"Les trois points de vue dans l'étude des espaces non holonomes," C. R. Acad. Sci. Paris **188** (1929), 973-976.

The three viewpoints in the study of non-holonomic spaces

Note [1] by **G. VRANCEANU**

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Translated by D. H. Delphenich

In a recent note [2], Horak showed that one can recover the curvature tensor on a non-holonomic manifold that Schouten gave by parallel translating an (internal) vector along an infinitesimal pentagon. I applied that pentagon method (in my Communication to the Congress at Bologna) and recovered my curvature tensor $\lambda_{hk,er}$ [3], which is different from Schouten's. The difference between the results illustrates the fact that the viewpoint that was adopted by Schouten in the study of non-holonomic spaces differs from my own, and I would like to show here how one can account for those two viewpoints in a very simple way, along with the viewpoint of Hadamard.

Suppose that we have a Pfaff system that is not generally completely integrable:

$$\sum_{a=1}^{n} c_{\alpha a}\, dx^a = 0 \qquad (\alpha = m + 1, \ldots, n) \tag{1}$$

on a Riemannian manifold V_n that has the variables $x^1, \ldots, x^n$ for coordinates. The manifold V_n, together with equations (1), is called a *non-holonomic manifold* V_n^m. In order to specify completely the properties of that manifold that are of interest to us, one last hypothesis is necessary.

Indeed, suppose, for the moment, that the system (1) is completely integrable. One knows that in that case, upon performing a suitable transformation of the variables x, one can give the system (1) the simple form:

$$dx^\alpha = 0 \qquad (\alpha = m + 1, \ldots, n). \tag{2}$$

Upon writing the metric on V_n in that case in the following way:

$$ds^2 = \sum_{i,j=1}^{m} a_{ij}\, dx^i dx^j - 2\sum_{i=1}^{m}\sum_{\alpha=m+1}^{n} a_{i\alpha}\, dx^i dx^\alpha + \sum_{\alpha,\beta=m+1}^{n} a_{\alpha\beta}\, dx^\alpha dx^\beta\,, \tag{3}$$

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[1] Session on 25 March 1929.
[2] Cf., Z. Horak, "Sur la courbure des variétés non holonomes," C. R. Acad. Sci. Paris **187** (1928), pp. 1273.
[3] See G. VRANCEANU, "Sur quelques tenseurs," C. R. Acad. Sci. Paris **186** (1928), pp. 995 [formula (61)].

one will see immediately that the integral manifolds of the system (2) define a family of ∞^{n-m} Riemannian manifolds V_m in V_n that have the first summation in formula (3) for their metrics.

One knows that one can pose the following two problems in regard to that family of Riemannian manifolds:

1. Find the intrinsic properties of each manifold V_m in the family.

It is obvious that those properties do not depend upon all of the coefficients $a_{i\alpha}$, $a_{\alpha\beta}$ ($i \leq m$, α, $\beta\gamma m$) of the metric (3) on V_n. One can express that fact by saying that the intrinsic properties of the family will remain the same when one considers the family to be embedded in another manifold V_n whose metric is obtained from (3) *by adding a form that is linear with respect to the left-hand side of equations* (2) *whose coefficients are arbitrary linear forms in all the dx.*

2. Find not only the intrinsic properties, but also the properties that are rigidly linked with the V_n.

In that case, the metric (3) will be an invariant of the problem.

One can add another problem to those two that is intermediate to them, but no less important, namely:

3. Determine the properties of the family that are common to not only the V_n, but also to all of the V_n whose metric is deduced from (3) *by adding an arbitrary quadratic form to the left-hand sides of equations* (2).

In that case, one immediately notes that if one chooses the variables x^i ($i \leq m$) to be orthogonal to the x^α ($\alpha > m$) then in order for one to have $a_{i\alpha} = 0$, they must remain orthogonal for all the V_n in that class. We already know a property of that nature (and I shall give some others in a later work): It is the one that is given by the Hadamard-Bompiani condition for the V_m to be totally geodesic manifolds in V_n ([1]).

Indeed, those conditions demand (in the case of $a_{i\alpha} = 0$) that the a_{ij} ($i, j \leq m$) are independent of the x^α, and if those conditions are fulfilled for the V_n then they will be fulfilled for all of the V_n in the same class.

One now poses those three problems in the case where the system (1) is not completely integrable. The first problem was already posed thirty years ago for the mechanical case of a non-holonomic system by Hadamard ([2]), while showing that the trajectories of the unforced system that are also geodesics of the corresponding non-holonomic manifold *will not remain the same* when one varies the *vis viva* of the system by an arbitrary linear form in the left-hand sides of the equations of non-holonomity.

([1]) See E. BOMPIANI, "Spazi riemanniani…," Rend. dei Lincei (5) **33** (1922), pp. 14.
([2]) See P. APPELL, "Les mouvements de roulement…," Scientia, Gauthier-Villars, pp. 47.

The second problem (viz., the one in which the metric on V_n is an invariant) was considered by Schouten, Synge ([1]), and Horak, whereas I considered the third problem (to be honest, I considered a problem that was equivalent to it and specified the geometry of a group of transformations of congruences), in which the metric on V_n can be varied by an arbitrary quadratic form in the left-hand sides of the equations of non-holonomity.

It should be pointed out that not only geodesics, but also some other remarkable geometric properties (e.g., parallelism, curvature, etc.), will no longer be invariants for the first problem as soon as the system (1) is not completely integrable; however, they will be invariants for the third problem.

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([1]) Cf., J. L. SYNGE, "Geodesics in non holonomic geometry," Math. Ann. **99** (1928), pp. 738.