beta x - \alpha y}{N},$$

in which a, b, c, α, β, γ mean constants. One finds that:

$$N\,\xi_1 = \xi\,(\gamma\,\eta - \beta\,\zeta), \qquad N\,\xi_2 = \gamma - \xi\,(\xi\,\gamma - \alpha\,\zeta), \qquad N\,\xi_3 = -\beta - \xi\,(\alpha\,\eta - \beta\,\xi),$$

$$N\,\eta_1 = -\gamma - \eta\,(\beta\,\zeta - \gamma\,\eta), \qquad N\,\eta_2 = \eta\,(\alpha\,\zeta - \gamma\,\xi), \qquad N\,\eta_3 = \alpha - \eta\,(\alpha\,\eta - \beta\,\xi),$$

$$N\,\zeta_1 = \beta - \zeta\,(\beta\,\zeta - \alpha\,\eta), \qquad N\,\zeta_2 = -\alpha - \zeta\,(\gamma\,\gamma - \alpha\,\zeta), \qquad N\,\zeta_3 = \zeta\,(\beta\,\xi - \alpha\,\eta)\,.$$

One recognizes the existence of equations (3) with the help of those formulas. Moreover, since:

$$e_1 = \frac{-1}{2N}\,[\alpha + \xi\,(\alpha\,\xi + \beta\,\eta + \gamma\,\zeta)], \qquad e_2 = \frac{-1}{2N}\,[\beta + \eta\,(\alpha\,\xi + \beta\,\eta + \gamma\,\zeta)],$$
$$e_3 = \frac{-1}{2N}\,[\gamma + \zeta\,(\alpha\,\xi + \beta\,\eta + \gamma\,\zeta)],$$

one will have:

$$\varepsilon = -\frac{\alpha\,\xi + \beta\,\eta + \gamma\,\zeta}{N} = -\frac{a\,\alpha + b\,\beta + c\,\gamma}{N^2}.$$

One will then have a normal family only when the line complex is special.

We exclude the possibility of equations (3) being true and ask what the largest and smallest values of $1 / h$ would be. If m and n mean two temporarily-undetermined functions then the partial derivatives with respect to u, v, w of the expression:

$$a_{11}\,u^2 + a_{22}\,v^2 + a_{33}\,w^2 + 2\,a_{12}\,u\,v + 2\,a_{23}\,v\,w + 2\,a_{13}\,u\,w + 2m\,(\xi\,u + \eta\,v + \zeta\,w) + n\,(u^2 + v^2 + w^2 - 1)$$

must be set to zero. In that way, the system will arise:

$$(2)\qquad \begin{cases} a_{11}\,u + a_{12}\,v + a_{13}\,w + m\,\xi + n\,u = 0, \\ a_{12}\,u + a_{22}\,v + a_{23}\,w + m\,\eta + n\,v = 0, \\ a_{13}\,u + a_{23}\,v + a_{33}\,w + m\,\zeta + n\,w = 0, \end{cases}$$

which gives the relation:

$$\xi\,u + \eta\,v + \zeta\,w = 0.$$

Equation (8) of the previous paragraph shows that the values u, v, w that appear here represent the direction cosines of the tangents to the lines of curvature of the first kind.

In order to see the meaning of the quantity m, one should consider that:

$$a_{11}\,\xi + a_{12}\,\eta + a_{13}\,\zeta = e_2\,\zeta - e_3\,\eta = \frac{1}{2}\left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2}\right),$$

$$a_{12}\,\xi + a_{22}\,\eta + a_{23}\,\zeta = e_3\,\xi - e_1\,\zeta = \frac{1}{2}\left(\frac{\lambda_1}{P_1} + \frac{\lambda_2}{P_2}\right),$$

$$a_{13}\,\xi + a_{23}\,\eta + a_{33}\,\zeta = e_1\,\eta - e_2\,\xi = \frac{1}{2}\left(\frac{\mu_1}{P_1} + \frac{\mu_2}{P_2}\right).$$

One will then have:

$$m = -\frac{1}{2P_1}$$

for $u = \kappa_1$, etc., and:

$$m = -\frac{1}{2P_2}$$

for $u = \kappa_2$, etc. The quantity n has the two values $1 / h_1$ and $1 / h_2$. They are the roots of the equation:

$$(3)\qquad \begin{vmatrix} a_{11} + \dfrac{1}{h} & a_{12} & a_{13} & \xi \\ a_{12} & a_{22} + \dfrac{1}{h} & a_{23} & \eta \\ a_{13} & a_{23} & a_{33} + \dfrac{1}{h} & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} = 0.$$

The determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \xi \\ a_{12} & a_{22} & a_{23} & \eta \\ a_{13} & a_{23} & a_{33} & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix}$$

shall be denoted by D, and the adjoints of its elements by $D_{\mu\nu}$, in which μ gives the column and ν gives the row. In place of (3), one will then get:

(4) $$\frac{1}{h^2} + \frac{\xi_1 + \eta_2 + \zeta_3}{h} - D = 0.$$

The connection between the determinants D and Δ is obtained as follows: One has:

$$a_{12} = \xi_2 + e_3 = \eta_1 - e_3 , \qquad a_{23} = \eta_3 + e_1 = \zeta_2 - e_1 , \qquad a_{13} = \zeta_1 + e_2 = \xi_3 - e_2 .$$

Thus:

$$D = \begin{vmatrix} \xi_1 & \xi_2 + e_3 & \xi_3 - e_2 & \xi \\ \eta_1 - e_3 & \eta_2 & \eta_3 + e_1 & \eta \\ \zeta_1 + e_1 & \zeta_2 - e_1 & \zeta_3 & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix}.$$

If one denotes the adjoints of the first three elements of the last column of this determinant by A, B, C, for the moment, then one will see that B emerges from A and C emerges from B by a simultaneous cyclic permutation of the symbols ξ, η, ζ, and the numbers 1, 2, 3. However, one has:

$$-A = \begin{vmatrix} \xi_2 & \xi_3 & \xi \\ \eta_2 & \eta_3 & \eta \\ \zeta_2 & \zeta_3 & \zeta \end{vmatrix} + \zeta\,(e_1\,\xi_2 + e_2\,\eta_2 + e_3\,\zeta_2) - \eta\,(e_1\,\xi_3 + e_2\,\eta_3 + e_3\,\zeta_3) - e_1\,\varepsilon ,$$

such that:

$$D = \Delta + \varepsilon^2.$$

We have denoted the two values of $1 / h$ that yield the points at which the tangent (ξ) will be cut by a neighboring one ($\xi + \delta\xi$) by $1 / \rho_1$ and $1 / \rho_2$. In order to determine them, we can appeal to the fact that here:

$$\sum \delta x\,(\eta\,\delta\zeta - \zeta\,\delta\eta) = 0 \qquad \text{and} \qquad \sum \xi\,\delta x = 0$$

or

$$\delta x : \delta y : \delta z = \delta\xi : \delta\eta : \delta\zeta .$$

If we take:

$$-\frac{\delta x}{h} = \delta\xi, \quad -\frac{\delta y}{h} = \delta\eta, \quad -\frac{\delta z}{h} = \delta\zeta$$

then the values of h that are compatible with these equations will coincide with ρ_1 and ρ_2. For the corresponding quantities u, v, w, one then has:

$$\begin{aligned}
\left(\xi_1 + \frac{1}{h}\right) u + \xi_2\, v + \xi_3\, w &= 0, \\
\eta_1\, u + \left(\eta_1 + \frac{1}{h}\right) v + \eta_3\, w &= 0, \\
\zeta_1\, u + \zeta_3\, v + \left(\zeta_3 + \frac{1}{h}\right) w &= 0.
\end{aligned}$$

They satisfy equation (9), § **13** and are the direction cosines of the tangents to the lines of curvature of the second kind. The quantities $1 / \rho_1$ and $1 / \rho_2$ are roots of the equation:

$$\begin{vmatrix} \xi_1 + \frac{1}{h} & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 + \frac{1}{h} & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 + \frac{1}{h} \end{vmatrix} = 0$$

or

$$\frac{1}{h^2} + \frac{\xi_1 + \eta_2 + \zeta_3}{h} + \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} + \begin{vmatrix} \xi_1 & \xi_3 \\ \zeta_1 & \zeta_3 \end{vmatrix} + \begin{vmatrix} \eta_2 & \eta_3 \\ \zeta_2 & \zeta_3 \end{vmatrix} = 0.$$

In regard to the conversion of the last term in this equation, one remarks that with the help of the relations:

$$\xi\,\xi_\nu + \eta\,\eta_\nu + \zeta\,\zeta_\nu = 0,$$

Δ_{14} can be brought into the form:

$$-\xi\,(\xi_1\,\eta_2 - \xi_2\,\eta_1 + \eta_2\,\zeta_3 - \eta_3\,\zeta_2 + \zeta_3\,\xi_1 - \zeta_1\,\xi_3)\,.$$

However, since:

$$\xi\,\Delta = \Delta_{14}\,,$$

the equation for the determination of ρ_1 and ρ_2 will be:

(5) $$\frac{1}{h^2} + \frac{\xi_1 + \eta_2 + \zeta_3}{h} - \Delta = 0.$$

We found the expression:

$$\mathfrak{r} = -\frac{\sum \delta x\, \delta\xi}{\sum \delta\xi^2}$$

for the abscissa $\mathfrak{r}$ of the shortest distance between the neighboring tangents (ξ) and ($\xi + \delta\xi$). We assume that the family of curves is a general one such that Δ is non-zero, and ask what the largest and smallest values of $\mathfrak{r}$ might be. We take:

$$a = \xi_1 u + \xi_2 v + \xi_3 w ,$$
$$b = \eta_1 u + \eta_2 v + \eta_3 w ,$$
$$c = \zeta_1 u + \zeta_2 v + \zeta_3 w .$$

Since:

$$\xi u + \eta v + \zeta w = 0,$$

it will follow conversely that:

$$(6) \qquad \begin{cases} u = \dfrac{1}{\Delta}(\Delta_{11}\, a + \Delta_{21}\, b + \Delta_{31}\, c), \\ v = \dfrac{1}{\Delta}(\Delta_{12}\, a + \Delta_{22}\, b + \Delta_{32}\, c), \\ w = \dfrac{1}{\Delta}(\Delta_{13}\, a + \Delta_{23}\, b + \Delta_{33}\, c). \end{cases}$$

If one then sets:

$$\tfrac{1}{2}(\Delta_{\mu\nu} + \Delta_{\nu\mu}) = b_{\mu\nu} = b_{\nu\mu}$$

then one will have:

$$(7) \qquad \mathfrak{r} = -\frac{b_{11}\, a^2 + b_{22}\, b^2 + b_{33}\, c^2 + 2\, b_{12}\, a\, b + 2\, b_{13}\, a\, c + 2\, b_{22}\, b\, c}{\Delta(a^2 + b^2 + c^2)} .$$

In order to determine the distinguished values $\mathfrak{r}_1$ and $\mathfrak{r}_2$, we set the derivatives with respect to a, b, c of the expression:

$$-\, \mathfrak{r}\, \Delta + 2m\, (\xi\, a + \eta\, b + \zeta\, c)$$

equal to zero, and when we further take:

$$a^2 + b^2 + c^2 = s,$$

we will get:

$$(8) \qquad \begin{cases} b_{11}\, a + b_{12}\, b + b_{13}\, c + \mathfrak{r}\, a\, \Delta + m\, s\, \xi = 0, \\ b_{12}\, a + b_{22}\, b + b_{23}\, c + \mathfrak{r}\, b\, \Delta + m\, s\, \eta = 0, \\ b_{13}\, a + b_{23}\, b + b_{33}\, c + \mathfrak{r}\, c\, \Delta + m\, s\, \zeta = 0, \end{cases}$$

which is added to the equation:

$$\xi\, a + \eta\, b + \zeta\, c = 0.$$

The values in question of $\mathfrak{r}$ then satisfy the relation:

$$\begin{vmatrix} b_{11}+\mathfrak{r}\,\Delta & b_{12} & b_{13} & \xi \\ b_{12} & b_{22}+\mathfrak{r}\,\Delta & b_{23} & \eta \\ b_{13} & b_{23} & b_{33}+\mathfrak{r}\,\Delta & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} = 0,$$

which is quadratic, the factor of $\mathfrak{r}^2\,\Delta^2$ is – 1, and that of $\mathfrak{r}\,\Delta$ is:

$$-(b_{11}+b_{22}+b_{33}) + \xi\,(b_{11}\,\xi + b_{12}\,\eta + b_{13}\,\zeta) + \eta\,(b_{12}\,\xi + b_{22}\,\eta + b_{23}\,\zeta) \\ + \zeta\,(b_{13}\,\xi + b_{23}\,\eta + b_{33}\,\zeta)\,,$$

while the absolute value has the form:

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} & \xi \\ b_{12} & b_{22} & b_{23} & \eta \\ b_{13} & b_{23} & b_{33} & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix}.$$

In order to convert those coefficients, we first remark that:

$$b_{11} = 2\,a_{23}\,\eta\,\zeta - a_{22}\,\zeta^2 - a_{33}\,\eta^2, \qquad b_{12} = -a_{13}\,\eta\,\zeta + a_{33}\,\eta\,\xi - a_{23}\,\zeta\,\xi + a_{12}\,\zeta^2,$$

$$b_{22} = 2\,a_{13}\,\xi\,\eta - a_{33}\,\xi^2 - a_{11}\,\zeta^2, \qquad b_{23} = -a_{12}\,\xi\,\zeta + a_{11}\,\eta\,\zeta - a_{13}\,\eta\,\xi + a_{23}\,\xi^2,$$

$$b_{33} = 2\,a_{12}\,\eta\,\zeta - a_{11}\,\eta^2 - a_{22}\,\xi^2, \qquad b_{13} = -a_{23}\,\eta\,\xi + a_{22}\,\zeta\,\xi - a_{12}\,\eta\,\zeta + a_{13}\,\eta^2.$$

If we now set:

$$\begin{aligned} \alpha &= a_{11} + a_{22} + a_{33}\,, \\ \alpha_1 &= a_{11}\,\xi + a_{12}\,\eta + a_{13}\,\zeta\,, \\ \alpha_2 &= a_{12}\,\xi + a_{22}\,\eta + a_{23}\,\zeta\,, \\ \alpha_3 &= a_{13}\,\xi + a_{23}\,\eta + a_{33}\,\zeta \end{aligned}$$

then it will follow that:

$$a_1\,\xi + a_2\,\eta + a_3\,\zeta = 0,$$

and

$$\begin{aligned} b_{11} &= \alpha\,(\xi^2 - 1) - 2\,\xi\,\alpha_1 + a_{11}\,, & b_{12} &= \eta\,\zeta\,\alpha - \eta\,\alpha_1 - \xi\,\alpha_2 + a_{12}\,, \\ b_{22} &= \alpha\,(\eta^2 - 1) - 2\,\eta\,\alpha_2 + a_{22}\,, & b_{23} &= \eta\,\zeta\,\alpha - \zeta\,\alpha_2 - \eta\,\alpha_3 + a_{23}\,, \\ b_{33} &= \alpha\,(\zeta^2 - 1) - 2\,\zeta\,\alpha_3 + a_{33}\,, & b_{13} &= \zeta\,\xi\,\alpha - \xi\,\alpha_3 - \zeta\,\alpha_1 + a_{13}\,. \end{aligned}$$

That shows that:

$$b_{11} + b_{22} + b_{33} = -\alpha, \quad b_{11}\,\xi + b_{22}\,\eta + b_{33}\,\zeta = 0 \qquad (\nu = 1, 2, 3),$$

such that the coefficient of $\mathfrak{r}\,\Delta$ will be equal to α.

The absolute value will become:

$$\begin{vmatrix} a_{11}-\alpha & a_{12} & a_{13} & \xi \\ a_{12} & a_{22}-\alpha & a_{23} & \eta \\ a_{13} & a_{33} & a_{33}-\alpha & \zeta \\ \xi & \eta & \zeta & 0 \end{vmatrix} = D$$

after a simple conversion.

The defining equation for $\mathfrak{r}_1$ and $\mathfrak{r}_2$ then has the form:

$$\mathfrak{r}^2 \, \Delta^2 - \alpha \, \mathfrak{r} \, \Delta - D = 0. \tag{9}$$

As in Part Two, (4), (5), and (9) will imply that:

$$\frac{1}{h_1}+\frac{1}{h_2}=\frac{1}{\rho_1}+\frac{1}{\rho_2}, \qquad \frac{1}{h_1 h_2}=\frac{1}{\rho_1 \rho_2}-\varepsilon^2, \qquad \mathfrak{r}_1=\frac{\rho_1 \rho_2}{h_2}, \qquad \mathfrak{r}_2=\frac{\rho_1 \rho_2}{h_1}.$$

One gets the following equation for the values of a, b, c that satisfy the system (8):

$$\begin{cases} (b_{11}\,a+b_{12}\,b+b_{13}\,c)(\eta\,c-\zeta\,b)+(b_{12}\,a+b_{22}\,b+b_{23}\,c)(\zeta\,a-\xi\,c) \\ \qquad +(b_{13}\,a+b_{23}\,b+b_{33}\,c)(\xi\,b-\eta\,a)=0. \end{cases} \tag{10}$$

If one takes:

$$a\,\alpha_1 + b\,\alpha_2 + c\,\alpha_3 = \alpha'$$

then that will imply that:

$$\begin{aligned} b_{11}\,a + b_{12}\,b + b_{13}\,c &= -\,\alpha\,a - \xi\,\alpha' + a\,a_{11} + b\,a_{12} + c\,a_{13}\,, \\ b_{12}\,a + b_{22}\,b + b_{23}\,c &= -\,\alpha\,b - \eta\,\alpha' + a\,a_{12} + b\,a_{22} + c\,a_{23}\,, \\ b_{13}\,a + b_{23}\,b + b_{33}\,c &= -\,\alpha\,c - \zeta\,\alpha' + a\,a_{13} + b\,a_{23} + c\,a_{33}\,. \end{aligned}$$

One will then get:

$$\begin{cases} (a_{11}\,a+a_{12}\,b+a_{13}\,c)(\eta\,c-\zeta\,b)+(a_{12}\,a+a_{22}\,b+a_{23}\,c)(\zeta\,a-\xi\,c) \\ \qquad +(a_{13}\,a+a_{23}\,b+a_{33}\,c)(\xi\,b-\eta\,a)=0, \end{cases} \tag{11}$$

in place of (10).

As a consequence of (8), § **13**, the system of values a, b, c that are established in that way will be proportional to the direction cosines of the tangents to the lines of curvature of the first type. We denote them by a', b', c', and a'', b'', c''. Now, according to (6), the system of values a', b', c' will determine a shift of the point (x, y, z) for which one has:

$$\delta x : \delta y : \delta z = u' : v' : w'.$$

That will determine a tangent that is close to the tangent (ξ, η, ζ). The direction cosines of the shortest distance between the two tangents are proportional to the quantities:

$$\eta\, c' - \zeta\, b', \qquad \zeta\, a' - \xi\, c', \qquad \xi\, b' - \eta\, a',$$

or the quantities a'', b'', c'', from (11). The shift that corresponds to the values a'', b'', c'' for which:

$$\delta x : \delta y : \delta z = u'' : v'' : w''$$

likewise determines a shortest distance between two neighboring tangents whose direction cosines are proportional to the values a', b', c'. In that way, the two shortest distances in question prove to be parallel to the tangents to the lines of curvature of the first kind.

To conclude this paragraph, let one of the ways of calculating the quantities κ_1, κ_2, etc., be emphasized. Those quantities must be proportional to the adjoints of the first three elements of the last column of the determinant in (3). When ν is one of the numbers 1 and 2, and p_ν means a proportionality factor, that will imply:

$$(12) \qquad p_\nu\, \kappa_\nu = -\begin{vmatrix} a_{12} & a_{13} & \xi \\ a_{22} + \dfrac{1}{h_\nu} & a_{23} & \eta \\ a_{23} & a_{33} + \dfrac{1}{h_\nu} & \zeta \end{vmatrix} = D_{41} - \frac{\xi}{h_\nu^2} + \frac{a_{12}\,\eta + a_{13}\,\zeta - (a_{22} + a_{33})\,\xi}{h_\nu}$$

$$= D_{41} - \xi\, D + \frac{\alpha_1}{h_\nu}.$$

Likewise:

$$p_\nu\, \lambda_\nu = D_{42} - \eta\, D + \frac{\alpha_2}{h_\nu}, \qquad p_\nu\, \mu_\nu = D_{43} - \zeta\, D + \frac{\alpha_3}{h_\nu},$$

and as a result:

$$p_1\, \kappa_1 - p_1\, \kappa_1 = \alpha_1 \left(\frac{1}{h_1} - \frac{1}{h_2} \right).$$

However, since it was found above that:

$$\alpha_1 = \frac{1}{2} \left(\frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2} \right),$$

one will have:

$$(13) \qquad p_1 = \frac{1}{2P_1} \left(\frac{1}{h_1} - \frac{1}{h_2} \right), \quad p_2 = \frac{-1}{2P_2} \left(\frac{1}{h_1} - \frac{1}{h_2} \right).$$

That shows that when only one of the quantities p_ν vanishes, the lines of curvature of the first kind will coincide with the principal normals and binormals, but that one will be dealing with a ray system when both quantities p_ν vanish. The way of determining κ_1, κ_2, … will then become entirely unusable for ray systems.

One further gets from (12) that:

$$p_2\ \kappa_1 = \zeta D_{42} - \eta D_{43} - \frac{e_1 - \varepsilon\,\xi}{h_2}.$$

However:

$$\zeta D_{42} - \eta D_{43} = \begin{vmatrix} a_{11} & a_{12} & \xi \\ a_{13} & a_{23} & \eta \\ \alpha_1 & \alpha_3 & 1 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \xi \\ \alpha_1 & \alpha_2 & 1 \\ a_{13} & \alpha_{23} & \zeta \end{vmatrix}$$

$$= a_{11}\,(\alpha_2\ \zeta - \alpha_3\ \eta) + a_{12}\,(\alpha_3\ \xi - \alpha_1\ \zeta) + a_{13}\,(\alpha_1\ \eta - \alpha_2\ \xi) = \varepsilon\,\alpha_1 - \sum e_\nu\,\xi_\nu\,.$$

If one sets:

$$N_{1\nu} = \xi_1\ \xi_\nu + \xi_2\ \eta_\nu + \xi_3\ \zeta_\nu, \quad N_{2\nu} = \eta_1\ \xi_\nu + \eta_2\ \eta_\nu + \eta_3\ \zeta_\nu, \quad N_{3\nu} = \zeta_1\ \xi_\nu + \zeta_2\ \eta_\nu + \zeta_3\ \zeta_\nu$$

then one will have:

$$\sum e_\nu\,\xi_\nu = \frac{N_{23} - N_{32}}{2} + \alpha\,e_1\,, \quad \sum e_\nu\,\eta_\nu = \frac{N_{31} - N_{13}}{2} + \alpha\,e_2\,, \quad \sum e_\nu\,\zeta_\nu = \frac{N_{12} - N_{21}}{2} + \alpha\,e_3\,,$$

and

$$p_2\ \kappa_1 = \frac{N_{32} - N_{23}}{2} - \alpha\,e_1 + \varepsilon\,\alpha_1 - \frac{e_1 - \varepsilon\,\xi}{h_2} = \frac{N_{32} - N_{23}}{2} + \frac{e_1}{h_1} + \varepsilon\left(\alpha_1 + \frac{\varepsilon}{h_2}\right). \tag{14}$$

Frobenius found the last expression for $p_2\ \kappa_1$ under the assumption that $\varepsilon = 0$. (J. f. reine angew. Math., Bd. 110, pp. 25, no. 26) It breaks down for a family of parallel surfaces.

§ 15. – The quantities P_1, P_2, R_1, R_2, and ϑ.

We infer the definitions of the derivatives of a function $\mathfrak{F}$ of x, y, z with respect to the arc-length of the curves of the system and the definitions of the lines of curvature of the first kind from equations (2), § **12** in the form:

$$(1)\qquad \begin{cases} g_1(\mathfrak{F}) = \kappa_1 \dfrac{\partial \mathfrak{F}}{\partial x} + \lambda_1 \dfrac{\partial \mathfrak{F}}{\partial y} + \mu_1 \dfrac{\partial \mathfrak{F}}{\partial z}, \\ g_2(\mathfrak{F}) = \kappa_2 \dfrac{\partial \mathfrak{F}}{\partial x} + \lambda_2 \dfrac{\partial \mathfrak{F}}{\partial y} + \mu_2 \dfrac{\partial \mathfrak{F}}{\partial z}, \\ g_3(\mathfrak{F}) = \xi \dfrac{\partial \mathfrak{F}}{\partial x} + \eta \dfrac{\partial \mathfrak{F}}{\partial y} + \zeta \dfrac{\partial \mathfrak{F}}{\partial z}, \end{cases}$$

and employ them in conjunction with (10), § **7** in order to calculate the quantities P_1, P_2, R_1, R_2, ϑ.

One has:

$$g_0(\xi) = \xi\,\xi_1 + \eta\,\xi_2 + \zeta\,\xi_3\,,$$

so, from (2), § **14**:

$$g_0(\xi) = 2\alpha_1 = 2\,(e_2\,\zeta - e_3\,\eta)\,.$$

However, since:

$$\frac{1}{P_1} = \sum \kappa_1\, g_0(\xi)\,, \qquad \frac{1}{P_2} = \sum \kappa_2\, g_0(\xi)\,,$$

one will have:

$$(2)\qquad \begin{cases} \dfrac{1}{P_1} = 2(\kappa_1\,\alpha_1 + \lambda_1\,\alpha_2 + \mu_1\,\alpha_3) = 2(e_1\,\kappa_2 + e_2\,\lambda_2 + e_3\,\mu_2), \\ \dfrac{1}{P_2} = 2(\kappa_2\,\alpha_1 + \lambda_2\,\alpha_2 + \mu_2\,\alpha_3) = -2(e_1\,\kappa_1 + e_2\,\lambda_1 + e_3\,\mu_1). \end{cases}$$

We infer the further expressions from (13), § **14**:

$$(3)\qquad \frac{1}{P_1} = \frac{2p_1}{\dfrac{1}{h_1} - \dfrac{1}{h_2}}\,, \qquad \frac{1}{P_2} = \frac{2p_2}{\dfrac{1}{h_2} - \dfrac{1}{h_1}}\,.$$

A rational expression in terms of ξ, η, ζ, and their derivatives shall be derived with their help whose vanishing will say that the lines of curvature of the first kind coincide with the principal normals and binormals of the family of curves.

(12), § **14** implies that:

$$p_1^2 = D_{41}^2 + D_{42}^2 + D_{43}^2 - D^2 + 2\,\frac{\alpha_1\,D_{41} + \alpha_2\,D_{42} + \alpha_3\,D_{43}}{h_1} + \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{h_1^2}\,.$$

One now sets:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = A$$

and denotes the adjoint of the element $a_{\mu\nu}$ by $A_{\mu\nu}$. One will then have:

$$\xi A = \alpha_1 A_{11} + \alpha_2 A_{12} + \alpha_3 A_{13}, \qquad \eta A = \alpha_1 A_{12} + \alpha_2 A_{22} + \alpha_3 A_{23},$$
$$\zeta A = \alpha_1 A_{13} + \alpha_2 A_{23} + \alpha_3 A_{33},$$
$$D_{4\nu} = -\alpha_1 A_{1\nu} - \alpha_2 A_{1\nu} - \alpha_3 A_{1\nu},$$

and as a result:

$$\sum \alpha_\nu D_{4\nu} = -A .$$

Furthermore:

$$\sum \alpha_\nu^2 = \xi \sum \alpha_\nu a_{1\nu} + \eta \sum \alpha_\nu a_{2\nu} + \zeta \sum \alpha_\nu a_{3\nu},$$

$$\sum \alpha_\nu a_{1\nu} = \xi (a_{11}\,\alpha - A_{22} - A_{33}) + \eta (a_{12}\,\alpha + A_{12}) + \zeta (a_{13}\,\alpha + A_{13})$$

$$= -D_{41} - \xi (A_{11} + A_{22} + A_{33}) + \alpha_1\,\alpha,$$

$$\sum \alpha_\nu a_{2\nu} = -D_{42} - \eta (A_{11} + A_{22} + A_{33}) + \alpha_2\,\alpha,$$

$$\sum \alpha_\nu a_{3\nu} = -D_{43} - \zeta (A_{11} + A_{22} + A_{33}) + \alpha_3\,\alpha,$$

so:

$$\sum \alpha_\nu^2 = -(A_{11} + A_{22} + A_{33} + D) .$$

Finally, one has:

$$\sum D_{4\nu}^2 + (A_{11} + A_{22} + A_{33})\, D$$

$$= \xi^2 (A_{12}^2 - A_{11} A_{22} + A_{13}^2 + A_{11} A_{33}) + \eta^2 (A_{12}^2 - A_{11} A_{22} + A_{23}^2 + A_{22} A_{33})$$
$$+ \zeta^2 (A_{13}^2 - A_{11} A_{33} + A_{23}^2 + A_{22} A_{33}) + 2\,\xi\,\eta (A_{13} A_{23} - A_{12} A_{33})$$
$$+ 2\,\eta\,\zeta (A_{12} A_{13} - A_{23} A_{11}) + 2\,\xi\,\zeta (A_{12} A_{23} - A_{23} A_{13})$$

$$= A\,[-\xi^2 (a_{22} + a_{33}) - \eta^2 (a_{11} + a_{33}) - \eta^2 (a_{11} + a_{22}) + 2\,\xi\,\eta\, a_{12}$$
$$+ 2\,\eta\,\zeta\, a_{23} + 2\,\xi\,\zeta\, a_{13}] = -\alpha A,$$

$$\sum D_{4\nu}^2 - D^2 = -\alpha A + D \sum \alpha_\nu^2 = A\left(\frac{1}{h_1} + \frac{1}{h_2}\right) - \frac{\sum \alpha_\nu^2}{h_1 h_2}.$$

As a result:

$$\text{(4)} \qquad \left\{ \begin{aligned} p_1^2 &= \left(\frac{1}{h_2} - \frac{1}{h_1}\right)\left(A - \frac{\sum \alpha_\nu^2}{h_1}\right), \\ p_2^2 &= \left(\frac{1}{h_2} - \frac{1}{h_1}\right)\left(A - \frac{\sum \alpha_\nu^2}{h_2}\right), \end{aligned} \right.$$

and

(5) $$\frac{1}{P_1^{\,2}P_2^{\,2}} = -\,16\,\frac{A^2 + (\alpha A - D)\sum \alpha_\nu^2}{\left(\frac{1}{h_1} - \frac{1}{h_2}\right)^2}.$$

If $\sum \alpha_\nu^2$ vanishes then the family of curves in question will be a ray system. If $\sum \alpha_\nu^2$ is non-zero, but the expression:

$$A^2 + (\alpha A - D)\sum \alpha_\nu^2$$

vanishes, then the lines of curvature of the first kind will coincide with the principal normals and binormals.

In order to determine the quantities R_1 and R_2, we start from the equation:

$$g_1\,(\kappa_1) = \kappa_1\frac{\partial\kappa_1}{\partial x} + \lambda_1\frac{\partial\kappa_1}{\partial y} + \mu_1\frac{\partial\kappa_1}{\partial z} = \frac{\kappa_2}{R_1} + \frac{\xi}{h_1}.$$

It yields:

$$\frac{1}{R_1} = \sum \kappa_2\, g_1(\kappa_1) = -\sum \kappa_1\, g_1(\kappa_2),$$

such that:

$$-\frac{1}{R_1} = \frac{\partial\kappa_2}{\partial x} + \frac{\partial\lambda_2}{\partial y} + \frac{\partial\mu_2}{\partial z} + \lambda_1\left(\kappa_1\frac{\partial\lambda_2}{\partial x} - \lambda_1\frac{\partial\kappa_2}{\partial x}\right) + \mu_1\left(\kappa_1\frac{\partial\mu_2}{\partial x} - \mu_1\frac{\partial\kappa_2}{\partial x}\right)$$
$$+\,\mu_1\left(\lambda_1\frac{\partial\mu_2}{\partial y} - \mu_1\frac{\partial\lambda_2}{\partial y}\right) + \kappa_1\left(\lambda_1\frac{\partial\kappa_2}{\partial y} - \kappa_1\frac{\partial\lambda_2}{\partial y}\right)$$
$$+\,\kappa_1\left(\mu_1\frac{\partial\kappa_2}{\partial z} - \kappa_1\frac{\partial\mu_2}{\partial z}\right) + \lambda_1\left(\mu_1\frac{\partial\lambda_2}{\partial z} - \lambda_1\frac{\partial\mu_2}{\partial z}\right).$$

However, since:

$$\kappa_1\,\lambda_2 - \lambda_1\,\kappa_2 = \zeta, \qquad \kappa_1\,\mu_2 - \mu_1\,\kappa_2 = -\,\eta,$$

one will have:

$$\lambda_1\left(\kappa_1\frac{\partial\lambda_2}{\partial x} - \lambda_1\frac{\partial\kappa_2}{\partial x}\right) + \mu_1\left(\kappa_1\frac{\partial\mu_2}{\partial x} - \mu_1\frac{\partial\kappa_2}{\partial x}\right) = \lambda_1\,\zeta_1 - \mu_1\,\eta_1$$

and

$$\lambda_1\,\zeta_1 - \mu_1\,\eta_1 + \mu_1\,\xi_2 - \kappa_1\,\zeta_2 + \kappa_1\,\eta_1 - \lambda_1\,\xi_3 = -\,2\,(\kappa_1\,e_1 + \lambda_1\,e_2 + \mu_1\,e_3).$$

As a result:

(6) $$\begin{cases} & -\dfrac{1}{R_1} = \dfrac{1}{P_2} + \dfrac{\partial \kappa_2}{\partial x} + \dfrac{\partial \lambda_2}{\partial y} + \dfrac{\partial \mu_2}{\partial z}, \\ \text{and correspondingly:} & \\ & -\dfrac{1}{R_2} = \dfrac{1}{P_1} + \dfrac{\partial \kappa_1}{\partial x} + \dfrac{\partial \lambda_1}{\partial y} + \dfrac{\partial \mu_1}{\partial z}. \end{cases}$$

The quantity ϑ is given by the equation:

$$\vartheta = \sum \kappa_2 \, g_0(\kappa_1).$$

If one applies the formulas:

$$\kappa_2 = \eta \, \mu_1 - \zeta \, \lambda_1 , \qquad \lambda_2 = \zeta \, \kappa_1 - \xi \, \mu_1 , \qquad \mu_2 = \xi \, \lambda_1 - \eta \, \kappa_1$$

then one will get:

$$\vartheta = \kappa_1 \, [\zeta \, g_0 \, (\lambda_1) - \eta \, g_0 \, (\mu_1)] + \lambda_1 \, [\xi \, g_0 \, (\mu_1) - \zeta \, g_0 \, (\kappa_1)] + \mu_1 \, [\eta \, g_0 \, (\kappa_1) - \xi \, g_0 \, (\lambda_1)] .$$

One replaces ξ^2 with $1 - \eta^2 - \zeta^2$, etc., here and takes:

(7) $$\varepsilon' = \kappa_1 \left(\frac{\partial \lambda_1}{\partial z} - \frac{\partial \mu_1}{\partial y} \right) + \lambda_1 \left(\frac{\partial \mu_1}{\partial x} - \frac{\partial \kappa_1}{\partial z} \right) + \mu_1 \left(\frac{\partial \kappa_1}{\partial y} - \frac{\partial \lambda_1}{\partial x} \right).$$

and one will get:

$$\vartheta = \varepsilon' + \sum \xi \, g_2(\kappa_1)$$

or

(8) $$\vartheta = \varepsilon' + \varepsilon .$$

If one replaces the quantities κ_1, λ_1, μ_1 in brackets in (7) with $\zeta \, \lambda_2 - \eta \, \mu_2$, $\xi \, \mu_2 - \zeta \, \kappa_2$, $\eta \, \kappa_2 - \xi \, \lambda_2$, resp., then since:

$$\sum \kappa_1 \, g_2(\xi) = - \varepsilon = \sum \xi \, g_1(\kappa_2) ,$$

that will imply the further equation:

$$\varepsilon' = \kappa_2 \left(\frac{\partial \lambda_2}{\partial z} - \frac{\partial \mu_2}{\partial y} \right) + \lambda_2 \left(\frac{\partial \mu_2}{\partial x} - \frac{\partial \kappa_2}{\partial z} \right) + \mu_2 \left(\frac{\partial \kappa_2}{\partial y} - \frac{\partial \lambda_2}{\partial x} \right).$$

The expression for ϑ that equation (8) yields is not rational in ξ, η, ζ, and their derivatives. One can find representations of ϑ that are rational in those quantities along various paths. The simplest of them seems to be the following: One starts from (14), § **14**, which will make:

$$N_{32} - N_{23} = 2\,p_2\,\kappa_1 - \frac{2e_1}{h_1} - 2\varepsilon\left(\alpha_1 + \frac{\xi}{h_2}\right).$$

From (13), § **14**, one has:

$$2\,p_2 = -\frac{1}{P_2}\left(\frac{1}{h_1} - \frac{1}{h_2}\right),$$

and from § **13** and § **14**, one has:

$$2\,e_1 = 2\,\varepsilon\,\xi + \frac{\kappa_2}{P_1} - \frac{\kappa_1}{P_2}, \qquad 2\,\alpha_1 = \frac{\kappa_1}{P_1} + \frac{\kappa_2}{P_2},$$

so:

$$N_{32} - N_{23} = \kappa_1\left(\frac{1}{h_1\,P_1} - \frac{\varepsilon}{P_1}\right) - \kappa_2\left(\frac{1}{h_1\,P_1} + \frac{\varepsilon}{P_2}\right) + 2\,\alpha\,\varepsilon\,\xi\,.$$

If one differentiates that equation with respect to x, the equation:

$$N_{13} - N_{31} = \lambda_1\left(\frac{1}{h_2\,P_2} - \frac{\varepsilon}{P_1}\right) - \lambda_2\left(\frac{1}{h_1\,P_1} + \frac{\varepsilon}{P_2}\right) + 2\,\alpha\,\varepsilon\,\xi$$

with respect to y, and the equation:

$$N_{21} - N_{12} = \mu_1\left(\frac{1}{h_2\,P_2} - \frac{\varepsilon}{P_1}\right) - \mu_2\left(\frac{1}{h_1\,P_1} + \frac{\varepsilon}{P_2}\right) + 2\,\alpha\,\varepsilon\,\xi$$

with respect to z and adds the results then when one recalls (11), § **7**, that will give:

$$(9)\qquad \left\{\begin{aligned} &\frac{\partial(N_{32}-N_{23})}{\partial x} + \frac{\partial(N_{13}-N_{31})}{\partial y} + \frac{\partial(N_{21}-N_{12})}{\partial z} = \alpha\,g_0(\varepsilon) - \frac{2g_1(\varepsilon)}{P_1} - \frac{2g_2(\varepsilon)}{P_2} \\ &\quad + \varepsilon\left[\alpha^2 + 2\left(\frac{1}{P_1^2} + \frac{1}{P_2^2}\right) + g_0(\alpha) - \frac{1}{h_1^2} - \frac{1}{h_2^2} + 2\varepsilon^2 + \vartheta\left(\frac{1}{h_1} - \frac{1}{h_2}\right)^2\right]. \end{aligned}\right.$$

However, since:

$$\frac{g_1(\varepsilon)}{P_1} + \frac{g_2(\varepsilon)}{P_2} = 2\left(\alpha_1\,\frac{\partial\varepsilon}{\partial x} + \alpha_2\,\frac{\partial\varepsilon}{\partial y} + \alpha_3\,\frac{\partial\varepsilon}{\partial z}\right),$$

$$\frac{1}{P_1^2} + \frac{1}{P_2^2} = 4(\alpha_1^2 + \alpha_2^2 + \alpha_3^2),$$

$$\frac{1}{h_1^2}+\frac{1}{h_2^2} = \alpha^2 + 2D, \qquad \left(\frac{1}{h_1}-\frac{1}{h_2}\right)^2 = \frac{\alpha^2+4D}{4},$$

the relation (9) will contain a representation of ϑ in the desired form. That will become important when $\varepsilon = 0$. One will then get the condition under which the family of surfaces whose orthogonal trajectories coincide with the families of curves considered to belong to a triply-orthogonal system of surfaces ($\vartheta = 0$) in the form:

$$\frac{\partial(N_{32}-N_{23})}{\partial x}+\frac{\partial(N_{13}-N_{31})}{\partial y}+\frac{\partial(N_{21}-N_{12})}{\partial z} = 0. \tag{10}$$

This form of the condition equation was derived by **Frobenius** in J. f. reine u. angew. Math., Bd. 110, pp. 23.

Equation (10) can be likewise given a very intuitive form of a different sort. One finds that:

$$N_{32} - N_{23} = 2\left(\frac{\partial\alpha_3}{\partial y}-\frac{\partial\alpha_2}{\partial z}\right) - 2\, g_0\,(e_1),$$

so one can replace (10) with:

$$\frac{\partial g_0(e_1)}{\partial x}+\frac{\partial g_0(e_2)}{\partial y}+\frac{\partial g_0(e_3)}{\partial z} = 0. \tag{11}$$

Finally, if one introduces the notation:

$$\delta_\nu\,(\mathfrak{F}) = \frac{\partial\mathfrak{F}}{\partial x}\xi_\nu+\frac{\partial\mathfrak{F}}{\partial y}\eta_\nu+\frac{\partial\mathfrak{F}}{\partial z}\zeta_\nu$$

and considers that one has:

$$\frac{\partial e_1}{\partial x}+\frac{\partial e_2}{\partial y}+\frac{\partial e_3}{\partial z} = 0$$

identically then one will find:

$$\delta_1\,(e_1) + \delta_2\,(e_2) + \delta_3\,(e_3) = 0, \tag{12}$$

in place of (11). That form of the condition equation was published by **Weingarten** in J. f. reine u. angew. Math., Bd. 83, pp. 9, and was also established also by **Frobenius** in the previously-cited paper on pp. 24.

§ 16. – Family of curves with a prescribed family of asymptotic lines.

If a family of curves possesses real asymptotic lines that are not straight then it will consist of the binormals to each of the two families of asymptotic lines. When a family of curves is given, that fact is closely related to the question of finding those families of curves (C) for which the former family of curves defines a family of asymptotic lines, and to further determine the second family of asymptotic lines of the family (C).

To that end, we define a family of curves by the equations:

$$dx : dy : dz = u : v : w \,, \tag{1}$$

in which u, v, w are functions of x, y, z that satisfy the equation:

$$u^2 + v^2 + w^2 = 1$$

and take $\partial u / \partial x = u_1$, $\partial v / \partial x = v_1$, etc.

The family of curves will be a ray system when two of the following conditions are fulfilled:

$$\begin{cases} u_1\, u + u_2\, v + u_3\, w = 0, \\ v_1\, u + v_2\, v + v_3\, w = 0, \\ w_1\, u + w_2\, v + w_3\, w = 0. \end{cases} \tag{2}$$

In that case, one understands ξ, η, ζ to mean three functions of x, y, z that satisfy the equations:

$$\xi u + \eta v + \zeta w = 0,$$
$$\xi^2 + \eta^2 + \zeta^2 = 1$$

and sets:

$$\xi' = v\, \zeta - w\, \eta, \qquad \eta' = w\, \xi - u\, \zeta, \qquad \zeta' = u\, \eta - v\, \xi\,.$$

If the family of curves in question is not a ray system then the direction cosines of its principal normals must be denoted by ξ', η', ζ', while those of its binormals might be denoted by ξ, η, ζ.

The differential equations:

$$dx : dy : dz = \xi : \eta : \zeta \tag{3}$$

then determine family of curves (C) for which a family of asymptotic lines is defined by equations (1).

The normal curvature of the orthogonal trajectories of the family of curves (c) that is characterized by (1) is, in fact:

$$- u\,(\xi_1\, u + \xi_2\, v + \xi_3\, w) - v\,(\eta_1\, u + \eta_2\, v + \eta_3\, w) - w\,(\zeta_1\, u + \zeta_2\, v + \zeta_3\, w)$$

or

$$\xi\,(u_1\, u + u_2\, v + u_3\, w) + \eta\,(v_1\, u + v_2\, v + v_3\, w) + \zeta\,(w_1\, u + w_2\, v + w_3\, w)\,.$$

If the family is a ray system then the coefficients of ξ, η, ζ will vanish here because of (2), but in any other case they will be proportional to the direction cosines of the principal normals, such that the normal curvature will possess the value zero in any case.

A family of orthogonal trajectories of the family (C) will be defined by the differential equations:

$$dx : dy : dz = m\,u + n\,\xi' : m\,v + n\,\eta' : m\,w + n\,\zeta',$$

in which one might have:

$$m^2 + n^2 = 1.$$

If we denote the direction cosines of their principal normals by ξ'', η'', ζ'' and their first curvature by $1 / r_1$ then we will have:

$$\frac{\xi''}{r_1} = \frac{\partial(mu+n\xi')}{\partial x}(mu+n\xi') + \frac{\partial(mu+n\xi')}{\partial y}(mv+n\eta') + \frac{\partial(mu+n\xi')}{\partial z}(mw+n\zeta').$$

In order for those principal normals to represent the second family of asymptotic lines of the family (C), we must have:

$$\xi''\xi + \eta''\eta + \zeta''\zeta = 0,$$

or

$$\begin{aligned}\left(m\sum \xi u_1 + n\sum \xi \xi_1'\right)(mu+n\xi') + \left(m\sum \xi u_2 + n\sum \xi \xi_2'\right)(mv+n\eta')& \\ + \left(m\sum \xi u_3 + n\sum \xi \xi_3'\right)(mw+n\zeta') &= 0.\end{aligned}$$

The factor of m^2 vanishes here, and what remains is:

$$(4)\qquad \left\{\begin{aligned} m\left(\xi'\sum \xi u_1 + \eta'\sum \xi u_2 + \zeta'\sum \xi u_3 + u\sum \xi \xi_1' + v\sum \xi \xi_2' + w\sum \xi \xi_3'\right)& \\ + n\left(\xi'\sum \xi \xi_1' + \eta'\sum \xi \xi_2' + \zeta'\sum \xi \xi_3'\right) &= 0.\end{aligned}\right.$$

That equation can be considered from two viewpoints according to whether the coefficients have a definite geometric meaning relative to the family of curves (C) or relative to the originally-given family of curves (1). If we remain in a neighborhood of the family (C) then we will set:

$$u = \alpha_1\,\kappa_1 + \alpha_2\,\kappa_2, \qquad v = \alpha_1\,\lambda_1 + \alpha_2\,\lambda_2, \qquad w = \alpha_1\,\mu_1 + \alpha_2\,\mu_2,$$

$$\xi' = -\alpha_2\,\kappa_1 + \alpha_1\,\kappa_2, \qquad \eta' = -\alpha_2\,\lambda_1 + \alpha_1\,\lambda_2, \qquad \zeta' = -\alpha_2\,\mu_1 + \alpha_1\,\mu_2,$$

so from § **10**:

$$\alpha_1 = \frac{\sqrt{-\frac{1}{h_2}}}{\sqrt{\frac{1}{h_1}-\frac{1}{h_2}}}, \qquad \alpha_2 = \frac{\sqrt{-\frac{1}{h_1}}}{\sqrt{\frac{1}{h_1}-\frac{1}{h_2}}}.$$

We will then have:

$$\begin{aligned}\xi\,u_1 + \eta\,v_1 + \zeta\,w_1 &= -u\,\xi_1 - v\,\eta_1 - w\,\zeta_1 \\ &= -\kappa_1\,[u\,g_1(\xi) + v\,g_1(\eta) + w\,g_1(\zeta)]\end{aligned}$$

$$- \kappa_2\,[u\, g_2\,(\xi) + v\, g_2\,(\eta) + w\, g_2\,(\zeta)]$$
$$- \;\xi\;[u\, g_3\,(\xi) + v\, g_3\,(\eta) + w\, g_3\,(\zeta)]$$

$$= \kappa_1\left(\frac{\alpha_1}{h_1} - \varepsilon\,\alpha_2\right) + \kappa_2\left(\varepsilon\,\alpha_1 + \frac{\alpha_2}{h_1}\right) - \xi\left(\frac{\alpha_1}{P_1} + \frac{\alpha_2}{P_2}\right),$$

and as a result:

$$\xi'\sum\xi\,u_1 + \eta'\sum\xi\,u_2 + \zeta'\sum\xi\,u_3 = -\,\alpha_1\,\alpha_2\left(\frac{1}{h_1} - \frac{1}{h_2}\right) + \varepsilon = -\sqrt{\frac{1}{h_1}}\sqrt{-\frac{1}{h_2}} + \varepsilon\,.$$

Furthermore:

$$\sum\xi\,\xi_1' = -\sum\xi'\,\xi_1 = \kappa_1\left(\frac{\alpha_2}{h_1} + \varepsilon\,\alpha_1\right) - \kappa_2\left(\varepsilon\,\alpha_2 - \frac{\alpha_1}{h_2}\right) - \xi\left(-\frac{\alpha_2}{P_1} + \frac{\alpha_1}{P_2}\right),$$

so:

$$u\sum\xi\,\xi_1' + v\sum\xi\,\xi_2' + w\sum\xi\,\xi_3' = -\,\alpha_1\,\alpha_2\left(\frac{1}{h_1} - \frac{1}{h_2}\right) - \varepsilon = -\sqrt{\frac{1}{h_1}}\sqrt{-\frac{1}{h_2}} - \varepsilon\,,$$

$$\xi'\sum\xi\,\xi_1' + \eta'\sum\xi\,\xi_2' + \zeta'\sum\xi\,\xi_3' = \frac{\alpha_2^2}{h_1} + \frac{\alpha_1^2}{h_2} = \frac{1}{h_1} + \frac{1}{h_2}\,.$$

We then get:

$$2\sqrt{\frac{1}{h_1}}\sqrt{-\frac{1}{h_2}}\,m - \left(\frac{1}{h_1} + \frac{1}{h_2}\right)n = 0.$$

That includes the theorems that were found before that the two families of asymptotic lines will coalesce into one when $\dfrac{1}{h_1\,h_2} = 0$ and that they will be perpendicular when $\dfrac{1}{h_1} + \dfrac{1}{h_2} = 0$.

If we refer equation (4) to the given family of curves (1), in the second place, then we will have to replace u, v, w with ξ, η, ζ, respectively, and further take:

$$\xi' = a_1 = \alpha_1\,\kappa_1 + \alpha_2\,\kappa_2\,, \qquad \eta' = b_1 = \alpha_1\,\lambda_1 + \alpha_2\,\lambda_2\,, \qquad \zeta' = c_1 = \alpha_1\,\mu_1 + \alpha_2\,\mu_2\,,$$

$$\xi = a_2 = -\,\alpha_2\,\kappa_1 + \alpha_1\,\kappa_2\,, \qquad \eta = b_2 = -\,\alpha_2\,\lambda_1 + \alpha_1\,\lambda_2\,, \qquad \zeta = c_2 = -\,\alpha_2\,\mu_1 + \alpha_1\,\mu_2\,.$$

If the family of curves (1) is a ray system then α_1 and α_2 will be subject to only the condition that:

$$\alpha_1^2 + \alpha_2^2 = 1.$$

In the other case, we will have:

$$\alpha_1 = \frac{\dfrac{1}{P_1}}{\sqrt{\dfrac{1}{P_1^2}+\dfrac{1}{P_2^2}}}, \qquad \alpha_2 = \frac{\dfrac{1}{P_2}}{\sqrt{\dfrac{1}{P_1^2}+\dfrac{1}{P_2^2}}},$$

such that, with the notations of § **10**, the curves $T_2 = 0$ will coincide with the principal normals, while the curves $T_1 = 0$ will coincide with the binormals to the family (1).

Equation (4) will now take the form:

$$m\left(a_1\sum a_2\,\xi_1 + b_1\sum a_2\,\xi_2 + c_1\sum a_2\,\xi_3 + \xi\sum a_2\frac{\partial a_1}{\partial x} + \eta\sum a_2\frac{\partial a_1}{\partial y} + \zeta\sum a_2\frac{\partial a_1}{\partial z}\right)$$

$$+\,n\left(a_1\sum a_2\frac{\partial a_1}{\partial x} + b_1\sum a_2\frac{\partial a_1}{\partial y} + c_1\sum a_2\frac{\partial a_1}{\partial z}\right) = 0.$$

Here, one has:

$$\sum a_2\,\xi_1 = \kappa_1\left(\frac{\alpha_2}{h_1} + \varepsilon\,\alpha_1\right) + \kappa_2\left(\varepsilon\,\alpha_2 - \frac{\alpha_1}{h_2}\right) + \xi\left(\frac{-\alpha_2}{P_1} + \frac{\alpha_1}{P_2}\right),$$

and therefore:

$$a_1\sum a_2\,\xi_1 + b_1\sum a_2\,\xi_2 + c_1\sum a_2\,\xi_3 = \alpha_1\,\alpha_2\left(\frac{1}{h_1} - \frac{1}{h_2}\right) + \varepsilon = \frac{1}{l_{T_1}}.$$

Moreover:

$$\xi\sum a_2\frac{\partial a_1}{\partial x} + \eta\sum a_2\frac{\partial a_1}{\partial y} + \zeta\sum a_2\frac{\partial a_1}{\partial z} = a_2\,g_0\,(a_1) + b_2\,g_0\,(b_1) + c_2\,g_0\,(c_1)$$

$$= -\,a_2\,g_0\,(\alpha_1) + \alpha_2\,g_0\,(\alpha_1) + \vartheta = \frac{1}{L_{T_1}}.$$

Finally:

$$a_1\sum a_2\frac{\partial a_1}{\partial x} + b_1\sum a_2\frac{\partial a_1}{\partial y} + c_1\sum a_2\frac{\partial a_1}{\partial z} = \alpha_1\sum a_2\,g_1(a_1) + \alpha_2\sum a_2\,g_2(a_1),$$

$$\sum a_2\,g_1(a_1) = -\,\alpha_2\,g_1\,(\alpha_2) + \alpha_1\,g_1\,(\alpha_2) + \frac{1}{R_1},$$

$$\sum a_2\,g_2(a_1) = -\,\alpha_2\,g_2\,(\alpha_1) + \alpha_1\,g_2\,(\alpha_2) - \frac{1}{R_2},$$

$$\alpha_1\sum a_2\,g_1(a_1) + \alpha_2\sum a_2\,g_2(a_1) = g_1\,(\alpha_2) - g_2\,(\alpha_1) + \frac{\alpha_1}{R_1} - \frac{\alpha_2}{R_2} = \frac{1}{R_{T_1}}.$$

Therefore:

$$\left(\frac{1}{l_{T_1}} + \frac{1}{L_{T_1}}\right) m + \frac{n}{R_{T_1}} = 0$$

will arise in place of equation (4). When $1 / R_{T_1}$ vanishes, m will be equal to zero, and that will imply the theorem:

If a family of curves possesses two mutually-perpendicular families (A_1) *and* (A_2) *of asymptotic lines then the family* (A_1) *will consist of geodetic lines of the family* (A_2), *and conversely.*

Voss carried out the determination of those families of curves for which the two families of curves coincide and are rectilinear in Math. Ann., Bd. 23, pp. 64.

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