"Ueber die Krümmung der Curvenschaaren," Math. Ann. **32** (1888), 545-565.

On the curvature of families of curves

By

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If one regards a surface as a simply-continuous manifold of curves then the most general curvature problem will relate to doubly-continuous manifolds of curves, which one can call "families of curves." Kummer's theory of rectilinear systems of rays falls within the scope of the problem envisioned, in which the curves are replaced with straight lines, and furthermore Lamé's theory of curvilinear coordinates, in which the three systems of surfaces that are implied by the analytical representation of a family of curves are assumed to intersect at right angles everywhere. In recent times, A. Voss has treated various general questions in regard to families of curves in his work on point-plane systems (Annalen, Bd. 16 and 23).

Above all, two viewpoints emerge in the curvature problem in question, which one can regard as the analogues of the curvature of a curve and that of a surface. The cross-section of a bundle of curves enters in place of a point on a curve, and the tangent bundle that belongs to the cross-section enters in place of the tangent, and it is to that tangent bundle that most of the concepts of the theory of rectilinear systems of rays finds immediate application.

The neighboring tangent to a curve tangent is replaced with the tangent bundle that belongs to a neighboring cross-section of a bundle of curves, which is why it will not be considered in what follows, since it initially comes down to the problem of discovering those curvature properties of a system of curves that remain preserved by a system of rays.

The following second viewpoint is to be emphasized: The ordinary theory of surfaces essentially consists of a theory of the curvature axes of the curves on a surface that go through a point, or in other words, the orthogonal trajectories of a normal system. If one extends that concept to the orthogonal trajectories of a family of curves then that will yield the proposed generalization with no difficulty.

In regard to the main analytic-geometric definitions, I must refer to my *Untersuchungen zur allgemeinen Theorie der krummen Oberflächen und geradlinigen Strahlensysteme* (Bonn, 1886), which I will cite by $\mathfrak{U}$, as before.

§ 1. – Notations. Tangent bundle.

A family of curves will be represented analytically when one expresses the rectangular coordinates *u*, *v*, *w* of a point in space by real continuous functions of three real mutually-independent variables *p*, *q*, *r*. *p* and *q* shall be considered to be constant along any curve of the family, while *r* will be considered to be variable. We would like to call the totality of curves that belong to a system of values (*p*, *q*) and the neighboring systems of values (*p* + *dp*, *q* + *dq*) a *bundle of curves*. We fix our attention on a regular point (*u*, *v*, *w*) on a curve that belongs to a system of values (*p*, *q*). The tangent to the curve at that point has the direction cosines ξ, η, ζ. The plane that is perpendicular to the ray (ξ, η, ζ) at the point (*u*, *v*, *w*), which is the normal plane to the curve that belongs to (*p*, *q*), shall be called a *normal plane* to the bundle of curves. It cuts out a "cross-section" from the bundle of curves. The latter is then to be regarded as an infinitely-small planar region that is bounded by a closed line, which does not generally degenerate into a line segment. At each point of that planar region, the curve of the bundle that goes through it will possess a well-defined tangent, and the totality of those tangents is called the *tangent bundle* that belongs to the point (*u*, *v*, *w*). A curve that does not belong to the family of curves shall be called a *trajectory* of the family when a curve of the family goes through each of its points, in general. If both tangents are always perpendicular to each other then the trajectory will be called an *orthogonal trajectory* of the family of curves.

With the use of the notations:

$$\sum\left(\frac{\partial u}{\partial p}\right)^2 = a_{11}, \qquad \sum\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = a_{12}, \qquad \sum\frac{\partial u}{\partial p}\frac{\partial u}{\partial r} = a_{13}, \qquad \sum\left(\frac{\partial u}{\partial q}\right)^2 = a_{22},$$

$$\sum\frac{\partial u}{\partial q}\frac{\partial u}{\partial r} = a_{23}, \qquad \sum\left(\frac{\partial u}{\partial r}\right)^2 = a_{33},$$

we will get:

$$\xi = \frac{\frac{\partial u}{\partial r}}{\sqrt{a_{33}}}, \qquad \eta = \frac{\frac{\partial v}{\partial r}}{\sqrt{a_{33}}}, \qquad \zeta = \frac{\frac{\partial w}{\partial r}}{\sqrt{a_{33}}},$$

in which $\sqrt{a_{33}}$ will be assumed to be non-zero and possess the sign of $\partial w / \partial r$. (Cf., 𝔘, § 1)

The point (*u* + *du*, *v* + *dv*, *w* + *dw*) will belong to the cross-section of the bundle of curves in question that goes through the point (*u*, *v*, *w*) as long as:

$$\sum \xi \, du = 0,$$

i.e., one has:

(1) $$a_{13}\, dp + a_{23}\, dq + a_{33}\, dr = 0 .$$

A trajectory of the family of curves will arise when two of the variables p, q, r can be regarded as functions of the third one at u, v, w . If equation (1) is fulfilled by that then one will be dealing with an orthogonal trajectory.

We shall understand the symbols δu, δv, δw, $\delta\xi$, $\delta\eta$, $\delta\zeta$ to mean the differentials of u, …, ζ that are defined under the condition (1), such that:

$$\delta u = \left(\frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}\frac{a_{13}}{a_{33}}\right) dp + \left(\frac{\partial u}{\partial q} - \frac{\partial u}{\partial r}\frac{a_{23}}{a_{33}}\right) dq ,$$

$$\delta\xi = \left(\frac{\partial \xi}{\partial p} - \frac{\partial \xi}{\partial r}\frac{a_{13}}{a_{33}}\right) dp + \left(\frac{\partial \xi}{\partial q} - \frac{\partial \xi}{\partial r}\frac{a_{23}}{a_{33}}\right) dq ,$$

in which we have set:

$$\delta u = u_p\, dp + u_q\, dq ,$$
$$\delta\xi = \xi_p\, dp + \xi_q\, dq ,$$

to abbreviate. Moreover, we introduce the notations:

$$\sum (u_p)^2 = \mathfrak{E} , \quad \sum u_p\, u_q = \mathfrak{F} , \quad \sum (u_q)^2 = \mathfrak{G} ,$$

$$\sum (\xi_p)^2 = \mathrm{H} , \quad \sum \xi_p\, \xi_q = \Phi , \quad \sum (\xi_q)^2 = \Psi ,$$

$$\sum \xi_p\, u_p = \sum \xi_p \frac{\partial u}{\partial p} = e_{11} , \qquad \sum \xi_p\, u_q = \sum \xi_p \frac{\partial u}{\partial q} = e_{12} ,$$

$$\sum \xi_q\, u_p = \sum \xi_q \frac{\partial u}{\partial p} = e_{21} , \qquad \sum \xi_q\, u_q = \sum \xi_q \frac{\partial u}{\partial q} = e_{22} .$$

The quantities $\mathfrak{E}$, $\mathfrak{G}$, H, Ψ, $\mathfrak{E}\,\mathfrak{G} - \Phi^2$, $\mathrm{H}\,\Psi - \Phi^2$ will be assumed to be non-zero.

I shall now link the tangent bundle in question with the concepts that Kummer put forth in his "allgemeine Theorie der geradlinigen Strahlensysteme" (J. reine und Angew. Math., Bd. 57). Let the abscissa of the point on the ray (ξ, η, ζ) that points away from the point (u, v, w) that meets the neighboring ray ($\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$) at the shortest distance be $\mathfrak{r}$. That will then produce the equation:

(2) $$\mathfrak{r} = -\frac{e_{11}\, dp^2 + (e_{12} + e_{21})\, dp\, dq + e_{22}\, dq^2}{\mathrm{H}\, dp^2 + 2\Phi\, dp\, dq + \Psi\, dq^2} .$$

The maximum $\mathfrak{r}_1$ and minimum $\mathfrak{r}_2$ satisfy the equation:

(3) $$(\mathrm{H}\,\Psi - \Phi^2)\,\mathfrak{r}^2 + [\mathrm{H}\,e_{22} - (e_{12} + e_{21})\,\Phi + e_{11}\,\Psi]\,\mathfrak{r} + e_{11}\,e_2 - \frac{(e_{12}+e_{22})^2}{4} = 0 ,$$

whose roots (which are always real) are assumed to be distinct. The values $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of the ratio dq / dp that the values $\mathfrak{r}_1$ ($\mathfrak{r}_2$, resp.) yield are implied by the equation:

(4) $$[e_{11}\,\Phi - \frac{e_{12}+e_{21}}{4}\mathrm{H}]\,dp^2 + [\Psi\,e_{11} - \mathrm{H}\,e_{22}]\,dp\,dq + [\Psi\,\frac{e_{12}+e_{21}}{4} - \Phi\,e_{22}]\,dq^2 = 0 .$$

The abscissas $\mathfrak{r}_1$ and $\mathfrak{r}_2$ of the focal points, which are imaginary under some circumstances, are the roots of the equation:

(5) $$(\mathrm{H}\,\Psi - \Phi^2)\,\mathfrak{r}^2 + [e_{11}\,\Psi + e_{22}\,\mathrm{H} - (e_{12} + e_{21})\,\Phi]\,\mathfrak{r} + e_{11}\,e_{22} - e_{12}\,e_{21} = 0 ,$$

and the values $\mathfrak{t}_3$ and $\mathfrak{t}_4$ of the ratio dq / dp that belong to $\mathfrak{r}_3$ and $\mathfrak{r}_4$ follow from the relation:

(6) $$(e_{21}\,\mathrm{H} - e_{11}\,\Phi)\,dp^2 + [e_{22}\,\mathrm{H} + (e_{21} - e_{12})\,\Phi - e_{11}\,\Psi]\,dp\,dq + (e_{22}\,\Phi - e_{12}\,\Psi)\,dq^2 = 0 .$$

The tangent bundle considered is called a *normal bundle* when the rays of the bundle are normal to one and the same surface. In that regard, it must be possible to determine $\mathfrak{r}$ as a function of p and q in such a way that equation (1) always exists, i.e., the right-hand side of the equation:

$$dr = -\frac{a_{13}}{a_{33}}\,dp - \frac{a_{23}}{a_{33}}\,dq$$

must be a complete differential, which emerges from the relation:

$$a_{33}\sum\frac{\partial u}{\partial p}\frac{\partial^2 u}{\partial q\,\partial r} - a_{23}\sum\frac{\partial u}{\partial p}\frac{\partial^2 u}{\partial r^2} - \frac{a_{13}}{2}\frac{\partial a_{33}}{\partial q} = a_{33}\sum\frac{\partial u}{\partial q}\frac{\partial^2 u}{\partial p\,\partial r} - a_{13}\sum\frac{\partial u}{\partial q}\frac{\partial^2 u}{\partial r^2} - \frac{a_{23}}{2}\frac{\partial a_{33}}{\partial p} .$$

That says nothing more than the equation:

$$e_{12} = e_{21} .$$

Namely, one has:

$$\xi_p = \frac{a_{33}\dfrac{\partial^2 u}{\partial p\,\partial r} - \dfrac{1}{2}\dfrac{\partial u}{\partial r}\dfrac{\partial a_{33}}{\partial p}}{a_{33}\sqrt{a_{33}}} - \frac{a_{33}\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{2}\dfrac{\partial u}{\partial r}\dfrac{\partial a_{33}}{\partial r}}{a_{33}^2\sqrt{a_{33}}}\,a_{13} ,$$

$$\xi_q = \frac{a_{33}\dfrac{\partial^2 u}{\partial p\,\partial r} - \dfrac{1}{2}\dfrac{\partial u}{\partial r}\dfrac{\partial a_{33}}{\partial q}}{a_{33}\sqrt{a_{33}}} - \frac{a_{33}\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{2}\dfrac{\partial u}{\partial r}\dfrac{\partial a_{33}}{\partial r}}{a_{33}^2\sqrt{a_{33}}}\,a_{23}\ .$$

Therefore:

$$e_{12} = \frac{a_{33}\sum\dfrac{\partial u}{\partial q}\dfrac{\partial^2 u}{\partial p\,\partial r} - a_{23}\dfrac{1}{2}\dfrac{\partial a_{33}}{\partial p}}{a_{33}\sqrt{a_{33}}} - \frac{a_{33}\sum\dfrac{\partial u}{\partial q}\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{2}a_{23}\dfrac{\partial a_{33}}{\partial r}}{a_{33}^2\sqrt{a_{33}}}\,a_{13}\,,$$

$$e_{21} = \frac{a_{33}\sum\dfrac{\partial u}{\partial p}\dfrac{\partial^2 u}{\partial p\,\partial r} - a_{23}\dfrac{1}{2}\dfrac{\partial a_{33}}{\partial q}}{a_{33}\sqrt{a_{33}}} - \frac{a_{33}\sum\dfrac{\partial u}{\partial p}\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{2}a_{13}\dfrac{\partial a_{33}}{\partial r}}{a_{33}^2\sqrt{a_{33}}}\,a_{23}\,.$$

The direction cosines of the shortest distance that belongs to $\mathfrak{r} = \mathfrak{r}_1$ shall be denoted by $\kappa_1, \lambda_1, \mu_1$, while those of the one that belongs to $\mathfrak{r} = \mathfrak{r}_2$ shall be denoted by $\kappa_2, \lambda_2, \mu_2$. It will then follow that:

$$\kappa_1 = \frac{\xi_p + \xi_q\,\mathfrak{t}_1}{V_2}\,, \qquad \kappa_2 = \frac{\xi_p + \xi_q\,\mathfrak{t}_1}{V_1}\,,$$

in which V_1 means the square root of $\mathrm{H} + 2\Phi\,\mathfrak{t}_1 + \Psi\,\mathfrak{t}_1^2$ with the sign of $\zeta_p + \zeta_q\,\mathfrak{t}_1$, while V_2 means the square root of $\mathrm{H} + 2\Phi\,\mathfrak{t}_2 + \Psi\,\mathfrak{t}_2^2$ with the sign of $\zeta_p + \zeta_q\,\mathfrak{t}_2$.

The following equations exist between the quantities $\xi, \kappa_1, \kappa_2, \ldots$:

$$\xi = \delta_0\,(\lambda_1\,\mu_2 - \mu_1\,\lambda_2)\,, \qquad \eta = \delta_0\,(\mu_1\,\kappa_2 - \kappa_1\,\mu_2)\,, \qquad \zeta = \delta_0\,(\kappa_1\,\lambda_2 - \lambda_1\,\mu_2)\,,$$

in which δ_0 is taken to be equal to + 1 or – 1 according to whether $\kappa_1\,\lambda_2 - \lambda_1\,\kappa_2 > 0$ or < 0 , resp.

The plane that goes through the point (u, v, w), whose normal is $(\kappa_1, \lambda_1, \mu_1)$ [$(\kappa_2, \lambda_2, \mu_2)$, resp.], shall be called the *principal plane* of the tangent bundle that belongs to $\mathfrak{r}_1$ [$\mathfrak{r}_2$, resp.].

If one sets:

$$\vartheta_1 = \frac{(e_{11} + \mathfrak{r}_2\,\mathrm{H})\,dp^2 + 2\left(\dfrac{e_{12}+e_{21}}{2} + \mathfrak{r}_2\Phi\right)dp\,dq + (e_{22} + \mathfrak{r}_2\,\Psi)\,dq^2}{\mathfrak{r}_2 - \mathfrak{r}_1}\,,$$

$$\vartheta_2 = \frac{(e_{11} + \mathfrak{r}_1\,\mathrm{H})\,dp^2 + 2\left(\dfrac{e_{12}+e_{21}}{2} + \mathfrak{r}_1\Phi\right)dp\,dq + (e_{22} + \mathfrak{r}_1\,\Psi)\,dq^2}{\mathfrak{r}_2 - \mathfrak{r}_1}$$

then one will have:

(7) $$\mathfrak{r} = \frac{\mathfrak{r}_1\,\vartheta_1 + \mathfrak{r}_2\,\vartheta_2}{\vartheta_1 + \vartheta_2}.$$

If one now introduces the angles τ and ψ with the help of the equations (cf., $\mathfrak{U}$, §§ 14 and 3):

$$\cos\,\tau = \frac{\sum \kappa_2\,\xi_p}{\sqrt{\mathrm{H}}}, \qquad \cos\,(\tau - \psi) = \frac{\sum \kappa_2\,\xi_p}{\sqrt{\Psi}},$$
$$\sin\,\tau = \frac{\sum \kappa_1\,\xi_p}{\sqrt{\mathrm{H}}}, \qquad \sin\,(\tau - \psi) = \frac{\sum \kappa_1\,\xi_p}{\sqrt{\Psi}},$$

in which $\sqrt{\mathrm{H}}$ possesses the sign ζ_p and $\sqrt{\Psi}$ has that of ζ_q, then ϑ_1 and ϑ_2 will be given as squares in the following form:

$$\vartheta_1 = [\sqrt{\mathrm{H}}\cos\,\tau\,dp + \sqrt{\Psi}\cos\,(\tau - \psi)\,dq]^2,$$
$$\vartheta_2 = [\sqrt{\mathrm{H}}\sin\,\tau\,dp + \sqrt{\Psi}\sin\,(\tau - \psi)\,dq]^2,$$

such that a comparison of the two values of $\mathfrak{r}$ in (2) and (7) implies the relations:

(8) $$\begin{cases} e_{11} = -\mathrm{H}\,(\mathfrak{r}_1\cos^2\tau + \mathfrak{r}_2\sin^2\tau), \\ e_{12} + e_{21} = -2\sqrt{\mathrm{H}}\sqrt{\Psi}\,[\mathfrak{r}_1\cos\tau\cos(\tau-\psi) + \mathfrak{r}_2\sin\tau\sin(\tau-\psi)], \\ e_{22} = -\Psi\,[\mathfrak{r}_1\cos^2(\tau-\psi) + \mathfrak{r}_2\sin^2(\tau-\psi)]. \end{cases}$$

Finally, let us point out the equations:

$$\xi_p = \sqrt{\mathrm{H}}\,(\kappa_1\sin\,\tau + \kappa_2\cos\,\tau),$$

$$\xi_q = \sqrt{\mathrm{H}}\,[\kappa_1\sin\,(\tau - \psi) + \kappa_2\cos\,(\tau - \psi)],$$

$$\kappa_1 = \quad \xi_p\,\frac{\cos(\tau-\psi)}{\sqrt{\mathrm{H}}\sin\psi} - \xi_q\,\frac{\cos\tau}{\sqrt{\Psi}\sin\psi},$$

$$\kappa_2 = -\,\xi_p\,\frac{\sin(\tau-\psi)}{\sqrt{\mathrm{H}}\sin\psi} + \xi_q\,\frac{\sin\tau}{\sqrt{\Psi}\sin\psi}.$$

§ 2. – Representation of the differentials δu, $\delta^2 u$, $\delta\kappa_1$, $\delta\kappa_2$. Volume element.

When one applies the relations:

$$\sigma_1 = e_{11}\frac{\cos(\tau-\psi)}{\sqrt{\mathrm{H}}\sin\psi} - e_{21}\frac{\cos\tau}{\sqrt{\Psi}\sin\psi},$$

$$\sigma_2 = e_{12}\frac{\cos(\tau-\psi)}{\sqrt{\mathrm{H}}\sin\psi} - e_{22}\frac{\cos\tau}{\sqrt{\Psi}\sin\psi},$$

$$\sigma_3 = -e_{11}\frac{\sin(\tau-\psi)}{\sqrt{\mathrm{H}}\sin\psi} + e_{21}\frac{\sin\tau}{\sqrt{\Psi}\sin\psi},$$

$$\sigma_4 = -e_{12}\frac{\sin(\tau-\psi)}{\sqrt{\mathrm{H}}\sin\psi} + e_{22}\frac{\sin\tau}{\sqrt{\Psi}\sin\psi},$$

$$\mathfrak{S}_1 = \sigma_1\, dp + \sigma_2\, dq, \quad \mathfrak{S}_2 = \sigma_3\, dp + \sigma_4\, dq,$$

one will get the linear system for δu, δv, δw :

$$\sum \xi\,\delta u = 0, \qquad \sum \kappa_1\,\delta u = \mathfrak{S}_1, \qquad \sum \kappa_2\,\delta u = \mathfrak{S}_2,$$

such that:

(1) $$\delta u = \kappa_1\,\mathfrak{S}_1 + \kappa_2\,\mathfrak{S}_2 .$$

δF, $\delta^2 F$ shall always be understood to mean the first (second, resp.) differentials of a function F of the three variables p, q, r that are defined under the condition:

$$a_{13}\, dp + a_{23}\, dq + a_{33}\, dr = 0 .$$

It then follows that:

$$\delta\kappa_1 = \left(\frac{\partial\kappa_1}{\partial p} - \frac{\partial\kappa_1}{\partial r}\frac{a_{13}}{a_{33}}\right)dp + \left(\frac{\partial\kappa_1}{\partial q} - \frac{\partial\kappa_1}{\partial r}\frac{a_{23}}{a_{33}}\right)dq = \kappa_{1p}\, dp + \kappa_{1q}\, dq,$$

$$\delta\kappa_2 = \left(\frac{\partial\kappa_2}{\partial p} - \frac{\partial\kappa_2}{\partial r}\frac{a_{13}}{a_{33}}\right)dp + \left(\frac{\partial\kappa_2}{\partial q} - \frac{\partial\kappa_2}{\partial r}\frac{a_{23}}{a_{33}}\right)dq = \kappa_{2p}\, dp + \kappa_{2q}\, dq,$$

as well as:

$$\sum \xi\,\kappa_{1p} = -\sum \kappa_1\,\xi_p = -\sqrt{\mathrm{H}}\sin\tau,$$

$$\sum \xi\,\kappa_{2p} = -\sum \kappa_2\,\xi_p = -\sqrt{\mathrm{H}}\cos\tau,$$

$$\sum \kappa_1\,\kappa_{2p} = -\sum \kappa_2\,\kappa_{1p},$$

$$\sum \xi\,\kappa_{1q} = -\sum \kappa_1\,\xi_q = -\sqrt{\Psi}\sin(\tau-\psi),$$

$$\sum \xi\,\kappa_{2q} = -\sum \kappa_2\,\xi_q = -\sqrt{\Psi}\cos(\tau-\psi),$$

$$\sum \kappa_1\,\kappa_{2q} = -\sum \kappa_2\,\kappa_{1q}.$$

We now take:

$$\sum \kappa_2\,\kappa_{1p} = -U_1, \qquad \sum \kappa_2\,\kappa_{1q} = -U_2$$

and get from the foregoing that:

$$\kappa_{1p} = \kappa_2\,U_1 - \xi\sqrt{\mathrm{H}}\sin\tau, \qquad \kappa_{1q} = \kappa_2\,U_2 - \xi\sqrt{\Psi}\sin(\tau-\psi),$$

$$\kappa_{2p} = -\kappa_1\,U_1 - \xi\sqrt{\mathrm{H}}\cos\tau, \qquad \kappa_{2q} = -\kappa_1\,U_2 - \xi\sqrt{\Psi}\cos(\tau-\psi).$$

Now, if one sets:

$$U = U_1\,dp + U_2\,dq,$$

$$H_1 = \sqrt{\mathrm{H}}\cos\tau\,dp + \sqrt{\Psi}\cos(\tau-\psi)\,dq,$$

$$H_2 = \sqrt{\mathrm{H}}\sin\tau\,dp + \sqrt{\Psi}\sin(\tau-\psi)\,dq$$

then that will give:

(2) $$\delta\kappa_1 = \kappa_2\,U - \xi\,H_2, \qquad \delta\kappa_2 = -\kappa_1\,U - \xi\,H_1.$$

It now follows that:

$$\delta^2 u = \kappa_1(\delta\mathfrak{S}_1 - \mathfrak{S}_2\,U) + \kappa_2(\delta\mathfrak{S}_2 + \mathfrak{S}_1\,U) - \xi(\mathfrak{S}_1\,H_2 + \mathfrak{S}_2\,H_1).$$

Here, we would like to denote the coefficients of κ_1, κ_2, ξ by V_1, V_2, V_0, such that we will have:

(3) $$\delta^2 u = \xi\,V_0 + \kappa_1\,V_1 + \kappa_2\,V_2.$$

The representation (1) of the differentials $\delta u, \delta v, \delta w$ shall be applied to the expression for the spatial element.

As is known, it has the form:

$$\begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} & \frac{\partial v}{\partial r} \\ \frac{\partial w}{\partial p} & \frac{\partial w}{\partial q} & \frac{\partial w}{\partial r} \end{vmatrix} dp\, dq\, dr\,.$$

We would like to denote the reciprocal value of the determinant that appears here by ϑ. The determinant itself can be written:

$$\begin{vmatrix} u_p & u_q & \xi \\ v_p & v_q & \eta \\ w_p & w_q & \zeta \end{vmatrix} \cdot \sqrt{a_{33}}\,,$$

and with the use of the equations that (1) implies:

$$u_p = \kappa_1\,\sigma_1 + \kappa_2\,\sigma_3\,, \qquad u_q = \kappa_1\,\sigma_2 + \kappa_2\,\sigma_4\,,$$

it will assume the form:

$$\delta_0\,(\sigma_1\,\sigma_4 - \sigma_2\,\sigma_3)\sqrt{a_{33}}\,,$$

or from the defining equations for the quantities σ:

$$\delta_0\,\frac{e_{11}\,e_{22} - e_{12}\,e_{21}}{Q_\xi}\sqrt{a_{33}}\,,$$

in which we have set:

$$Q_\xi = \sqrt{\mathrm{H}}\sqrt{\Psi}\,\sin\psi\,.$$

Therefore, from (5), § 1, one gets the following expression for the determinant in question:

$$\delta_0\,Q_\xi\,\mathfrak{r}_3\,\mathfrak{r}_4\,\sqrt{a_{33}}\,,$$

and the following equation for ϑ:

$$\vartheta = \frac{\delta_0}{\mathfrak{r}_3\,\mathfrak{r}_4\,Q_\xi\sqrt{a_{33}}}\,.$$

If one is dealing with a ray system, so all of the curves of the family are straight lines, then *u*, *v*, *w* can be represented in the form:

$$u = x + r\,\xi\,, \qquad v = y + r\,\eta\,, \qquad w = z + r\,\zeta\,,$$

in which x, y, z (viz., the coordinates of the points on the surface) and ξ, η, ζ (viz., the direction cosines of the rays of the system) depend upon only p and q. Hence, $\sqrt{a_{33}}$ will be equal to 1. The factor δ_0 / Q_ξ does not change along a ray, and after one drops that factor, one will get Kummer's density measure $\bar{\vartheta} = 1 / \mathfrak{r}_3\, \mathfrak{r}_4$, which will coincide with the Gaussian curvature of a normal system for a surface r = const. when x, y, z is chosen such that $\sum \xi\, dx = 0$.

The area of a cross-section of a bundle of curves will be given by the absolute value of the expression:

$$(\sigma_1\, \sigma_4 - \sigma_2\, \sigma_3)\, dp\, dq = \mathfrak{r}_3\, \mathfrak{r}_4\, Q_\xi\, dp\, dq .$$

As is known, the surface element of r = const. has the value $\sqrt{\sum \left(\frac{\partial v}{\partial p} \frac{\partial w}{\partial q} - \frac{\partial v}{\partial q} \frac{\partial w}{\partial p} \right)^2}\, dp\, dq$ at the point (u, v, w), which can be expressed by $h\, \vartheta^{-1}\, dp\, dq$ with the help of the first Lamé differential parameter:

$$h = \sqrt{\left(\frac{\partial r}{\partial u}\right)^2 + \left(\frac{\partial r}{\partial v}\right)^2 + \left(\frac{\partial r}{\partial w}\right)^2} .$$

If one denotes the angle between the positive part of the tangent to the curve (p, q = const.) and the normal to the surface (r = const.) by χ then one will have:

$$\frac{dr}{h} = dr \sqrt{a_{33}} \cdot \cos \chi ;$$

i.e., dr / h is the projection of the element of that curve onto the normal that surface. $\chi = 0$ will imply the meaning of the quotient dr / h that Lamé gave.

§ 3. – Curvature of the orthogonal trajectories of a family of curves.

We would like to understand $\cos \alpha$, $\cos \beta$, $\cos \gamma$ ($\cos l$, $\cos m$, $\cos n$, resp., $\cos a$, $\cos b$, $\cos c$, resp.) to mean the direction cosines of the tangent (binormal, resp., principal normal, resp.) of a space curve at a regular point (u, v, w).

One then gets:

$$\cos \alpha = \frac{du}{\sqrt{du^2 + dv^2 + dw^2}} ,$$

in which the square root, which shall be denoted briefly by ds, is determined such that $\cos \gamma$ proves to be positive. (Cf., $\mathfrak{U}$, § 1)

Moreover, one will have:

$$\cos l = \frac{dv\,d^2w - dw\,d^2v}{\sqrt{\sum (dv\,d^2w - dw\,d^2v)^2}},$$

in which the square root, which will be called D, is determined in such a way that cos n will be positive. Finally, it emerges that:

$$\cos a = \varepsilon(\cos m \cos \gamma - \cos n \cos \beta),$$

where ε equals + 1 or – 1 according to whether:

$$\cos l \cos \beta - \cos m \cos \alpha$$

is greater than zero or less than zero, respectively.

The center of curvature (i.e., the point of intersection of the principal normal with the projection of the neighboring principal normal onto the osculating plane) has the abscissa ρ relative to (u, v, w). What then arises is:

$$\rho = \varepsilon \frac{ds^2}{D}.$$

The expressions considered shall now be formed for an orthogonal trajectory to a family of curves, such that $du\ d^2u$, … are replaced with $\delta u\ \delta^2 u$, …, resp.

In that way, one gets:

$$\cos \alpha = \frac{\kappa_1 \mathfrak{S}_1 + \kappa_2 \mathfrak{S}_2}{\sqrt{\mathfrak{S}_1^2 + \mathfrak{S}_2^2}}, \qquad \cos l = \delta_0 \frac{\xi(\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1) - (\kappa_2 \mathfrak{S}_1 - \kappa_1 \mathfrak{S}_2) V_0}{\sqrt{(\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1)^2 + (\mathfrak{S}_1^2 + \mathfrak{S}_2^2) V_0^2}},$$

$$ds = \sqrt{\mathfrak{S}_1^2 + \mathfrak{S}_2^2}, \qquad D = \sqrt{(\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1)^2 + (\mathfrak{S}_1^2 + \mathfrak{S}_2^2) V_0^2},$$

where the roots are determined from the condition cos $\gamma > 0$ (cos $n > 0$, resp.).

Furthermore:

$$\cos a = \xi \frac{(\kappa_2 \mathfrak{S}_1 - \kappa_1 \mathfrak{S}_2)(\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1) + \xi(\mathfrak{S}_1^2 + \mathfrak{S}_2^2) V_0}{\sqrt{(\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1)^2 + (\mathfrak{S}_1^2 + \mathfrak{S}_2^2)^2 V_0^2}},$$

$$\rho = \varepsilon \frac{\left(\sqrt{\mathfrak{S}_1^2 + \mathfrak{S}_2^2}\right)^3}{\sqrt{(\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1)^2 + (\mathfrak{S}_1^2 + \mathfrak{S}_2^2)^2 V_0^2}}.$$

The curvature axis is perpendicular to the osculating plane at the center of curvature and is parallel to the binormal. We would like to denote the coordinates of the point at which that curvature axis cuts the normal plane of the bundle of curves by u', v', w' and likewise denote the direction cosines of the line that goes through (u, v, w) and (u', v', w') by $\cos\bar{\alpha}$, $\cos\bar{\beta}$, $\cos\bar{\gamma}$. It then follows that:

$$u' - u = \frac{ds^2}{\mathfrak{S}_1 V_2 - V_1 \mathfrak{S}_2} (\kappa_2\, \mathfrak{S}_1 - \kappa_1\, \mathfrak{S}_2) ,$$

$$\cos\bar{\alpha} = \varepsilon' \frac{\kappa_2\, \mathfrak{S}_1 - \kappa_1\, \mathfrak{S}_2}{ds} ,$$

in which ds has the previous meaning, and ε' is taken to be equal + 1 or – 1 in order to make $\cos\bar{\gamma}$ positive. If we now take R to be the abscissa of the point (u', v', w') relative to the point (u, v, w) then we will have:

$$R = \varepsilon' \frac{ds^3}{\mathfrak{S}_1 V_2 - V_1 \mathfrak{S}_2} .$$

One can call R the *geodetic radius of curvature* of an orthogonal trajectory, and likewise, one can call an orthogonal trajectory whose coordinates satisfy the equation $\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1 = 0$ a *geodetic line.* For such a thing, one can show that it yields the shortest connection between two sufficiently close points among all orthogonal trajectories, as well as that it will be described by a point upon which no forces act that is constrained to move on an orthogonal trajectory of a family of curves.

The abscissa of the point at which the curvature axis in question cuts the ray (ξ, η, ζ) shall be called h. That will then imply that:

$$h = \frac{ds^2}{V_0} .$$

Since only the first differentials dp and dq enter into h, h will remain unchanged for all orthogonal trajectories with the same tangents, just as h possesses the meaning of the radius of curvature for the geodetic line with that tangent, since for $\mathfrak{S}_1 V_2 - \mathfrak{S}_2 V_1 = 0$, one will have:

$$\cos a = \xi , \qquad D = \varepsilon\, ds\, V_0 ,$$

such that ρ will then go to h. One then finds the relationship between h and ρ that is expressed by Meusnier's theorem for the radii of curvature of all planar sections of a surface that have the same tangent.

In order to discover the analogue of Euler's theorem, we represent V_0 as a quadratic form in $\mathfrak{S}_1$ and $\mathfrak{S}_2$. If one introduces the values of dp and dq that are expressed by $\mathfrak{S}_1$, $\mathfrak{S}_2$ and the quantities σ into the expressions for H_1 and H_2 then that will give:

$$H_1 = \frac{\mathfrak{S}_1[\sigma_4\sqrt{\mathrm{H}}\cos\tau - \sigma_3\sqrt{\mathrm{H}}\cos(\tau-\psi)] - \mathfrak{S}_2[\sigma_2\sqrt{\mathrm{H}}\cos\tau - \sigma_1\sqrt{\mathrm{H}}\cos(\tau-\psi)]}{\sigma_1\sigma_4 - \sigma_2\sigma_3},$$

$$H_2 = \frac{\mathfrak{S}_1[\sigma_4\sqrt{\mathrm{H}}\sin\tau - \sigma_3\sqrt{\mathrm{H}}\sin(\tau-\psi)] - \mathfrak{S}_2[\sigma_2\sqrt{\mathrm{H}}\sin\tau - \sigma_1\sqrt{\mathrm{H}}\sin(\tau-\psi)]}{\sigma_1\sigma_4 - \sigma_2\sigma_3}.$$

When one applies the defining equations for the quantities σ and equations (8), § 1, one will now get:

$$\sigma_4\sqrt{\mathrm{H}}\ \cos\tau - \sigma_3\sqrt{\mathrm{H}}\cos(\tau-\psi) = \frac{e_{12}-e_{21}}{2} = e_0,$$

$$\sigma_2\sqrt{\mathrm{H}}\ \cos\tau - \sigma_1\sqrt{\mathrm{H}}\cos(\tau-\psi) = \mathfrak{r}_0\, Q_\xi,$$

$$\sigma_4\sqrt{\mathrm{H}}\ \sin\tau - \sigma_3\sqrt{\mathrm{H}}\cos(\tau-\psi) = -\,\mathfrak{r}_1\, Q_\xi,$$

$$\sigma_2\sqrt{\mathrm{H}}\ \sin\tau - \sigma_1\sqrt{\mathrm{H}}\sin(\tau-\psi) = \frac{e_{12}-e_{21}}{2} = e_0.$$

Therefore:

$$H_1 = \frac{e_0\,\mathfrak{S}_1 - \mathfrak{r}_2\,Q_\xi\,\mathfrak{S}_2}{\mathfrak{r}_3\,\mathfrak{r}_4\,Q_\xi}, \qquad H_2 = \frac{-\,\mathfrak{r}_1\,Q_\xi\,\mathfrak{S}_1 - e_0\,\mathfrak{S}_2}{\mathfrak{r}_3\,\mathfrak{r}_4\,Q_\xi},$$

and

$$V_0 = -\,(\mathfrak{S}_1 H_2 + \mathfrak{S}_2 H_1) = \frac{\mathfrak{r}_1\mathfrak{S}_1^2 + \mathfrak{r}_2\mathfrak{S}_2^2}{\mathfrak{r}_3\,\mathfrak{r}_4},$$

such that:

$$h = \mathfrak{r}_3\,\mathfrak{r}_4\,\frac{\mathfrak{S}_1^2 + \mathfrak{S}_2^2}{\mathfrak{r}_1\mathfrak{S}_1^2 + \mathfrak{r}_2\mathfrak{S}_2^2}.$$

Therefore, the maximum h_1 of h belongs to the tangent $(\kappa_2, \lambda_2, \mu_2)$ and has the value $h_1 = \mathfrak{r}_3\,\mathfrak{r}_4 / \mathfrak{r}_2$, while the minimum h_2 of h belongs to the tangent $(\kappa_1, \lambda_1, \mu_1)$ and has the value $\mathfrak{r}_3\,\mathfrak{r}_4 / \mathfrak{r}_1$. If λ is the angle between the positive parts of the tangents that are determined by δu, δv, δw, and $\mathfrak{S}_2 = 0$ then one will have ([1]):

$$\frac{1}{h} = \frac{\sin^2\lambda}{h_1} + \frac{\cos^2\lambda}{h_2},$$

which is an equation that has the form of Euler's theorem.

([1]) The existence of that relation, which A. Voss mentioned in Math. Ann., Bd. 23, pp. 70, was probably first pointed out by Hamilton. The endpoint of h that differs from the point (u, v, w) is precisely what Hamilton called "focus by projection." (Trans. Roy. Irish Acad. v. XVI, Part I, Science, pp. 47)

If $\mathfrak{r}_1$ and $\mathfrak{r}_2$ have opposite signs, so the point (u, v, w) lies between the limit points of the shortest distance, then there will exist two orthogonal trajectories that yield infinite values of h, which will then play the role of asymptotic lines.

If the tangent bundle in question is a normal bundle then one will have:

$$e_0 = 0\,, \quad \mathfrak{r}_1 = \mathfrak{r}_3\,, \quad \mathfrak{r}_2 = \mathfrak{r}_4\,, \quad H_1 = -\frac{\mathfrak{S}_1}{\mathfrak{r}_1}\,, \quad H_2 = -\frac{\mathfrak{S}_1}{\mathfrak{r}_2}\,, \quad h_1 = \mathfrak{r}_1\,, \quad h_2 = \mathfrak{r}_2\,.$$

One must still consider the generalization of the concept of "conjugate tangents to a surface" ([1]). Let the direction cosines of the line perpendicular to the curve tangents (ξ, η, ζ) and $(\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta)$ be $\cos \alpha'$, $\cos \beta'$, $\cos \gamma'$, such that:

$$\cos \alpha' = \frac{\kappa_1 H_1 - \kappa_2 H_2}{\sqrt{H_1^2 + H_2^2}}\,,$$

in which the root is determined in such a way that $\cos \gamma'$ will be positive.

A line that is laid through the point (u, v, w) and whose direction cosines are $\cos \alpha'$, $\cos \beta'$, $\cos \gamma'$ might be called the *tangent adjoint to the tangent* $(\cos \alpha, \cos \beta, \cos \gamma)$.

If one expresses $\cos \alpha$ in terms of H_1 and H_2 then one will have:

$$\cos \alpha = \frac{\kappa_1 (e_0 H_1 - \mathfrak{r}_2 Q_\xi H_2) - \kappa_2 (\mathfrak{r}_1 Q_\xi H_1 + e_0 H_2)}{\sqrt{(e_0 H_1 - \mathfrak{r}_2 Q_\xi H_2)^2 + (\mathfrak{r}_1 Q_\xi H_1 + e_0 H_2)^2}}\,.$$

If the values of H_1 and H_2 that belong to $\cos \alpha'$, $\cos \beta'$, $\cos \gamma'$ are denoted by H_1' and H_2', resp., then that will imply the further expression for $\cos \alpha'$:

$$\cos \alpha' = \frac{\kappa_1 (e_0 H_1' - \mathfrak{r}_2 Q_\xi H_2') - \kappa_2 (\mathfrak{r}_1 Q_\xi H_1' + e_0 H_2')}{\sqrt{(e_0 H_1' - \mathfrak{r}_2 Q_\xi H_2')^2 + (\mathfrak{r}_1 Q_\xi H_1' + e_0 H_2')^2}}\,.$$

Let the tangent $(\cos \alpha'', \cos \beta'', \cos \gamma'')$ be adjoint to the tangent $(\cos \alpha', \cos \beta', \cos \gamma')$. It then follows that:

$$\cos \alpha'' = \frac{\kappa_1 H_1' - \kappa_2 H_2'}{\sqrt{H_1'^2 + H_2'^2}} = \frac{\kappa_1 (e_0 H_1 + \mathfrak{r}_2 Q_\xi H_2) - \kappa_2 (-\mathfrak{r}_1 Q_\xi H_1 + e_0 H_2)}{\sqrt{(e_0 H_1 + \mathfrak{r}_2 Q_\xi H_2)^2 + (\mathfrak{r}_1 Q_\xi H_1 - e_0 H_2)^2}}\,.$$

It then follows from this that when a tangent (2) is adjoint to a tangent (1), it is only when $e_0 = 0$ that one will also have that (1) is always adjoint to (2). (1) and (2) will then have the same relationship to each other that the conjugate tangents to a surface have.

([1]) Cf., A. Voss, Math. Ann., Bd. 23, pp. 46.

If we exclude the case of $e_0 = 0$ then (1) will be adjoint to (2) when (1) coincide with one of the two tangents, which will imply the equation:

$$\mathfrak{r}_1 H_1^2 + \mathfrak{r}_2 H_2^2 = 0 .$$

They will be real only when $\mathfrak{r}_1\,\mathfrak{r}_2$ is negative, and they thus prove to be the tangent to the two asymptotic lines of the family of curves that go through the point (u, v, w). The latter intersect those neighboring rays $(\xi, \xi + \delta\xi)$ whose shortest distance falls in the normal plane to the bundle of curves. However, since one now has:

$$\cos\alpha = \cos\alpha', \qquad \cos\beta = \cos\beta', \qquad \cos\gamma = \cos\gamma',$$

the tangents in question will be adjoint to each other.

The tangent (2) is perpendicular to (1) when the relation exists that (1):

$$e_0 (H_1^2 + H_2^2) + (\mathfrak{r}_1 - \mathfrak{r}_2)\, Q_\xi\, H_1\, H_2 = 0 ,$$

and since:

$$e_0^2 = (\mathfrak{r}_3\,\mathfrak{r}_4 - \mathfrak{r}_1\,\mathfrak{r}_2)\; Q_\xi^2 ,$$

it will decompose into the following one:

$$2\, e_0\, H_2 + Q_\xi\, [\mathfrak{r}_1 - \mathfrak{r}_2 \pm (\mathfrak{r}_3 - \mathfrak{r}_4)]\, H_1 = 0 .$$

The corresponding tangents (1) are therefore real only when the focal points of the bundle (ξ, η, ζ) are real, and they will intersect the neighboring rays to the ray (ξ, η, ζ) that yield those two focal points.

Finally, (2) will coincide with $(\kappa_1, \lambda_1, \mu_1)$ [$(\kappa_2, \lambda_2, \mu_2)$, resp.] when H_2 (H_1, resp.) vanishes. (1) will then meet those neighboring rays to (ξ, η, ζ) that belong to the values $\mathfrak{r}_1$ ($\mathfrak{r}_2$, resp.) of $\mathfrak{r}$, i.e., to the boundary points of the shortest distance.

If the tangent $(\cos\alpha', \cos\beta', \cos\gamma')$ defines the angles φ_1 and φ_2 with the tangents $(\kappa_1, \lambda_1, \mu_1)$ [$(\kappa_2, \lambda_2, \mu_2)$, resp.] then:

$$\cos\varphi_1 = \frac{H_1}{\sqrt{H_1^2 + H_2^2}}, \qquad \cos\varphi_2 = \frac{-H_2}{\sqrt{H_1^2 + H_2^2}},$$

and one will have:

$$\delta\cos\varphi_1 = \cos\varphi_2\, \frac{H_1\,\delta H_2 - H_2\,\delta H_1}{H_1^2 + H_2^2},$$

(1) That is the equation for Voss's "lines of curvature." I have not retained that terminology because the curves in question do not need to always be real.

$$-\delta \cos \varphi_2 = \cos \varphi_1 \frac{H_1\,\delta H_2 - H_2\,\delta H_1}{H_1^2 + H_2^2},$$

so when one sets:

$$\frac{H_1\,\delta H_2 - H_2\,\delta H_1}{H_1^2 + H_2^2} - U = S,$$

the following equation will arise:

$$\delta \cos \alpha' = S\,(\kappa_1 \cos \varphi_2 - \kappa_2 \cos \varphi_1).$$

One sees from this that neighboring adjoint tangents intersect. Let the abscissa of the point of intersection relative to the point (u, v, w) by $\bar{R}$. One then gets the relation:

$$\bar{R} = \frac{\mathfrak{S}_1 H_2 + \mathfrak{S}_2 H_1}{S\sqrt{H_1^2 + H_2^2}} = -\frac{\mathfrak{r}_1 \mathfrak{S}_1^2 + \mathfrak{r}_2 \mathfrak{S}_2^2}{\mathfrak{r}_3\,\mathfrak{r}_4\,S\sqrt{H_1^2 + H_2^2}} = -\frac{\mathfrak{S}_1^2 + \mathfrak{S}_2^2}{h \cdot S \cdot \sqrt{H_1^2 + H_2^2}}.$$

Let it be pointed out that:

$$H_1\,\delta H_2 - H_2\,\delta H_1 = 0$$

is the differential equation for those orthogonal trajectories of the family of curves whose adjoint tangents define constant angles with the tangent (κ_1, λ_1, μ_1). Along such a trajectory, one will have:

$$\bar{R} = \frac{\mathfrak{S}_1^2 + \mathfrak{S}_2^2}{h \cdot U \cdot \sqrt{H_1^2 + H_2^2}},$$

which is a value that will be denoted by $\mathfrak{R}$ in what follows.

§ 4. – Lines of curvature.

Those orthogonal trajectories to a family of curves that lie completely in principal planes ([1]) demand special attention. Such a curve, whose tangents possess the direction cosines κ_1, λ_1, μ_1 (κ_2, λ_2, μ_2, resp.), shall be called the *lines of curvature* that belong to $\mathfrak{r}_2$ ($\mathfrak{r}_1$, resp.). The one that belongs to $\mathfrak{r}_1$ will then imply the equation $\mathfrak{S}_1 = 0$, while the one that belongs to $\mathfrak{r}_2$ will imply the equation $\mathfrak{S}_2 = 0$.

We would like to understand an *isogonal* trajectory to the lines of curvature to mean an orthogonal trajectory of the family of curves whose tangents define constant angles with the tangents to the lines of curvature. The differential equation for those isogonal trajectories is then:

([1]) They overlap with the curves that A. Voss first mentioned in Math. Ann., Bd. 23, pp. 70 in § 5.

$$\mathfrak{S}_1\ \delta\mathfrak{S}_2 - \mathfrak{S}_2\ \delta\mathfrak{S}_1 = 0\ .$$

In order to simplify what follows, it is useful to express the quantity *U*, which took the form of a linear form in *dp* and *dq*, in terms of $\mathfrak{S}_1$, $\mathfrak{S}_2$, as well as H_1, H_2. One gets:

$$U = \frac{(U_1\,\sigma_4 - U_2\,\sigma_3)\,\mathfrak{S}_1 - (U_1\,\sigma_2 - U_2\,\sigma_1)\,\mathfrak{S}_2}{\sigma_1\,\sigma_4 - \sigma_2\,\sigma_3}\,,$$

$$U = \frac{1}{Q_\xi}[(-U_1\sqrt{\Psi}\sin(\tau-\psi) + U_2\sqrt{\mathrm{H}}\sin\tau\,H_1 + (U_1\sqrt{\Psi}\cos(\tau-\psi) - U_2\sqrt{\mathrm{H}}\cos\tau\,H_2]\,,$$

and one might set:

$$U = u_1\,\mathfrak{S}_1 + u_2\,\mathfrak{S}_2 = \frac{Q_{\kappa_1}H_1 - Q_{\kappa_2}H_2}{Q_\xi}\,.$$

We first take $\mathfrak{S}_1 = 0$. We will then have:

$$\cos\alpha = \kappa_2\,, \qquad ds = \mathfrak{S}_2\,,$$

$$\cos l = \delta_0\frac{\xi\,u_2 + \dfrac{\kappa_2}{h_1}}{\sqrt{u_2^2 + \left(\dfrac{1}{h_1}\right)^2}}\,, \qquad \cos a = \varepsilon_1\frac{\xi\cdot\dfrac{1}{h_1} - \kappa_1\,u_2}{\sqrt{u_2^2 + \left(\dfrac{1}{h_1}\right)^2}}\,,$$

$$\rho = \rho' = \varepsilon_1\frac{1}{\sqrt{u_2^2 + \left(\dfrac{1}{h_1}\right)^2}}\,, \qquad \varepsilon' = -1, \qquad R = R_2 = \frac{-1}{u_2}\,,$$

where:

$$\sqrt{u_2^2 + \left(\frac{1}{h_1}\right)^2} \quad \text{and} \quad \varepsilon_1 = \pm 1$$

are determined in such a way that cos *n* and cos *c* will prove to be positive.

If we next take $\mathfrak{S}_2 = 0$ then it will follow that:

$$\cos\alpha = \kappa_1\,, \qquad ds = \mathfrak{S}_1\,,$$

$$\cos l = \delta_0 \frac{\xi u_1 - \dfrac{\kappa_2}{h_2}}{\sqrt{u_1^2 + \left(\dfrac{1}{h_2}\right)^2}}, \qquad \cos a = \varepsilon_2 \frac{\xi \cdot \dfrac{1}{h_2} + \kappa_2 u_1}{\sqrt{u_1^2 + \left(\dfrac{1}{h_2}\right)^2}},$$

$$\rho = \rho' = \frac{\varepsilon_2}{\sqrt{u_1^2 + \left(\dfrac{1}{h_2}\right)^2}}, \qquad \varepsilon' = 1, \qquad R = R_1 = \frac{1}{u_1},$$

where:

$$\sqrt{u_1^2 + \left(\frac{1}{h_2}\right)^2} \quad \text{and} \quad \varepsilon_2 = \pm 1$$

are determined in such a way that cos *n* and cos *c* will take positive values.

Those formulas now imply that:

$$U = \frac{\mathfrak{S}_1}{R_1} - \frac{\mathfrak{S}_2}{R_2}, \qquad U_1 = \frac{\sigma_1}{R_1} - \frac{\sigma_3}{R_2}, \qquad U_2 = \frac{\sigma_2}{R_1} - \frac{\sigma_4}{R_2},$$

$$\rho' = \varepsilon_1 \frac{1}{\sqrt{\left(\dfrac{1}{R_2}\right)^2 + \left(\dfrac{1}{h_1}\right)^2}}, \qquad \rho'' = \varepsilon_2 \frac{1}{\sqrt{\left(\dfrac{1}{R_1}\right)^2 + \left(\dfrac{1}{h_2}\right)^2}}.$$

Moreover, since the geodetic radius of curvature of an isogonal trajectory of the lines of curvature possesses the value:

$$R = \varepsilon' \frac{ds}{U},$$

if c_1 and c_2 mean the cosines of the angles that the tangents to the isogonal trajectories make with the tangents $(\kappa_1, \lambda_1, \mu_1)$ [$(\kappa_2, \lambda_2, \mu_2)$, resp.] then this equation will follow:

$$R = \frac{\varepsilon'}{\sqrt{\dfrac{c_1}{R_1} - \dfrac{c_2}{R_2}}}.$$

The quantities R_1 and R_2 find further employment as a result of the following argument:

The family of curves in question is intersected by all of the trajectories that go through the point (*u*, *v*, *w*) in a doubly-infinite manifold of points. Each point of that manifold corresponds to a system of values $\xi, \eta, \zeta, \kappa_1, \lambda_1, \mu_1, \kappa_2, \lambda_2, \mu_2$. If one now draws the radii to the unit sphere that are parallel to the directions ξ, η, ζ [κ_1, λ_1, μ_1, resp., κ_2, λ_2, μ_2, resp.] then one will get the following values for the surface element of the sphere (ξ, η, ζ) [(κ_1, λ_1, μ_1), resp., (κ_2, λ_2, μ_2), resp.]:

$$\sqrt{\mathrm{H}\Psi - \Phi^2}\; dp\, dq\,,$$

$$\sqrt{\sum(\kappa_{1p})^2 \sum(\kappa_{1q})^2 - \left(\sum \kappa_{1p}\kappa_{1q}\right)^2} \cdot dp\, dq\,, \text{ resp.,}$$

$$\sqrt{\sum(\kappa_{2p})^2 \sum(\kappa_{2q})^2 - \left(\sum \kappa_{2p}\kappa_{2q}\right)^2} \cdot dp\, dq\,, \text{ resp.,}$$

in which the roots are taken to be positive.

The first of these roots is equal to the absolute value of Q_ξ. As for the remaining ones, it follows that:

$$\sum(\kappa_{1p})^2 \sum(\kappa_{1q})^2 - \left(\sum \kappa_{1p}\kappa_{1q}\right)^2 = \left[U_1\sqrt{\Psi}\sin(\tau - \psi) - U_2\sqrt{\mathrm{H}}\sin\tau\right]^2 = Q_{\kappa_1}^2\,,$$

$$\sum(\kappa_{2p})^2 \sum(\kappa_{2q})^2 - \left(\sum \kappa_{2p}\kappa_{2q}\right)^2 = \left[U_1\sqrt{\Psi}\cos(\tau - \psi) - U_2\sqrt{\mathrm{H}}\cos\tau\right]^2 = Q_{\kappa_2}^2\,,$$

and when one applies the expressions that were found for U_1 and U_2, the quantities Q_{κ_1}, Q_{κ_2} will now take the values:

$$Q_{\kappa_1} = \frac{e_0}{R_1} + \frac{\mathfrak{r}_1\, Q_\xi}{R_2}\,,$$

$$Q_{\kappa_2} = \frac{\mathfrak{r}_2\, Q_\xi}{R_1} - \frac{e_0}{R_2}\,.$$

In the case of $e_0 = 0$, one then gets:

$$\frac{Q_{\kappa_1}}{Q_\xi} = \frac{\mathfrak{r}_1}{R_2}\,, \qquad \frac{Q_{\kappa_2}}{Q_\xi} = \frac{\mathfrak{r}_2}{R_2}\,.$$

In addition to the two points on the tangents (κ_1, λ_1, μ_1) [(κ_2, λ_2, μ_2), resp.] whose abscissas are R_2 [R_1, resp.], we get two more distinguished points from the values of $\mathfrak{R}$ that were given at the end of § **3**. Namely, $H_2 = 0$ [$H_1 = 0$, resp.] are the equations of the orthogonal trajectories of the family of curves whose adjoint tangents coincide with the tangents (κ_1, λ_1, μ_1) [(κ_2, λ_2, μ_2), resp.]. Thus, from § **3**, the equations in question will yield a point on the tangent (κ_1, λ_1, μ_1) whose abscissa is:

$$\mathfrak{R} = R'' = \frac{\mathfrak{r}_1\, Q_\xi}{Q_{\kappa_1}}\,,$$

and a point on the tangent (κ_2 , λ_2 , μ_2) whose abscissa is:

$$\mathfrak{R} = R' = \frac{\mathfrak{r}_2\, Q_\xi}{Q_{\kappa_2}} .$$

Those relations then define the generalization of equations (2) and (4) that were presented in volume 31 of these Annalen on page pp. 87.

By a line of reasoning like the one that was pursued in *loc. cit.* for surfaces, we will see that R' and R'' are the abscissas of the focal points of certain ray bundles. One of those bundles will arise when one imagines drawing the lines:

$$(\kappa_1 , \lambda_1 , \mu_1) \quad \text{and} \quad (\kappa_1 + \delta\kappa_1, \lambda_1 + \delta\lambda_1, \mu_1 + + \delta\mu_1)$$

that correspond to each of the points:

$$(u, v, w) \quad \text{and} \quad (u + \delta u, v + \delta v, w + \delta w) ,$$

resp. In order to find their main properties, we set:

$$\sum \kappa_{1p}^2 = L_1 , \quad \sum \kappa_{1p}\, \kappa_{2q} = M_1 , \qquad \sum \kappa_{1q}^2 = N_1 ,$$

and

$$\sum \kappa_{1p}\, u_p = e_1 , \qquad \sum \kappa_{1p}\, \kappa_q = f_1 , \sum \kappa_{1q}\, \kappa_q = f_1' , \qquad \sum \kappa_{1q}\, u_q = g_1 ,$$

moreover. In that way, one gets:

$$e_1 = \sigma_3\, U_1 , \quad f_1 = \sigma_4\, U_1 , \quad f_1' = \sigma_3\, U_2 , \quad g_1 = \sigma_4\, U_2 .$$

Since:

$$f_1 - f_1' = \frac{\sigma_1\, \sigma_4 - \sigma_2\, \sigma_3}{R_1} ,$$

one can be dealing with a normal bundle only when R_1 is infinitely large, which will be excluded along with the vanishing of the quantity:

$$L_1\, N_1 - M_1^2 = Q_{\kappa_1}^2 .$$

The abscissa of one focal point will be zero, while that of the other will prove to be $\mathfrak{R}''$, which is a quantity that coincides with R_2 for $e_0 = 0$. The equation that implies the abscissas $\mathfrak{r}_1''$ and $\mathfrak{r}_2''$ for the limit points of the shortest distance from the ray to its neighboring ray assumes the form:

$$\mathfrak{r}''^2 - \mathfrak{R}''\,\mathfrak{r}'' - \frac{1}{4}\frac{h_2^2\,\mathfrak{R}''^2}{R_1^2} = 0\,,$$

such that:

$$\mathfrak{r}_1'' - \mathfrak{r}_2'' = \mathfrak{R}''\,h_2\sqrt{\frac{1}{R_1^2}+\frac{1}{h_1^2}}\,,$$

i.e., one has:

$$\rho'' = \frac{\mathfrak{R}''\,h_2}{\mathfrak{r}_1''-\mathfrak{r}_2''}\,,$$

up to sign.

One gets the quantity $\mathfrak{R}'$ as the abscissa of the focal point that does not coincide with the point (u, v, w) for the ray bundle (u, v, w, κ_2, λ_2, μ_2) in an analogous way, and when one denotes the abscissas of the limit points of the shortest distance by $\mathfrak{r}_1'$ and $\mathfrak{r}_2'$, it will once more follow that:

$$\rho' = \frac{\mathfrak{R}'\,h_1}{\mathfrak{r}_1'-\mathfrak{r}_2'}\,,$$

up to sign. The equations that were presented for ρ' and ρ'' define the generalization of the theorem that was published in vol. 31 of these Annalen on pp. 92 as (4).

§ 5. – The curves of the family are straight lines.

When the curves of the family considered are straight lines, the u, v, w can always be put into the form:

$$u = x + r\,\xi\,, \qquad v = y + r\,\eta\,, \qquad w = z + r\,\zeta\,,$$

in which x, y, z, ξ, η, ζ are functions of only p and q, and ξ, η, ζ have the same meanings as before. If one calculates the abscissa r of the point on a ray of the surface (x, y, z) and employs the Kummer relations:

$$e = \sum\frac{\partial x}{\partial p}\frac{\partial \xi}{\partial p}\,, \qquad f = \sum\frac{\partial x}{\partial q}\frac{\partial \xi}{\partial p}\,, \qquad f' = \sum\frac{\partial x}{\partial p}\frac{\partial \xi}{\partial q}\,, \qquad g = \sum\frac{\partial x}{\partial q}\frac{\partial \xi}{\partial q}$$

then one will have:

$$e_{11} = e + r\,\mathrm{H}, \quad e_{12} = f + r\,\Phi, \quad e_{21} = f' + r\,\Phi, \quad e_{22} = g + r\,\Psi\,.$$

If α denotes one of the numbers 1, 2, 3, 4 then one will further have:

$$\mathfrak{r}_\alpha = r_\alpha - r\,,$$

when one denotes the abscissas of the limit points of the shortest distance by r_1, r_2 and the abscissas of the focal points by r_3, r_4. At the same:

$$e = -\mathrm{H}\,(r_1 \cos^2 \tau + r_2 \sin^2 \tau)\,,$$

$$f + f' = -\,2\sqrt{\mathrm{H}}\sqrt{\Psi}\,[r_1 \cos \tau \cos(\tau - \psi) + r_2 \sin \tau \sin(\tau - \psi)]\,,$$

$$g = -\Psi\,[r_1 \cos^2(\tau - \psi) + r_2 \sin^2(\tau - \psi)]\,.$$

If one then sets:

$$s_1 = e\frac{\cos(\tau - \psi)}{\sqrt{\mathrm{H}}\sin\psi} - f'\frac{\cos\tau}{\sqrt{\Psi}\sin\psi}\,,$$

$$s_2 = f\frac{\cos(\tau - \psi)}{\sqrt{\mathrm{H}}\sin\psi} - g\frac{\cos\tau}{\sqrt{\Psi}\sin\psi}\,,$$

$$s_3 = -\,e\frac{\sin(\tau - \psi)}{\sqrt{\mathrm{H}}\sin\psi} + f'\frac{\sin\tau}{\sqrt{\Psi}\sin\psi}\,,$$

$$s_4 = -\,f\frac{\sin(\tau - \psi)}{\sqrt{\mathrm{H}}\sin\psi} + g\frac{\sin\tau}{\sqrt{\Psi}\sin\psi}$$

then one will have:

$$\sigma_1 = s_1 + r\sqrt{\mathrm{H}}\,\sin\tau\,, \qquad \sigma_2 = s_2 + r\sqrt{\Psi}\,\sin(\tau - \psi)\,,$$

$$\sigma_3 = s_3 + r\sqrt{\mathrm{H}}\,\cos\tau\,, \qquad \sigma_4 = s_4 + r\sqrt{\Psi}\,\cos(\tau - \psi)\,,$$

$$\mathfrak{S}_1 = s_1\,dp + s_2\,dq + r\,H_2\,, \qquad \mathfrak{S}_2 = s_3\,dp + s_4\,dq + r\,H_1\,.$$

One then gets the following expressions for the quantities $\mathfrak{R}'$ and $\mathfrak{R}''$:

$$\mathfrak{R}' = \frac{(r_2 - r)\,Q_\xi}{Q_{\kappa_1}}\,, \qquad \mathfrak{R}'' = \frac{(r_1 - r)\,Q_\xi}{Q_{\kappa_2}}\,,$$

such that the endpoints of the abscissas $\mathfrak{R}'$ and $\mathfrak{R}''$ define two straight lines that lie in the principal plane and go through the limit points of the shortest distance. Things are different with the quantities R_1 and R_2. Namely, one then has:

$$Q_{\kappa_1} = \frac{f - f'}{2R_1} + \frac{(r_1 - r)\,Q_\xi}{R_2}\,,$$

$$Q_{\kappa_2} = \frac{(r_2 - r)\,Q_\xi}{R_1} - \frac{f - f'}{2R_2},$$

such that when one considers the relations:

$$\frac{(f - f')^2}{4} = Q_\xi^2\,(r_3\,r_4 - r_1\,r_2)\,, \qquad r_1 + r_2 = r_3 + r_4$$

one will get:

$$R_1 = \frac{(r_3 - r)(r_4 - r)\,Q_\xi^2}{Q_{\kappa_1}\,\frac{f - f'}{2} + Q_{\kappa_1}\,Q_\xi\,(r_1 - r)},$$

$$R_2 = \frac{(r_3 - r)(r_4 - r)\,Q_\xi^2}{Q_{\kappa_1}\,Q_\xi\,(r_2 - r) - Q_{\kappa_2}\,\frac{f - f'}{2}}.$$

It follows from this that, except for the case of $f = f'$, the endpoints of the abscissas R_1 and R_2 define two hyperbolas that lie in the principal plane, each of which intersects an asymptote at the point:

$$r = \frac{Q_{\kappa_1}(f - f') + 2\,Q_{\kappa_2}\,Q_\xi\,r_1}{2\,Q_{\kappa_2}\,Q_\xi}$$

or

$$r = \frac{2\,Q_{\kappa_1}\,Q_\xi\,r_2 - Q_{\kappa_2}(f - f')}{2\,Q_{\kappa_1}\,Q_\xi}$$

resp., on the ray (x, h, z).

The expressions that were presented for the quantities $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\kappa_1$, $\delta\lambda_1$, $\delta\mu_1$, $\delta\kappa_2$, $\delta\lambda_2$, $\delta\mu_2$ must now be complete differentials. That implies nine relations that nonetheless reduce to three, as one easily sees:

$$Q_{\kappa_1} = -\frac{\partial\sqrt{\mathrm{H}}\cos\tau}{\partial q} + \frac{\partial\sqrt{\Psi}\cos(\tau - \psi)}{\partial p},$$

$$Q_{\kappa_2} = \frac{\partial\sqrt{\mathrm{H}}\sin\tau}{\partial q} - \frac{\partial\sqrt{\Psi}\sin(\tau - \psi)}{\partial p},$$

(1) $$Q_\xi = \frac{\partial U_1}{\partial q} - \frac{\partial U_2}{\partial p}.$$

When one considers the equations:

$$Q_{\kappa_1} = -\,U_1\sqrt{\Psi}\sin(\tau - \psi) + U_2\sqrt{\mathrm{H}}\sin\tau\,,$$
$$Q_{\kappa_2} = -\,U_1\sqrt{\Psi}\cos(\tau - \psi) + U_2\sqrt{\mathrm{H}}\cos\tau\,,$$

the first two of the former relations will imply the formulas:

$$U_1 = \frac{\partial(\tau - \psi)}{\partial p} - \frac{\dfrac{\partial\sqrt{\mathrm{H}}}{\partial q} - \dfrac{\partial\sqrt{\Psi}}{\partial p}\cos\psi}{\sqrt{\Psi}\sin\psi} ,$$

$$U_2 = \frac{\partial\tau}{\partial q} + \frac{\dfrac{\partial\sqrt{\Psi}}{\partial p} - \dfrac{\partial\sqrt{\mathrm{H}}}{\partial q}\cos\psi}{\sqrt{\mathrm{H}}\sin\psi} ,$$

which are derived directly in $\mathfrak{U}$, pp. 73.

Finally, we consider the case of $f = f'$, in which a normal system is present. It then follows that:

$$s_1 = -\, r_2\sqrt{\mathrm{H}}\sin\tau , \qquad s_2 = -\, r_2\sqrt{\Psi}\sin(\tau - \psi) ,$$
$$s_3 = -\, r_1\sqrt{\mathrm{H}}\cos\tau , \qquad s_4 = -\, r_1\sqrt{\Psi}\cos(\tau - \psi) ,$$

such that:

$$\sigma_1 = \sqrt{\mathrm{H}}\sin\tau\cdot(r - r_1) , \quad \sigma_2 = \sqrt{\Psi}\sin(\tau - \psi)(r - r_2) ,$$
$$\sigma_3 = \sqrt{\mathrm{H}}\cos\tau\cdot(r - r_2) , \quad \sigma_4 = \sqrt{\Psi}\cos(\tau - \psi)(r - r_1) .$$

The surfaces r = const. now define a family of parallel surfaces.

The expressions:

$$du = (\kappa_1\, \sigma_1 + \kappa_2\, \sigma_3)\, dp + (\kappa_1\, \sigma_2 + \kappa_2\, \sigma_4)\, dq , \ldots$$

that appear for an arbitrary surface of the family must be complete differentials here. In that way, three integrability conditions will come about, which nonetheless reduce to the following two:

$$\text{(2)} \qquad \begin{cases} \sigma_4 U_1 - \sigma_3 U_2 = \dfrac{\partial\sigma_2}{\partial p} - \dfrac{\partial\sigma_1}{\partial q} , \\[2ex] \sigma_2 U_1 - \sigma_1 U_2 = \dfrac{\partial\sigma_3}{\partial q} - \dfrac{\partial\sigma_4}{\partial p} . \end{cases}$$

If one denotes the determinant that appears here by – D then it will follow that:

$$-\, D\, U_1 = -\, \sigma_1\frac{\partial\sigma_2}{\partial p} + \sigma_1\frac{\partial\sigma_1}{\partial q} + \sigma_3\frac{\partial\sigma_3}{\partial q} - \sigma_3\frac{\partial\sigma_4}{\partial p} ,$$

$$-\, D\, U_2 = -\, \sigma_2\frac{\partial\sigma_2}{\partial p} + \sigma_2\frac{\partial\sigma_1}{\partial q} + \sigma_4\frac{\partial\sigma_3}{\partial q} - \sigma_4\frac{\partial\sigma_4}{\partial p} .$$

If one now takes:

$$du^2 + dv^2 + dw^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$

then one will have:

$$D^2 = E\,G - F^2$$

and:

$$-D\,U_1 = \frac{1}{2}\frac{\partial E}{\partial q} - \frac{1}{2}\frac{F}{G}\frac{\partial G}{\partial p} - D\,\frac{\partial \arctan \frac{\sigma_2}{\sigma_4}}{\partial p},$$

$$-D\,U_2 = -\frac{1}{2}\frac{\partial G}{\partial p} - \frac{1}{2}\frac{F}{G}\frac{\partial G}{\partial q} + \frac{\partial F}{\partial q} - D\,\frac{\partial \arctan \frac{\sigma_2}{\sigma_4}}{\partial q},$$

from which it is clear that equation (1) agrees with Liouville's expression for the curvature.

Let it be further remarked that equations (2) are identical to the relations (9) that were presented in 𝔘, pp. 18, when $r = 0$.

Bonn in February 1888.