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PARTIAL DIFFERENTIAL EQUATIONS

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I. – General properties of differential systems.

1. Existence of solutions. – A relation between the independent variables $x_1, \ldots, x_m$, the unknown functions $z_1, \ldots, z_n$ of those variables, and a finite number of their partial derivatives with respect to the x is called a *partial differential equation*. One refers to the order of the highest derivatives that appear in it as its *order*. A system of partial differential equations ([1]):

$$\varphi_k(x_1, \ldots, x_m, z_1, \ldots, z_m, z_{1,1,0\cdots 0}, \ldots, z_{i,\alpha_1\cdots\alpha_m} \ldots) = 0 \qquad (k = 1, 2, \ldots) \tag{1}$$

will be called, briefly, a *differential system*. We assume that the derivatives of z_i up to order r inclusive are included in equations (1), so the left-hand sides of those equations depend upon the variables:

$$x_1, \ldots, x_m, z_1, \ldots, z_n, z_{i,\alpha_1\cdots\alpha_m} \qquad \left(\sum \alpha \le r_i;\ i = 1, \ldots, n\right). \tag{2}$$

If there are n analytic functions $z_1, \ldots, z_n$ of the variables x that can be developed into an ordinary power series ([1.a])

$$z_i = \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} \frac{z^0_{i\alpha_1\cdots\alpha_m}}{\alpha_1! \cdots \alpha_m!} (x_1 - x_1^0)^{\alpha_1} \cdots (x_m - x_m^0)^{\alpha_m} \tag{3}$$

at a location $x_1^0, \ldots, x_m^0$ and satisfy the differential system (1) identically then they will be referred to collectively as an *integral* or a *solution* of (1), and (1) will be called an *integrable system*. Any solution of (1) also fulfills the infinitude of equations that one obtains when one repeatedly partially differentiates equations (1) with respect to the $x_1, \ldots, x_m$ under the assumption ([2]) that the z_i and its derivatives are functions of x.

The question of the existence of integrals that was posed and solved in some special cases by *A. Cauchy* ([3]) was resolved by *Sophie von Kowalevski* ([4]) for a differential system (1) that consists of n equations and can be solved in the form ([5]):

([1]) We have set $z_{i\alpha_1\cdots\alpha_m} \equiv \dfrac{\partial^{\alpha_1+\cdots+\alpha_m} z_i}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$.

([1.a]) From the standpoint of the analysis of *real* quantities, only a few special categories of differential problems have been investigated up to now (II A 7). An adaptation of the *Cauchy-Lipschitz* existence theorem (II A 4 a, no. 3) to arbitrary partial differential systems has still not been carried out yet.

([2]) In what follows, we shall call such differentiations "derivations," and the equations that arise in that way, the "derived equations" of (1).

([3]) Esp., C. R. Acad. Sci. Paris **15** (1842), pps. 44, 85, 141 = *Oeuvres* (1) **7**, pp. 17, 33, 62.

([4]) J. f. Math. **80** (1875), pp. 1; see also *G. Darboux*, C. R. Acad. Sci. Paris **80** (1875), pps. 101, 317; *L. Königsberger*, J. f. Math. **109**, pp. 261; *ibid.*, **112**, pp. 181; Math. Ann. **42**, pp. 485; *P. Stäckel*, J. f. Math. **119**, pp. 339.

([5]) For $n = 1$, that form can always be exhibited, possibly by a complete linear transformation of the x, but not for $n > 1$; see, *C. Bourlet*, Ann. éc. norm. sup. (1891), Suppl., pp. 48.

$$\frac{\partial^{r_i} z_i}{\partial x_1^{r_i}} = U_i\,(x_1, \ldots, x_m, z_1, \ldots, z_m, \ldots, z_{k,\beta_1\cdots\beta_m} \cdots) \qquad (i = 1, \ldots, n). \tag{4}$$

For an algebraic system ([6]), one conveniently appeals to a generalization of the normal form that *C. Weierstrass* used for ordinary algebraic differential equations ([7]) in order to achieve that solution. Equations (4) possess one and only one system of regular integral functions z_i at the location $x_1^0, \ldots, x_m^0$ such that z_i, $\frac{\partial z_i}{\partial x_1}$, $\frac{\partial^2 z_i}{\partial x_1^{\,2}}$, …, $\frac{\partial^{r_i-1} z_i}{\partial x_1^{\,r_i-1}}$ go to regular, but otherwise arbitrary, functions χ_i, $\chi_i^{(1)}, \ldots, \chi_i^{(r_i-1)}$ for $x_1 = x_1^0$ (at the location $x_2^0, \ldots, x_m^0$, resp.). In so doing, one assumes that the functions U_i are all regular at the location:

$$x_1^0, \ldots, x_m^0, (\chi_1)^0, \ldots, (\chi_m)^0, \; z^0_{k,\beta_1,\ldots,\beta_m} = \left(\frac{\partial^{\beta_2+\cdots+\beta_m}\, \chi_k^{(\beta_1)}}{\partial x_2^{\beta_2} \cdots \partial x_m^{\beta_m}}\right)^0 .$$

Namely, with the "initial conditions" that were posed, one will know all of the coefficients $z^0_{i,\alpha_1,\ldots,\alpha_m}$ in the series development (3) for which $\alpha_1 < r_i$. Equations (4) and their successive derivatives will then yield all of the remaining coefficients. The power series (3) is then formally determinate ([8]). From a principle that goes back to *Cauchy* ("Calcul des limites," cf., footnote 3 and II A 4 a, no. **9**) its convergence will be proved when one replaces the system (1) with a similar one that is integrable in closed form and shows the existence of system of integral functions for it that are developable in powers of the quantities $x_h - x_h^0$ and whose coefficients have moduli that are all greater then the corresponding ones in the series (3).

2. Continuation. Passive systems. – The question of the existence of solutions of a system (1) with arbitrarily-many equations, unknowns $z_1, \ldots, z_m$, and independent variables $x_1, \ldots, x_m$ was first solved under special assumptions by *C. Méray* and *C. Riquer* ([9]), as well as *C. Bourlet* ([10]), and then in general by *Riquier* ([11]) and *A. Tresse* ([12]).

According to *Tresse*, any system (1) can take on a *canonical form* by solving for certain derivatives (2) in which only the equations:

([6]) *Königberger* [J. f. Math. **111**, pp. 1, 156; Berl. Ber. (1894), pp. 989; Math. Ann. **20**, pp. 587, *ibid.* **39**, pp. 285] had adapted the concept of irreducibility in algebra to such systems. *J. Drach* [Ann. éc. norm. sup. (1898), pp. 243] based his "logical integration" of an algebraic differential system, which was patterned on *Galois* theory, on a (far-reaching) definition of that concept for arbitrary algebraic differential systems; cf., II A 4 b, nos. **36**, **38**.

([7]) *Kowalevski*, *loc. cit.*; *Königberger*, J. f. Math. **109**, pp. 263.

([8]) The fact that the convergence is not guaranteed by that alone when the system does not possess the form (4) was shown by *S. v. Kowalevski* in an example (*loc. cit.*, pp. 22).

([9]) Ann. éc. norm. sup. (1890), pp. 23.

([10]) Cf., footnote 5.

([11]) Ann. éc. norm. sup. (1893), pps. 65, 123, 167; Paris sav. [étr.] **32**.

([12]) Acta math. **18** (1894), pp. 1.

(5) $$z_{i,\alpha_1\cdots\alpha_m} = \psi_{i,\alpha_1\cdots\alpha_m}(x_1, \ldots, x_m, z_1, \ldots, z_m, z_{k,\beta_1\cdots\beta_m})$$

with the following character appear: None of the quantities $z_{k,\beta_1\cdots\beta_m}$ that enter into the right-hand side appear on the left-hand side. For each of the quantities $z_{k,\beta_1\cdots\beta_m}$ that occur in $\psi_{i,\alpha_1\cdots\alpha_m}$, one has $\sum \beta_i \leq \sum \alpha_i$. If the equality sign is true then $k \geq i$, and in the case $k = i$, the first-nonvanishing difference $\beta_1 - \alpha_1$, $\beta_2 - \alpha_2$, … will be positive. *Riquier* ([13]) gave another type of solution that includes the foregoing one as a special case. *Riquier* called the quantities ([14]) on the left-hand side of equations (5) and their infinitude of derivatives with respect to the *x principal*, and all of the remaining quantities $z_{k,\beta_1\cdots\beta_m}$ *parametric*. Each of the former can be expressed as functions of the parametric quantities and the *x* by means of the system (5) and its derivatives, but possibly in several different ways. The system (1) or (5) is called *passively involutory (Méray-Riquier)* or an *involutory system (Lie)* when those different representations are identical for every principal quantity. In order for that to be true, it is necessary and sufficient that this should be true for a certain finite number of principal derivatives ([15]). The conditions of passivity then find their expression in a system of partial differential equations that the $\psi_{i,\alpha_1\cdots\alpha_m}$ have to satisfy as functions of the *x* and the parametric quantities, or also, for the unsolved form (1) of the differential system, in a number of partial differential equation that the functions φ_k must satisfy as a result of (1).

In order for a system (1) with *n* unknowns and $p \leq n$ equations to be passive, it is sufficient that certain functional determinants (I B 1 b, no. **20**) of the φ_k with respect to the highest derivatives do not vanish ([16]) because of (1), and $n - p$ of its integral functions z_i can then be taken to be arbitrary, in general.

If a differential system *S* that has been put into the canonical form (5) is not passive then a system S' will follow from each single derivation of equations (5) with respect to $x_1, \ldots, x_m$ that possibly yields two different representations for some of the derivatives on the left-hand side. A comparison of it will lead to new relations S''. If one gives the differential system (S, S', S'') the canonical form and proceeds with it as one did with *S*, etc., then after a finite number of steps, one will arrive at either a contradiction, namely, possible relations between the *x* alone, and *S* will not be integrable then, or to a passive system whose integration is reduced to that of *S*.

A passive system (5) possesses one and only one system of regular integral functions $z_1, \ldots, z_n$ at the location $x_1^0, \ldots, x_m^0$ that have the property that the parametric quantities $z_{k,\beta_1\cdots\beta_m}$ reduce to arbitrarily-chosen constants $z^0_{k,\beta_1\cdots\beta_m}$ for $x_1 = x_1^0, \ldots, x_m = x_m^0$. In that way, one assumes that the ψ behave regularly at the location $x_1^0, \ldots, x_m^0, \ldots, z^0_{k,\beta_1\cdots\beta_m}, \ldots$ and that the arbitrarily-chosen series:

([13]) Ann. éc. norm. sup. (1893), pp. 66; He called his canonical form *harmonic* or *orthonormal*.

([14]) One can also find quantities $z_{k,0,\ldots,0} \equiv z_0$ among them.

([15]) *Riquier*, Ann. éc. norm. sup. (1893), pp. 77, *et seq.*

([16]) Cf., e.g., the article by *Stäckel* that was cited in footnote 4.

$$\sum_{\beta_1,\ldots,\beta_m} \frac{z^0_{k,\beta_1\cdots\beta_m}}{\beta_1!\cdots\beta_m!}(x_1-x_1^0)^{\beta_1}\cdots(x_m-x_m^0)^{\beta_m}$$

possesses a finite domain of convergence. In fact, the power series (3) is formally determined uniquely by the assumptions that were made and with the help of equations (5) and their derivatives. *Riquier* ([17]) carried out the proof of convergence using *Cauchy*'s *calcul des limites* by first reducing the given differential system to a differential system of a special form ("système franc") by introducing certain derivatives of the z_i as the new unknowns. *E. Delassus* ([18]) achieved the same thing for a modification of *Tresse*'s normal form (5) that he gave by reducing the problem to a series of *Kowalevski* systems (4). One can give a form to the initial condition into which only a finite number of constants and arbitrary functions of the x enter ([19]).

From the above, the *Cauchy problem* (*Darboux*), i.e., the problem of obtaining an integral by means of the initial conditions that are necessary and sufficient for its determination, can always be solved by a series development. For the latter, up to now, it is only in some special classes of *linear* ([20]) differential systems and under certain assumptions that one can give limits on convergence and carry out analytic continuations that are independent of the choice of initial conditions ([21]).

3. Mayer systems. – If all derivatives of the z_i from a certain order s on can be expressed in terms of the x, z and their derivatives up to order $s - 1$ by means of the passive system (5) and its derivatives, in other words, if the number of parametric quantities is finite ($= \nu$), then the most general solution will depend upon a finite number ν of arbitrary constants, and (1) or (5) will be called a *Mayer system* (cf., no. **16**). Conversely, any system of functions z_i into which a finite number of parameters enter will be the most general solution of one and only one *Mayer* system.

From the above, the most-general solution of any other passive system will contain an unbounded, uncountable (I A 5, no. **2**) set of arbitrary constants.

4. The general integral. – *A. M. Ampère* ([22]) called a system of functions $z_1, \ldots, z_n$ that is defined by a system of equations (K) between the variables $x_1, \ldots, x_m, z_1, \ldots, z_n$ that depends upon parameters and arbitrary functions in any way the *general integral* of a differential system (1) when it fulfills no other differential equations besides equations (1) and its derivatives (as long as the arbitrary elements are not subject to condition equations), i.e., when all relations (1) and its derivatives, and only them, can be obtained from (K) by infinitely-repeated derivation with respect to the x and elimination of the arbitrary elements. *Darboux* ([23]) replaced that requirement with the

([17]) Ann. éc. norm. sup. (1893), pp. 123.
([18]) Ann. éc. norm. sup. (1896), pp. 421.
([19]) *Delassus, loc. cit.*; *Riquier*, Ann. éc. norm. sup. (1897), pp. 259.
([20]) I. e., ones that are linear with respect to the unknowns and their derivatives.
([21]) *J. Horn*, Acta math. **12**, pp. 113; **14**, pp. 337; *Königsberger*, J. f. Math. **112**, pp. 199.
([22]) J. éc. polyt. **10**, cah. **17** (1815), pp. 549; cf., *Imschenetsky*, Arch. Math. **54**, pp. 209, Chap. 1.
([23]) *Surfaces* 2, pp. 98.

far-reaching one that every solution of the *Cauchy* problem should be obtained from (K) by specializing the arbitrariness that is contained in it ([24]). An integral that is general in that sense is also general in the sense of *Ampère*, but not conversely ([25]). One often refers to the relations (K) themselves as the *general integral* (or also: the *general integral equations*) of the differential system (1). Any solution $z_1, \ldots, z_n$ that is obtained from the general integral equations (K) establishing condition equations for the arbitrary elements is called a *particular integral*.

For a partial differential equation of order n with one unknown z and two independent variables x, y, in the simplest case, the general integral is defined by $n + 1$ relations between x, y, z, n variables $\rho_1, \ldots, \rho_n$, and n groups of ν_i functions:

$$\varphi_{i1}(\rho_i),\ \varphi_{i2}(\rho_i),\ \ldots,\ \varphi_{i,\nu_i}(\rho_i) \qquad (i = 1, \ldots, n)$$

[in which the functions φ_{ik} in each group each satisfy a system of $\nu_i - 1$ ordinary differential equations with the independents ρ_i ([26]) such that just *one* function from each group will remain arbitrary], as well as a finite number of derivatives of the φ. *Ampère* ([27]) combined the differential equations with that integral form (e.g., all first-order partial differential equations with one unknown and two independent variables) into a *first class*.

Arbitrary functions can also occur in the general integral equations of a differential system (1) in the form of *partial quadratures* whose integrand includes other variables besides the integration variables ([28]). The theory of linear differential equations offers numerous examples of that ([29]). Thus, following an idea of *Delassus* ([30]), *É. Borel* represented the set of all integrals that are regular in a certain domain of a partial differential equation with m independent variables x that is linear with respect to z and its derivatives in terms of a single expression $\int_0^{2\pi} f(x_1, \ldots, x_m, \alpha)\,\varphi(\alpha)\,d\alpha$ that depended upon the arbitrary function φ.

5. Singular integrals. – A system of functions:

$$z_i = f_i(x_1, x_2, \ldots, x_m) \qquad (i = 1, 2, \ldots, n) \tag{6}$$

([24]) Meanwhile, different formal systems (K) might be necessary for different domains of values for the x in order for that to be true.

([25]) *Delassus*, Darb. Bull. **1** (1895), pp. 37; *Goursat B* **2**, art. 178.

([26]) According to *M. Lévy* [C. R. Acad. Sci. Paris **75** (1872), pp. 1094] the numbers ν_i can reduce to 2 or 1. Some of the arguments of the φ_{ik} can also be identical. Cf., also *Goursat B* **2**, Chap. 8, where somewhat-more-general integral forms were considered (no. **51**).

([27]) *Loc. cit.*, pp. 568; his definition is somewhat narrower in scope.

([28]) *Ampère*, loc. cit., pp. 557.

([29]) Cf., e.g., *B. Brisson*, J. éc. Poly. **7**, cah. 14 (1808), pp. 191; *S. D. Poisson*, *ibidem* 12, cah. 19, (1823), pp. 215); *A. Weiler*, J. f. Math. **51** (1856), pp. 105, and other places.

([30]) Darb. Bull. **1** (1895), pp. 122; *Delassus*, *ibidem*, pp. 37.

is called a *singular integral* of the differential system S when equations (6) define nothing but systems of values $x_1, x_2, \ldots, x_m, z_1, \ldots, z_{i,\alpha_1\cdots\alpha_m}, \ldots$ that indeed fulfill equations (1), but for which the right-hand sides of no possible canonical solution (5) of the system S is regular without exception. Therefore, a singular integral cannot be obtained directly by the method in no. **2** with the help of series development. However, there is always a system of equations S_1 in the variables (2) that subsumes the relations S, will be obtained from them by certain differentiations with respect to the variables $z_{i,\alpha_1\cdots\alpha_m}$, and will be satisfied by all possible singular integrals S, and only them ([31]). $S1$ can itself again possess singular integrals, etc., and one will thus arrive at the distinction between simple, double, … singular integrals ([32]).

If one starts with a differential system S of order n with *one* unknown z and replaces the latter with $z + \varepsilon z'$, and correspondingly replaces $z_{\alpha_1\cdots\alpha_m}$ with $z_{\alpha_1\cdots\alpha_m} + \varepsilon z'_{\alpha_1\cdots\alpha_m}$ and develops in powers of ε then upon setting all coefficients of the first powers of ε equal to zero, what will result is the *auxiliary system* S', which are n^{th}-order partial differential equations in z' and its derivatives that defines all solutions of S that are infinitely-close to a given solution z ([33]). z will be a singular solution ([34]) when the associated equations S' either possess a general integral z' that has a lower degree of generality than an arbitrary solution z or do not remain independent of z' and its derivatives, and are possibly fulfilled identically ([35]). The conditions for that, together with S, will produce the previously-defined differential system S_1, and for *one* n^{th}-order partial differential equation:

$$f(x_1, \ldots, x_m, z, \ldots, z_{\alpha_1\cdots\alpha_m}, \ldots) = 0 \qquad \left(\sum \alpha_i \leq n\right) \tag{7}$$

it will read as follows:

$$f = 0\,, \quad \frac{\partial f}{\partial x_i} + \sum_{\beta_1\cdots\beta_m}^{0,1,\cdots,n-1} z_{\beta_1\cdots\beta_{i+1}\cdots\beta_m} \frac{\partial f}{\partial z_{\beta_1\cdots\beta_m}} = 0\,, \quad \frac{\partial f}{\partial z_{\alpha_1\cdots\alpha_m}} = 0$$

$$(i = 1, \ldots, m\,;\ \sum \beta < n\,;\ \alpha_1, \alpha_2, \ldots = 0, 1, \ldots, n\,;\ \sum \alpha = n).$$

A singular integral will not generally take on any form by specialization. However, there can be solutions that are singular, as well as particular.

6. Intermediate integrals. – One refers to a differential system S' with the same unknown and independent variables as a differential system S as an *intermediate integral* when the general

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([31]) See, e.g., *Königsberger*, J. f. Math. **109**, pp. 290.

([32]) *Delassus*, art. 22.

([33]) *Darboux*, C. R. Acad. Sci. Paris **96**, pp. 766; *Surfaces* 4, Note XI.

([34]) This definition of singular solutions was given by *Poisson*, J. éc. polyt. **7**, cah. 13 (1806), pp. 114 as a generalization of an idea of *A. M. Legendre* [Paris mém. (1790), pp. 218]; cf., *Boole*, Suppl. vol., pp. 70

([35]) The fact that this will happen for any integral z can be avoided by an algebraic transformation of S.

integral of S' also depends upon a finite ([36]) number of constants and fulfills the system S, without being identical to its general integral. S will then be an algebraic consequence of S' and its derivatives. One also frequently calls only the equations of a differential system S' with the stated properties that do not follow from S an "intermediate integral." A differential system does not necessarily need to possess intermediate integrals.

If the relations S' include enough parameters and arbitrary functions that any non-singular integral of S fulfills at least *one* particular form of the system S' then S' will be called a *general intermediate integral*.

A partial differential equation of order ν with *one* unknown is an intermediate integral [or $(n - \nu)^{\text{th}}$ *integral*] of an equation of the form (7) in the case $\nu < n$ when it is fulfilled identically by means of the ν^{th}-order equation and its derivatives. All possible k^{th} integrals of an equation (7) will be ascertained by integrating a system of k^{th}-order partial differential equations with *one* unknown ([37]). In particular, one will obtain all possible first integrals of (7) by seeking the common integrals of a system of first-order partial differential equations in which the quantities x_i, z, and the derivatives of z up to order $n - 1$ inclusive figure as independent variables ([38]).

7. Complete integrals. – If the differential system (1) includes the derivatives of z_i up to order r_i inclusive then one understands a *complete integral* of the system (1) to mean a system of functions:

$$z_i = f_i\,(x_1, x_2, \dots, x_m, c_1, c_2, \dots, c_\nu) \qquad (i = 1, 2, \dots, n) \tag{8}$$

that depends upon the arbitrary constants $c_1, \dots, c_\nu$ and has the property that eliminating the c_i from (8) and the equations:

$$z_{i,\alpha_1\cdots\alpha_m} = \frac{\partial^{\alpha_1+\cdots+\alpha_m}}{\partial x_1^{\alpha_1}\cdots\partial x_m^{\alpha_m}}\, f_i \qquad \left[\sum \alpha \le r_i\,; i = 1, \dots, n\right] \tag{9}$$

will produce all of the relations (1), and only them. The number n of arbitrary constants is then equal to the number of quantities:

$$z_1, \dots, z_n,\ z_{i,\alpha_1\cdots\alpha_m} \qquad (\alpha_1, \dots, \alpha_m = 1, \dots, r_i\,;\ \sum \alpha \le r_i\ ;\ i = 1, 2, \dots, n), \tag{10}$$

minus the number of equations (1) ([39]).

([36]) From time to time, it is useful to drop that restriction, e.g., in order to speak of the intermediate integrals of a *Mayer* system. *Delassus*, Ann. éc. norm. sup. (1897), pp. 195.

([37]) *A. V. Bäcklund*, Math. Ann. **11** (1877), pp. 240.

([38]) For intermediate integrals of an n^{th}-order equation that have order higher than n, resp., cf., e.g., no. **50**.

([39]) When n functions z_i with $m \cdot n$ arbitrary constants will define the complete integral of a system of n first-order partial differential equations with n unknowns z_i and m independent variables was shown by *Königsberger*, Math. Ann. **44**, pp. 17.

The concepts of "complete" and "general" integral will coincide for a *Mayer* system (no. **3**), and only for such a thing. Any other system in involution (no. **2**) S will possess infinitely-many different complete systems. If one adds the derived equations of S up to any order to S then that will produce a system in involution S' whose complete integral will include more than ν arbitrary constants. A complete integral of that system in involution that arises from the differential equation (7) by adding all of its derivatives up to order $n + k$ inclusive will also be referred to as *complete integral of rank k* of the equation (7)([40]). A complete integral of rank k of the second-order differential equation:

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\,\partial y}, \frac{\partial^2 z}{\partial y^2}\right) = 0$$

will then include $2k + 5$ arbitrary constants ([40]).

Lie ([41]) called a differential system that possesses at least *one* complete integral *unrestricted integrable:* Such a system (1) is characterized by the fact that it is included in a system in involution that subsumes no equation in the variables (2) alone besides (1).

8. Different forms for the most general differential system. – Any differential system S can be replaced with a differential system S' with the same independent variables x_i such that only the *first* derivatives include the unknowns in such a way that one considers certain derivatives $z_{i,\alpha_1\cdots\alpha_m}$ to be further unknowns. When one adds the derivatives of S up to a certain order to S and then introduces certain derivatives of the z_i as further unknowns, S can be replaced, in particular, by a differential system that is *linear* in the first derivatives:

$$\sum_{k=1}^{m}\sum_{s=1}^{p} a_{iks}\frac{\partial u_s}{\partial x_k} + a_i = 0 \qquad (i = 1, 2, \ldots), \tag{11}$$

in which $u_1, \ldots, u_p$ mean the unknowns, and the a_i, a_{iks} are functions of $x_1, \ldots, x_m, u_1, \ldots, u_p$. *Riquier* showed that any arbitrary passive system S can be replaced by a passive system of the form (11) ([42]).

According to *Riquier* ([11]), one can further replace any passive system S with the unknowns $z_1, \ldots, z_n$ with n differential systems $S_1, S_2, \ldots, S_n$ in such a way that S_ν include only the x_i and the unknowns $z_1, \ldots, z_{\nu-1}$ and their derivatives, and will be converted into a passive system with the *one* unknown z_ν after substituting an arbitrary integral $z_1, \ldots, z_{\nu-1}$ of the differential system ($S_1, S_2, \ldots, S_{\nu-1}$). The integration of S is thereby reduced to a series of differential systems with m independent variables and *one* unknown in each case.

([40]) *J. König*, Math. Ann. **24**, pp. 505; cf., *Jacobi*'s integrals with "excess" constants, *Werke* 5, pp. 404.

([41]) Leipziger Ber. (1895), pp. 71.

([42]) Ann. éc. norm. sup. (1893), pp. 359. *Bourlet* also considered systems of that form; cf., footnote 5, as well as the special cases that were treated by *Bäcklund*, Math. Ann. **17**, pp. 321, *et seq.* and *v. Weber*, J. f. Math. **118**, pp. 154, *et seq.*

If one sets $z = z_1\,\xi_1 + \dots + z_n\,\xi_n$ then one can further express the derivatives $z_{i,\alpha_1\cdots\alpha_m}$ in terms of the derivatives of z by means of the equations $z_i = \partial z / \partial\xi_i$, and adds the relations $\frac{\partial^2 z}{\partial\xi_i\,\partial\xi_k} = 0$ to the system (1) then the integration of the differential system (1) will be reduced to that of a differential system with the independent variables $x_1, \dots, x_m, \xi_1, \dots, \xi_n$, and *one* unknown z. In particular, any arbitrary differential system can then be replaced with a second-order differential system with *one* unknown that is linear in the second derivatives ([43]).

The theory of systems of total differential equations provides another formulation of the most general differential equations.

One says: k relations between the variables $x_1, x_2, \dots, x_\nu$ *satisfies* the system of linearly-independent total differential equations:

$$\sum_{s=1}^{\nu} \xi_{is}(x_1, x_2, \dots, x_\nu)\, dx_s = 0 \qquad (i = 1, 2, \dots, r\,;\ r < \nu) \tag{12}$$

or defines an *integral equivalent (integral)* of it when the values of k of the variables x in (12) that one infers from them produce identities when they are substituted for the remaining x and their derivatives. If one writes the k relations in the form $x_i = f_i\,(x_{k+1}, \dots, x_n)$ then one will get a system S of $r\,(\nu - k)$ first-order partial differential equations for the k unknown functions f_i. Thus, if the coefficients ξ_{is} are subject to no condition equations then k cannot be smaller than $\nu\, r : (r + 1)$. Therefore, $k\,(r + 1) - r\,\nu$ of the f_i can be chosen arbitrarily, while the other ones define an integral of S ([44]).

Conversely, the integration of any differential system S can be reduced to that of a system of total differential equations. S will include, e.g., only *one* unknown z and its derivatives $z_{\alpha_1\cdots\alpha_m}$ up to order n inclusive, so let N be the number of quantities:

$$x_1, \dots, x_m,\ z_{1,0,\dots0},\ \dots,\ z_{\alpha_1\cdots\alpha_m},\ \dots \qquad \left(\sum \alpha_i < n; \alpha_1, \dots, \alpha_m = 0, 1, \dots, n\right). \tag{13}$$

Any integral of S will then be defined by an $N - m$-parameter ([45]) system of equations between the variables (13) that subsumes the relations S, fulfills all total differential equations of the form:

$$dz_{\beta_1\cdots\beta_m} = \sum_{i=1}^{m} z_{\beta_1\cdots\beta_{i-1},\beta+1,\cdots\beta_m}\, dx_i \tag{14}$$
$$\left(\sum \beta_i < n-1; \beta_1, \dots, \beta_m = 0, 1, \dots, n-1\right),$$

([43]) *J. Drach*, C. R. Acad. Sci. Paris **125** (1897), pp. 598.

([44]) *Forsyth, Theory of diff. eq.* 1, Chap. 13.

([45]) I. e., one that consists of $N - m$ independent equations. The definition of integral above was first given for the case of $n = 1$ by *J. F. Pfaff* (footnote 104).

and can be solved for the N – m variables z, $z_{\alpha_1 \cdots \alpha_m}$.

9. Lie's generalization of the concept of integral. – *Lie* ([46]) freed the definition of an integral from the latter restriction by introducing the concept of n^{th}-*order surface element* in the space R_{n+1} with the points coordinates $z, x_1, \ldots, x_m$; one understands that to mean an arbitrary system of values (13). The N quantities (13) themselves are called the *coordinates* of the surface element. Two infinitely-close elements $x_i, z, z_{\alpha_1 \cdots \alpha_m}$ and $x_i + dx_i, z + dz, \ldots, z_{\alpha_1 \cdots \alpha_m} + dz_{\alpha_1 \cdots \alpha_m}, \ldots$ are called *united* when they satisfy the equations (14). A ν-fold-extended family of n^{th}-order surface elements that satisfy equations (14) such that each element of the family is then united with all of its neighboring elements is called an *element-*M_ν *(element manifold, union),* and in particular, an *element-*M_ν^ρ when the manifold of associated "points" $z, x_1, \ldots, x_m$ is r-fold-extended. One has $\rho \leq \nu$; $\nu \leq m$ ([46.a]). An element-M_ν^ρ is then defined by an $N - \nu$-fold system of relations between the N variables (13) that satisfy the total differential equations (14), among which one will find $m - \rho + 1$ equations in $z, x_1, \ldots, x_m$. The most general system of equation of that type will be found with no integration. Naturally, the $m - \rho + 1$ relations in $z, x_1, \ldots, x_m$ can be chosen arbitrarily. An element-M_0^0 or an element-M_1^1 (i.e., a system of a simple infinitude of n^{th}-order surface elements, each of which is united with the infinitely-close ones) is called an n^{th}-order *strip*, and an element-M_m^m is called a *surface* in the space R_{n+1} .

A *first*-order surface element ([46]) $z, x_1, \ldots, x_m, p_1, \ldots, p_m$ is defined by a point $z, x_1, \ldots, x_m$ in R_{m+1} and a plane that goes through it:

$$\zeta - z = p_1 (\xi_1 - x_1) + \ldots + p_m (\xi_m - x_m) ,$$

in which ζ, ξ_i mean the running coordinates. The most-general element-M_m^ρ that consists of first-order surface elements, i.e., the most-general integral equivalent to the total differential equation:

$$dz - p_1\, dx_1 - \ldots - p_m\, dx_m = 0$$

that consists of $m + 1$ equations will be obtained when one adds to $m - \rho + 1$ arbitrary relations that are not free of z:

([46]) For $n = 1$, see *Transform.* 2, Chap. 4; *Goursat A*, Chap. 10; *v Weber*, Chap. 7. For $n > 1$, cf., the work of *Bäcklund* (footnote 55); *E. v. Weber*, Math. Ann. **44**, pp. 458; *O. Biermann*, Leipziger Ber. (1896), pp. 665; *J. Beudon*, Ann. éc. norm. sup/ (1896), Suppl. The foundations of the geometric interpretation of partial differential equations were addressed by *G. Monge*, *O. Bonnet* (footnote 181, 186, no. **32**) and *P. Du Bois-Reymond* (*Beiträge*); for a historical survey of that topic, see *Lie-Scheffers*, *Berührungs.*, pp. 514, *et seq.*

([46.a]) The latter inequality is always true for $n = 1$, but true for $n > 1$ only when certain trivial systems of equations are ignored (e.g., unions whose n^{th}-order elements include all elements of order $n - 1$). m-fold-extended unions of n^{th}-order whose associated first-order elements likewise define an element-M_m^ρ cannot be represented in the coordinates (13) in the case of $\rho < m$. The relevant modifications of the element coordinates were given by *F. Engel* (footnote 251) for the case of $m = n = 2$.

$$\Omega_i\,(z, x_1, \ldots, x_m) = 0 \qquad\qquad (i = 1, 2, \ldots, m - \rho + 1)$$

those ρ equations that arise by eliminating the λ from the system:

$$1 = \sum \lambda_i \frac{\partial \Omega_i}{\partial z}, \qquad - p_k = \sum \lambda_i \frac{\partial \Omega_i}{\partial x_k} \qquad (k = 1, \ldots, m).$$

Corresponding to the values $\rho = 0, 1, \ldots, m$, there are then $m + 1$ categories of element-M_m's. One can convert any element-M_m^ρ into an element-M_m^m by a contact transformation (cf., the next no.).

Any element-M_ν that consists of n^{th}-order surface elements that satisfy an n^{th}-order differential system S in *one* unknown, in other words, any $N - \nu$-parameter system of equations in the N variables (13) that includes the relations S and satisfies the total differential equations (14), is called an *integral-*M_ν (and also an *integral structure, -manifold, -union*); in the case $\nu = m$, it is also briefly called an *integral* of the differential system S. In contrast to that, one dealt with only *integral surfaces* in the older definition of an integral.

A system of values (13) that satisfies the relations S is called a *singular* or *non-singular* surface element of the differential system S according to whether it does or does not fulfill all of the equations of the system S_1 that was defined in no. **5**, respectively. An integral-M_ν of S is called *singular* when it satisfies the equations S_1 .

An n^{th}-order equation with *one* unknown and m independent variables possesses one and only one integral M_m that includes an arbitrarily-chosen non-singular integral-M_{m-1} ([47]). The latter determination can be brought into the form of the initial conditions in no. **1** by introducing new variables ([48]). The same thing will be true for any n^{th}-order system in involution with one unknown, while the most-general integral-M_{m-1} of *one* n^{th}-order equation will be found with no integration (when one adds the $N - m$ defining equations of an arbitrary M_m to the n^{th}-order equation), but the determination of an integral-M_{m-1} for a *system* of n^{th}-order equations will generally require certain integrations.

According to *Lie* ([49]), an n^{th}-order system in involution with one unknown and m independent variables is characterized by the property that each of its integral-M_q ($q < m$) belongs to at least *one* integral-M_m .

Now, one says a *complete integral* of a system of equations S in the N variables (13) to mean any non-singular family of integral-M_m that includes enough parameters for every non-singular surface element of S to contain one and only one M_m of the family, in other words, any $N - m$-term system of equations in the N variables (13) and ν arbitrary constants that fulfill the total differential equations (14), and which will yield all relations S by eliminating the constants, and only them. A complete integral generally consists of *surfaces*, so it can take the form (8), (9). The number ν of arbitrary constants is defined as it was in no. **7**.

([47]) See, e.g., *Bäcklund*, Math. Ann. **13**, 414.
([48]) E.g., *Goursat B* 1, pp. 26.
([49]) Leipziger Ber. (1895), pp. 71.

If one, with *Lie* (50), understands a *surface element* to mean a system of values (51):

$$x_1, \ldots, x_m, z_1, \ldots, z_m, p_{11}, \ldots, p_{mn} \qquad \left(p_{ik} = \frac{\partial z_i}{\partial x_k}\right),$$

and one calls two neighboring surface elements *united* when they fulfill the equations:

$$dz_i = \sum p_{ik}\, dx_k \qquad (i = 1, \ldots, n),$$

then the concepts of *element-*M_ν ($\nu \leq m$), *strips*, *integral*, *complete integral*, as well as *Lie*'s definition of a system in involution (49) can be adapted to first-order differential systems with n unknowns with no further discussion, so since any differential system can be reduced to that form (no. **8**), to arbitrary differential systems. A closely-related generalization of that Ansatz leads to the investigation of systems of equations that include the coordinates of the first-order surface elements $z, x_i, p_i, z', x'_i, p'_i, \ldots$ in *several* $m + 1$-dimensional spaces, which was begun by *Bäcklund* in the simplest cases (52).

10. Transformations of the differential system. – One understands a *contact transformation* of the $2m + 1$ variables $z, x_1, \ldots, x_m, p_1, \ldots, p_m$ or the space R_{m+1} $(z, x_1, \ldots, x_m)$ to mean (III D 7) a transformation:

$$(15) \qquad z' = Z(z, x_1, \ldots, x_m, p_1, \ldots, p_m)\,; \quad x'_i = X_i(z, \ldots, p_m)\,; \quad p'_i = P_i(z, \ldots, p_m) \; (i = 1, \ldots, m)$$

whose right-hand side satisfies an identity of the form:

$$dZ - P_1\, dX_1 - \ldots - P_m\, dX_m \equiv \rho\,(z, x_1, \ldots, p_m)\,(dz - \sum p_i\, dx_i) \qquad (\rho \equiv 0)$$

for any arbitrary system of values of the z, x_i, p_i and their differentials. By an $(n - 1)$-fold extension (53) of the transformation (15), one will get formulas of the form:

$$z'_{\alpha_1 \cdots \alpha_m} = P_{\alpha_1 \cdots \alpha_m}(z, x_1, \ldots, x_m, z_{\beta_1 \cdots \beta_m}, \ldots)$$

$$(\alpha_1, \ldots, \alpha_m = 0, 1, 2, \ldots, n\,;\ \sum \alpha_i \leq n\,;\ \sum \beta_i \leq \sum \alpha_i)$$

(50) Leipziger Ber. (1895), pp. 111.

(51) Such a thing is also called an M_m*-element* in the space $x_1, \ldots, x_m, z_1, \ldots, z_n$ (*Bäcklund*, Math. Ann. **17**, pp. 286), and a *line element* in the space $x, z_1, \ldots, z_m$ in the case of $m = 1$; *Lie-Scheffers*, *Berührungs.*, Chap. 2.

(52) Math. Ann. **17**, pp. 285, esp., pp. 305; *ibid.* **19**, pp. 387.

(53) *Transform.* 2, pp. 378-383.

that will represent a transformation of the N variables (13) when they are combined with (15).

The $(n-1)$-fold-extended contact transformations are characterized by the fact that they will convert any two neighboring, united n^{th}-order surface elements in the space R_{m+1} $(z, x_1, \ldots, x_m)$ into two neighboring united elements in $R'_{m+1}(z', x'_1, \ldots, x'_m)$ again. They are the only invertible finitely multivalued transformations of the N variables (13) under which every element M_ν will once more go to an element-M_ν (54), and any system in involution S will again go to a system in involution.

By contrast, *Bäcklund* (55) considered *surface transformations:*

$$x'_i = X_i(z, x_1, \ldots, x_m, z_1, \ldots, z_{\beta_1\cdots\beta_m}, \ldots); \qquad z' = Z(x_1, \ldots) \qquad \left(\sum \beta_i \leq n\right),$$

by means of which every M_m in the space $z, x_1, \ldots, x_m$ will correspond to an M_m of first-order elements in the space $z', x'_1, \ldots, x'_m$, but every M_m in the latter will correspond to a certain family of M_m in the former, and he examined the relations between differential systems in both spaces that were mediated by them.

Relations can exist between the surface elements of two n^{th}-order systems in involution with *one* unknown under which every integral of the one system will correspond to one and only one integral in the other (56). By contrast, in the event that $n > 1$, in addition to possible extended contact transformations *invertible finitely-multivalued* relations of that type will exist only when each of the two systems in involution consist of more than one equation (57).

II. – Linear first-order partial differential equations with one unknown.

11. Linear first-order partial differential equations. – If $\xi_1, \xi_2, \ldots, \xi_m$ are any functions of the variables $x_1, x_2, \ldots, x_m$ and one has put the general integral equations for the simultaneous system of ordinary differential equations (II A 4):

$$dx_1 : dx_2 : \ldots : dx_m = \xi_1 : \xi_2 : \ldots : \xi_m \tag{16}$$

into the form:

$$f_1(x_1, x_2, \ldots, x_m) = c_1, \ldots, \qquad f_{m-1}(x_1, x_2, \ldots, x_m) = c_{m-1} \tag{17}$$

(54) *Bäcklund*, Math. Ann. **9**, pp. 297; *Engel*, Leipziger Ber. (1890), pp. 203; see III D 7.

(55) Math. Ann. **9**, pp. 297; *ibid.*, **11**, pp. 199; *ibid.*, **13**, pp. 69; *ibid.*, **15**, pp. 39; *ibid.*, **17**, pp. 285; *ibid.*, **19**, pp. 387. The simplest case was considered already by *Du Bois-Reymond*, pp. 166.

(56) Cf., no. **52**. There, one will also find examples of relations between two differential systems under which any integral of the one will correspond to a *family* of integrals of the other. The most general transformation principle of that type was formulated by *Delassus*, Ann. éc. norm. sup. (1897), pp. 238.

(57) *Bäcklund*, Math. Ann. **9**, pp. 312; *ibid.*, **19**, pp. 399-406.

then one calls the f_i *integrals* of the system (16). They are characterized by the property that their total differential df_i vanishes due to (16), so they are particular integrals of the *linear homogeneous partial differential equation:*

$$X\,f \equiv \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_m \frac{\partial f}{\partial x_m} = 0\,. \tag{18}$$

The general integral of that equation is an arbitrary function $f_1, \ldots, f_{m-1}$.

Conversely, if one knows $m - 1$ independent particular solutions f_i of (18) then the general integral equations of the system (16) can be written in the form (17). The two integration problems (16), (18) will be equivalent then ([58]). The simultaneous system (16) is said to be *adjoint* to the partial differential equation (18) (cf., no. **13**, as well as II A 4 b, no.**1**).

If the quotients $\xi_i : \xi_1$ are regular at the location $x_1^0, \ldots, x_m^0$ then equation (18) will possess one and only one system of $m - 1$ particular solutions ([59]) $h_1, h_2, \ldots, h_{m-1}$ that are regular at the location x^0 and go to $x_2, x_3, \ldots, x_m$, resp., when $x_1 = x_1^0$, so they possess the form:

$$h_{i-1} = x_i + (x_1 - x_1^0)\,\mathfrak{P}_i\,(x_1 - x_1^0, x_2 - x_2^0, \ldots, x_m - x_m^0) \qquad (i = 2, \ldots, m), \tag{19}$$

in which the $\mathfrak{P}_i$ mean ordinary power series in the $x_k - x_k^0$. The h_i are called the *principal integrals* ([60]) of equation (18) with respect to $x_1 = x_1^0$. If the function $\Phi\,(x_2, \ldots, x_m)$ is regular at the location $x_2^0, \ldots, x_m^0$ then $\Phi\,(h_1, \ldots, h_{m-1})$ will be the integral function of (18) that is regular at the location x^0 and goes to $\Phi\,(x_2, \ldots, x_m)$ when $x_1 = x_1^0$ (no. **1**). If the h_i are also regular at the location $x_1^0, x_2', \ldots, x_m'$ then one can solve the $m - 1$ equations $h_i = x_{i+1}'$ as follows:

$$x_i = x_i' + (x_1 - x_1^0)\,\mathfrak{P}_i\,(x_1 - x_1^0, x_2' - x_2^0, \ldots, x_m' - x_m^0) \qquad (i = 2, \ldots, m).$$

The right-hand sides are those integral functions of the simultaneous system (16) that reduce to the prescribed constants $x_2', \ldots, x_m'$ when $x_1 = x_1^0$.

According to *Lagrange* ([61]), an arbitrary relation between the f_i (or h_i) defines the general integral x_m of the *inhomogeneous linear partial differential equation* ([58]):

$$\xi_1 \frac{\partial x_m}{\partial x_1} + \cdots + \xi_{m-1} \frac{\partial x_m}{\partial x_{m-1}} = \xi_m\,, \tag{20}$$

([58]) *Lagrange*, Berl. nouv. mém. (1779), pp. 121 = *Oeuvres* 4, pp. 585, 624; *ibidem* (1785), pp. 174 = *Oeuvres* 5, pp. 543; *Jacobi*, "Dilucidationes, etc.," J. f. Math. **23** = *Werke* 4, pp. 147; Historical surveys are in *Mansion*, §§ 5 and 6; *Lie-Scheffer*, *Berührungs.*, pp. 514, *et seq.*

([59]) *Jacobi*, *Werke* 4, pp. 196.

([60]) *Natani*, J. f. Math. **58**, pp. 302.

([61]) Berl. nouv. mém. (1779), pp. 152 = *Oeuvres* 4, pp. 624; *ibidem* (1785), pp. 174 = *Oeuvres* 5, pp. 543.

with the unknowns x_m and the independent variables $x_1, \ldots, x_{m-1}$ ([62]). k such relations will yield the general integral $x_{m-k+1}, \ldots, x_m$ of the differential system ([63]):

$$\xi_1 \frac{\partial x_s}{\partial x_1} + \cdots + \xi_{m-1} \frac{\partial x_s}{\partial x_{m-1}} = \xi_s \qquad (s = m - k + 1, \ldots, m) \tag{21}$$

with the unknowns $x_{m-k+1}, \ldots, x_m$ and the independent variables $x_1, \ldots, x_{m-1}$.

The $(m - 1)$-fold infinitude of curves in the space R_m $(x_1, x_2, \ldots, x_m)$ that are defined by (16) are called the *integral curves* ([64]) of the simultaneous system (16) or also the *characteristics* ([65]) of the partial differential equation (18) or (20). The most-general integral surface of (20) will then be generated by any ∞^{m-2} characteristics of the differential system (21), and the most general integral manifold of the differential system (21) will be generated by any ∞^{m-k-1} ([66]). A surface in R_m whose points have some or all ξ_i possessing algebraic branches is generally the locus of singularities (vertices) of the characteristics ([67]), and in special cases, it will be a singular ([68]) integral surface of equation (20).

Lie ([69]) called the determination of a particular integral f_1 of equation (18) an *operation* $m - 1$. If one introduces k known solutions f_i in place of just as many x in (18) then $X f$ will be converted into an expression with $m - k$ terms, and the search for a $(k + 1)^{\text{th}}$ integral requires an operation $m - k - 1$. By introducing the $m - 1$ integrals f_i as new independent variables and performing a quadrature, $X f$ will go to the symbol for an infinitesimal translation $\partial f / \partial x$ (II A 6) of R_m . Conversely, the integration of (18) will be achieved with that reduction ([70]). *Lie* showed some of the uses that one can derive from the fact that the infinitesimal transformation $X f$ belongs to a group that is either present in finite ([71]) form or only given by its defining equations ([70]) and infinitesimal transformations.

12. The Jacobi multiplier. – The functional determinant of $f, f_1, \ldots, f_{m-1}$ is identical to $\rho \cdot X f$ for any f. The function ρ, which depends upon the choice of the solutions f_i, is called a

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([62]) An adaptation of that theorem to a certain class of first-order partial differential equations with several unknowns was given by *M. Hamburger*, J. f. Math. **100**, pp. 399; vgl., no. **58**.

([63]) *Jacobi*, J. f. Math. 2, pp. 317 = *Werke* 4, pp. 1; *Werke* 4, pp. 229; cf., no. **58**. The generalization to complete systems (no. **13**) was given by *N. Saltykow*, J. de math. (5) **3** (1897), pp. 423; *Mayer*, Leipziger Ber. (1899), pp. 16.

([64]) Or "trajectories" of the one-parameter group *X f*, *Transform.* 1, pp. 99; cf., II A 6.

([65]) Cf., no. **32**, footnote 182.

([66]) *Lie-Scheffers*, *Berührungs.*, pp. 516.

([67]) *Goursat*, Am. J. Math. **11**, pp. 329.

([68]) *Du Bois-Reymond*, *Beiträge.*, pp. 31, *et seq.*; *Darboux*, "Sol. sing.," pp. 77. For the definition of the singular solutions by means of multipliers (next no.), cf., *Jacobi*, *Werke* 4, pp. 358-364; *H. Weber*, J. f. Math. **66**, pp. 233, *et seq.* Cf., II A 4 a, no. **22**.

([69]) Math. Ann. **11**, pp. 530.

([70]) See, e.g., Leipziger Ber. (1895), pp. 269.

([71]) *Ibidem*, (1889), pp. 287; *Transform.* 3, Chap. 26.

Jacobi multiplier ([72]) of equation (18) or the simultaneous system (16), and it satisfies the inhomogeneous linear partial differential equation:

$$\frac{\partial(\rho\,\xi_1)}{\partial x_1}+\frac{\partial(\rho\,\xi_2)}{\partial x_2}+\cdots+\frac{\partial(\rho\,\xi_m)}{\partial x_m}=0\,. \tag{22}$$

Conversely, if ρ fulfills that equation then there will always be $m - 1$ solutions f_i of (18) such that the functional determinant of $f, f_1, \ldots, f_{m-1}$ is identical to $\rho\,X\,f$. The quotient of two multipliers is constant or a solution of (18). If $X\,f$ goes to $X'\,f$ when one introduces new variables $y_1, \ldots, y_m$ whose functional determinant relative to the x is denoted by ∇ then $\rho : \nabla$ will be a multiplier of $X'\,f = 0$ when it is expressed in terms of the variables y_i . If one knows a multiplier ρ and $m - 2$ independent particular solutions $f_1, \ldots, f_{m-2}$ then one will get f_{m-1} by a quadrature (viz., *the principle of the last multiplier*) ([73]). For *Lie* ([74]), the fact that no further advantage can be gained by the knowledge of ρ alone in the determination of $f_1, \ldots, f_{m-2}$ follows from the fact that due to (22), $\rho X\,f$ is the most general infinitesimal transformation of the (infinite) group of R_m that does not change the volume.

13. Complete system. – A common integral to several (linearly-independent) equations ([75]):

$$X_i\,f \equiv \sum_{k=1}^{m} \xi_{ik}\,\frac{\partial f}{\partial x_k}=0 \qquad (i = 1, 2, \ldots, \mu) \tag{23}$$

also fulfills all of the equations that arise from them under *bracket operations* (II A 6, no. **5**) ([76]):

$$(X_i\,X_k) \equiv \sum_s (X_i\,\xi_{ks} - X_k\,\xi_{is})\,\frac{\partial f}{\partial x_s}=0\,, \tag{24}$$

which follow by eliminating the second derivatives of f from the first derivatives of (23). The expressions $(X_i\,X_k)$ for each f are linear combinations of $X_s\,f$ if and only if the system (23) is passive (no. **2**), and it will be called a μ-parameter *complete system (Clebsch).* The general case will be reduced to that one by adding the relations (24) and once more forming the brackets, etc. The property of completeness of the differential system (23) will remain preserved under arbitrary transformations of the x_i, as well as when one replaces the $X_i\,f$ with any μ independent linear

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([72]) "Theoria novi multiplicatoris, etc.," J. f. Math. **27**, pp. 199; *ibidem*, **29**, pp. 213, 333 = *Werke* 4, pp. 317; cf., *Boole*, Suppl.-Vol. Chap. 31. *L. Boltzmann*, Math. Math. **42**, pp. 374.

([73]) For its applications to dynamics, see *Jacobi*, 10-18, *Vorl. über Dynamik*, cf., also II A 4 b, no. **12**.

([74]) Leipziger Ber. (1895), pp. 293.

([75]) *Jacobi*, *Werke* 5, pp. 39, *et seq.*; *Boole*, Trans. London Math. Soc. (1862), pp. 437; *Boole*, Suppl. Vol., Chap. 25, 26; *A. Clebsch*, J. f. Math. **65** (1865), pp. 257; *Transform.* 1, Chap. 5.

([76]) The system of equations (23), (24) is *invariantly coupled* with the system (23), *Engel*, Leipziger Ber. (1889), pp. 165; cf., no. **60**.

combinations $Y_i f$. They can be chosen in such a way that they define a "Jacobi system" (*Clebsch*), i.e., that all $(Y_i Y_k)$ are identically zero. For example, when one solves the system (23) as follows:

$$0 = Y_k f \equiv \frac{\partial f}{\partial x_k} + \sum_{h=1}^{m-\mu} a_{kh} \frac{\partial f}{\partial x_{\mu+h}} \qquad (k = 1, 2, \ldots, \mu), \tag{25}$$

the conditions for completeness will then be:

$$Y_i a_{ks} - Y_k a_{is} \equiv 0 \qquad (i, k = 1, \ldots, \mu ; s = 1, 2, \ldots, m - \mu). \tag{26}$$

The most general *Jacobi* system $Y_1 f = 0, \ldots, Y_\mu f = 0$ that is equivalent to the complete system (23) will be obtained ([77]) when one chooses μ functions $\varphi_1, \ldots, \varphi_m$ arbitrarily, but in such a way that the determinant of the μ^2 expressions $X_i \varphi_k$ is not identically zero, and solves the equations:

$$X_i f = \sum_{k=1}^{\mu} X_i \varphi_k \cdot Y_k f \qquad (i = 1, 2, \ldots, \mu)$$

for the $Y_k f$. $Y_k \varphi_i$ will then be equal to 1 to 0 according to whether indices i, k are equal or different, resp. The assumption $\varphi_1 = x_1, \ldots, \varphi_\mu = x_\mu$ will lead to (25).

If $\mu = m$ then the system (23) will be fulfilled by only a constant f. In the case of $\mu < m$, if one introduces the integrals $x_2, \ldots, x_\mu, u_{\mu+1}, \ldots, u_m$ of the first equation of the *Jacobi* system (25), along with x_1, as new independent variables in the $\mu - 1$ other equations (25) then it will be converted into a $(\mu - 1)$-parameter *Jacobi* system that is free of x_1. A $(\mu - 1)$-fold repetition of that process will lead to *one* equation with $m - \mu + 1$ independent variables whose integral will either yield $m - \mu$ particular solutions $f_1, \ldots, f_{m-\mu}$ of the complete system (23) that are independent with respect to $x_{\mu+1}, x_{\mu+2}, \ldots, x_m$ when they are expressed in terms of the x or it will produce (25). Conversely, if the system (23) possesses $m - \mu$ independent solutions then it will be complete. The general integral of (23) is an arbitrary function of the f_i. Therefore, the complete system (23) defines a *decomposition* of the space R_m $(x_1, \ldots, x_m)$ ([78]) into μ-fold-extended point manifolds:

$$f_1 (x_1, x_2, \ldots, x_m) = c_1, \ \ldots, \qquad f_{m-\mu} (x_1, x_2, \ldots, x_m) = c_{m-\mu} \tag{27}$$

that are called the *characteristics* of the complete system (23). Conversely, a well-defined complete system (23) will belong to any decomposition of space (27). Any point P of the R_m lies on one and only one characteristic that also includes the characteristics (no. **11**) of all partial differential equations of the form $\sum \rho_s X_s f = 0$ that go through P. Any $\infty^{m-\mu-1}$ characteristics of the complete system generate the most-general "integral surface":

$$\varphi (f_1, f_2, \ldots, f_{m-1}) = 0$$

([77]) *Clebsch*, J. f. Math. **65**, pp. 257.
([78]) *Lie-Engel*, *Transform.* 1, Chaps. 6 and 7.

of the system (23).

The infinitesimal transformations $X_1 f$, …, $X_\mu f$ leave the functions f_1, …, $f_{m-\mu}$ invariant, so it will also leave all characteristics and integral surfaces of the complete system (23) individually invariant ([78]).

One says that the complete system (23) *admits the infinitesimal transformation* $A f$ ([79]) when $A f$ takes any solution, characteristic, the integral surface of (23) to another such thing. In order for that to be true, it is necessary and sufficient that all of the bracket expressions $(X_i\, A)$ should be linear combinations of the $X_s f$. *Lie* ([80]) has shown the simplifications that known infinitesimal transformations (groups, resp.) can impart to the integration of complete systems.

The $m-\mu$-rowed functional matrix of the f_i and the μ-rowed matrix of the coefficients ξ_{ik} are corresponding matrices ([81]). The quotient ρ of two complementary determinants of those things is called a *Lie multiplier* ([82]) of the complete system (23). For $\mu = 1$, it is identical to the *Jacobi* multiplier, and for $\mu = m - 1$, it is identical to an *Euler* multiplier (next no.). If (23) is a *Jacobi* system then ρ will be a *Jacobi* multiplier of each of equations (23) ([83]). If one knows $m - \mu - 1$ solutions and a *Lie* multiplier of a *Jacobi* system then the last solution will follow by a quadrature ([84]).

14. Systems of total differential equations. – There are $m - \mu$ linearly-independent systems of functions η_{k1}, …, η_{km} such that:

$$0 \equiv \sum_s \xi_{is}\, \eta_{ks} \qquad (i = 1, \dots, \mu\,;\ k = 1, \dots, m - \mu).$$

The differential system (23) and the system of total differential equations:

(28) $$\nabla_k \equiv \sum_s \eta_{ks}\, dx_s = 0 \qquad (k = 1, 2, \dots, m - \mu)$$

determine each other reciprocally ([85]) and are called *adjoint*. If the *Jacobi* system (25) is equivalent to (23) then the equations:

(29) $$dx_{\mu+h} = \sum_k a_{kh}\, dx_k \qquad (h = 1, 2, \dots, m - \mu)$$

([79]) *Lie*, Math. Ann. **11**, pp. 494; *ibid.*, **24**, pp. 542; *Transform.* 1, Chap. 8; *Mayer*, Leipziger Ber. (1885), pp. 343; *ibid.* (1895), pp. 506; cf., *Lie-Scheffers*, *Vorlesungen über Differentialgleichungen*, etc., Leipzig, 1891.

([80]) Math. Ann. **11**, pp. 494, *et seq.*; *ibid.*, **25**, pp. 71; Leipziger Ber. (1893), pp. 343; *ibid.* (1895), pp. 506; cf., *Lie-Scheffers, Vorlesungen über Differentialgleichungen*, etc., Leipzig, 1891.

([81]) Cf., e.g., *Gordan-Kerschensteiner*, *Invariantentheorie*, Leipzig, 1885, 1, pp. 95.

([82]) Math. Ann. **11** (1877), pp. 501.

([83]) *Mayer*, Math. **12**, pp. 132.

([84]) *Ibidem*, pp. 140. For the connection between the solutions, Lie multipliers, and the infinitesimal transformations that a complete system admits, cf., *Lie*, Math. Ann. **11**, pp. 506; Leipziger Ber. (1895), pp. 313.

([85]) That connection weas first developed by *Boole*, Trans. London Math. Soc. (1862), pp. 437; *Boole*, Suppl. Vol. Chap. 25; cf., *Mayer*, Math. Ann. **5**, pp. 448; for its conceptual interpretation, see *Engel*, Leipziger Ber. (1889), pp. 159.

is equivalent to (28). If f is a solution to the differential system (23) then df will be a linear *integrable combination* $\sum \rho_s \nabla_s$ of the differential expressions ∇_s, and conversely. f will also be an *integral of the total differential equations* (28) then. (23) is complete if and only if the system (28) possesses $m - \mu$ independent integrals (or integrable combinations) and will be called *unrestricted integrable*. For arbitrary values of c_i, the relations (27) define an integral equivalent (no. **8**) of the system (28) and will be referred to as its *general integral*. In particular, if $\mu = m - 1$ then the system that is adjoint to (23) will consist of *one* total differential equation that is *exact*, i.e., its left-hand side will take the form df_1 when one multiplies it by a function ρ, namely, the *Euler multiplier* ([86]).

If $X f = \sum \xi_i \frac{\partial f}{\partial x_i}$ is an arbitrary infinitesimal transformation (II A 6, no. **4**) and $\nabla \equiv \sum \eta_i \, dx_i$ is any differential expression then the characteristic function $\Lambda \equiv \sum \xi_i \, \eta_i$ will be a simultaneous invariant of the infinitesimal transformation $X f$ and ∇ ([87]). In other words, if $X f$ goes to $X' f = \sum \xi'_i \frac{\partial f}{\partial x'_i}$ and ∇ goes to $\sum \eta'_i \, dx'_i$ under any arbitrary transformation of variables then at the same time, Λ will go to $\sum \xi'_i \eta'_i$. As a result, the system (28) will be coupled with its adjoint system (23), or what amounts to the same thing, it is invariantly coupled with the *adjoint family of infinitesimal transformations:*

$$\rho_1 X_1 f + \ldots + \rho_1 X_1 f . \tag{30}$$

If the symbol $X \nabla$ has the meaning:

$$X \nabla \equiv \sum \eta_i \, d\xi_i + \sum X \, \eta_i \, dx_i \equiv d\Lambda + \sum\sum \xi_i \left(\frac{\partial \eta_k}{\partial x_i} - \frac{\partial \eta_i}{\partial x_k} \right) dx_k$$

then one will say that the differential expression ∇ (the equation $\nabla = 0$, resp.) *admits the infinitesimal transformation $X f$* ([88]) when $X \nabla$ vanishes identically (is representable in the form $\rho \cdot \nabla$, resp.). One likewise says that the system (28) admits the infinitesimal transformation when all expressions $X \nabla_s$ can be represented in the form $\sum \rho_{sk} \nabla_k$. The system (28) is unrestricted integrable if and only if it admits all infinitesimal transformations of the adjoint family (30). For the solved form (29), the conditions for that find their expression in the identities (26) ([89]), and for the unsolved form, they consist of the condition that the $m - \mu$ bilinear forms:

([86]) *J. Collett*, Ann. éc. norm. sup. (1870), pp. 59; *H. Laurent*, Nouv. Ann. de math. (3) **6**, pp. 19; *Forsyth A*, Chap. 1.

([87]) *Engel*, Leipziger Ber. (1896), pp. 414.

([88]) *Lie*, Norw. Arch. **2** (1877), pp. 156; Leipziger Ber. (1896), pp. 405; *Engel*, *ibidem*, pp. 413, *et seq.*, and 1889, pp. 157, *et seq.*

([89]) They were given by *F. Deahna*, J. f. Math. **20** (1840), pp. 340.

$$\sum_{i=1}^{m}\sum_{k=1}^{m} \mathrm{H}_{ik}^{(s)}\, u_i\, v_k \qquad \left(s = 1, \ldots, m-\mu;\ \mathrm{H}_{ik}^{(s)} \equiv \frac{\partial \eta_{si}}{\partial x_k} - \frac{\partial \eta_{sk}}{\partial x_i} \right)$$

will vanish as a result of the relations [90]:

$$\sum_i \eta_{si}\, u_i = 0\,, \quad \sum_k \eta_{sk}\, v_k = 0 \qquad (s = 1, \ldots, m-\mu).$$

In other words, all $2m - 2\mu + 2$-rowed principal determinants in the $m - \mu$ skew-symmetric matrices:

$$\left\| \begin{matrix} 0 & \mathrm{H}_{12}^{(s)} & \mathrm{H}_{13}^{(s)} & \cdots & \mathrm{H}_{1m}^{(s)} & \eta_{11} & \cdots & \eta_{m-\mu,1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \mathrm{H}_{m1}^{(s)} & \mathrm{H}_{m2}^{(s)} & \mathrm{H}_{m3}^{(s)} & \cdots & 0 & \eta_{1m} & \cdots & \eta_{m-\mu,m} \\ -\eta_{11} & & & \cdots & -\eta_{1m} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\ -\eta_{m-\mu,1} & \cdots & \cdots & \cdots & -\eta_{m-\mu,m} & 0 & \cdots & 0 \end{matrix} \right\| \qquad (s = 1, 2, \ldots, m-\mu)$$

will vanish identically.

15. Jacobi's integration method. [91] – If one has replaced the complete system (23) with an equivalent *Jacobi* system:

$$Y_1 f = 0\,, \qquad Y_2 f = 0\,, \qquad \ldots, \qquad Y_\mu f = 0\,, \tag{31}$$

by means of the arbitrary functions $\varphi_1, \varphi_2, \ldots, \varphi_\mu$, according to no. **13**, and if ψ_1 means an integral of $Y_1 f = 0$ that is not a function of $\varphi_2, \ldots, \varphi_\mu$ then all functions:

$$\psi_2 \equiv Y_2\ \psi_1\,; \quad \psi_3 \equiv Y_2\ \psi_2\,; \quad \ldots \quad ;\ \psi_\nu \equiv Y_2\ \psi_{\nu-1}$$

will also be integrals of $Y_1 f = 0$. If ψ_ν is the first of those functions such that $Y_2\ \psi_\nu$ can be represented in the form $\chi(\varphi_2, \varphi_3, \ldots, \varphi_\mu, \psi_1, \psi_2, \ldots, \psi_\nu)$ then there will always be a function χ_1 of the same quantities that is also fulfilled by the equation $Y_2 f = 0$. It is defined to be an arbitrary integral of the simultaneous system:

$$\varphi_2 : d\psi_1 : \ldots : d\psi_\nu = 1 : \psi_2 : \psi_3 : \ldots : \psi_\nu : \chi\,.$$

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[90] *G. Frobenius*, J. f. Math. **82** (1876), pp. 267; a geometric interpretation for $m = 3$, $m = 2$ was given by *A. Voss*, Math. Ann. **16**, pp. 556.

[91] J. f. Math. **60**, pp. 26 = *Werke* 5, pp. 29, 33. *Vorl. über Dynamik* ; *Clebsch*, J. f. Math. **65**, pp. 260.

$\chi_2 \equiv Y_3\, \chi_1$, $\chi_3 \equiv Y_3\, \chi_2$, etc. will also be integrals of the first two equations (31) then. By repeating the previous argument, one will arrive at a common integral to the first three equations (31), etc. Ultimately, when one starts from the function ψ_1, one will obtain at least *one* solution to the *Jacobi* system (31) by integrating certain simultaneous systems.

A. Weiler ([92]) gave a similar, generally advantageous, integration process that was based upon a special normal form for the complete system (23).

16. The principal integral. – If one employs the principal integral ([93]) of the first equation in (25) relative to $x_1 = x_1^0$ (no. **11**) in the first of the reductions by which one will conclude the existence of solutions for the *Jacobi* system (25) as in no. **13**, and one proceeds analogously with the further reductions then, in the event that the a_{ik} are regular at the location $x_1^0, \dots, x_m^0$, one will arrive at $m - \mu$ integrals $h_1, h_2, \dots, h_{m-\mu}$ that are regular at that location and will reduce to $x_{\mu+1}, \dots, x_m$, resp., for:

$$(32) \qquad x_1 = x_1^0, \qquad \dots, \qquad x_\mu = x_\mu^0 ,$$

and they are called the *principal integrals* of the complete system (23) [or of the adjoint system (28)] *with respect to* $x_1 = x_1^0, \dots, x_\mu = x_\mu^0$. $\Phi(h_1, \dots, h_{m-\mu})$ is then that solution of (23) or (25) that reduces to the arbitrarily-prescribed function $\Phi(x_{\mu+1}, \dots, x_m)$ by means of (32) ([94]). If one solves the equations:

$$f_i(x_1, \dots, x_m) = f_i(x_1^0, \dots, x_m^0) \qquad (i = 1, 2, \dots, m - \mu)$$

for the characteristic of the complete system (23) (no. **13**) that goes through the $x_1^0, \dots, x_m^0$ in the form:

$$(33) \qquad h_s(x_1, \dots, x_m, x_1^0, \dots, x_\mu^0) = x_{\mu+s}^0 \qquad (s = 1, 2, \dots, m - \mu)$$

then the h_s will be principal integrals. The functions:

$$x_{\mu+s} = h_s(x_1^0, \dots, x_m^0, x_1, \dots, x_\mu)$$

([92]) Zeit. Math. Phys. **8** (1863), pp. 264; *ibid.* **20** (1875), pp. 271; *ibid.* **39** (1894), pp. 355; cf., *Clebsch, loc. cit.*; *Mayer*, Math. Ann. **9**, pp. 347.

([93]) *H. Grassmann* (no. **19**); *L. Natani*, J. f. Math. **58**, pp. 502; *Lie*, *Transf.* 1, Chap. 5.

([94]) Cf., also *Méray*, Ann. éc. norm. sup. (1890), pp. 217. For the connection between the principal integral and the finite equations of the μ-parameter group $Y_1 f, \dots, Y_\mu f$ (II A 6), cf., *F. Schur*, J. f. Math. **108** (1891), pp. 313.

of the variables $x_1, \ldots, x_\mu$ are regular at the location $x_1^0, \ldots, x_m^0$ and reduce to the constants $x_{\mu+s}^0$, resp., by means of (32). They fulfill the system (29) identically, so when the $x_{\mu+s}^0$ mean arbitrary constants, they will define the general of the differential system ([95]):

$$\frac{\partial x_{\mu+h}}{\partial x_k} = a_{kh} \qquad (h = 1, 2, \ldots, m - \mu\,;\ k = 1, \ldots, \mu), \tag{34}$$

which is *passive*, due to (26). Any *Mayer system* (no. **3**) can be reduced to the form (34), so to the integration of a complete system ([96]).

17. The Lie-Mayer transformation. – If one follows *Lie* ([97]) and *Mayer* ([98]) and introduces y's in place of $x_1, \ldots, x_\mu$ in the *Jacobi* system (25) by means of the formulas:

$$x_1 = x_1^0 + y_1\,, \quad x_2 = x_2^0 + y_1 y_2\,, \qquad \ldots, \qquad x_\mu = x_\mu^0 + y_1 y_\mu\,, \tag{35}$$

by which a_{ik} will go to $[a_{ik}]$, then:

$$\frac{\partial f}{\partial y_1} + \sum_{h=1}^{m-\mu} ([a_{1h}] + y_2 [a_{2h}] + \cdots + y_\mu [a_{\mu h}]) \frac{\partial f}{\partial x_{\mu+h}} = 0 \tag{36}$$

will be one of the transformed equations. If one follows *Lie* and interprets $y_2, \ldots, y_\mu$ as parameters then $y_1, x_{\mu+1}, \ldots, x_m$ will be the coordinates of an arbitrary point on one of the $\infty^{\mu-1}$ planar, $m - \mu + 1$-fold extended point-manifolds $M_{m-\mu+1}$ whose equations follow from (35) by eliminating y_1 and define a *pencil* with the *axis* (32). If one further determines the characteristics of the associated equation (36) on any $M_{m-\mu+1}$ that include a fixed point P of the axis (no. **11**) then those $\infty^{\mu-1}$ curves will generate the characteristic of the *Jacobi* system (25) ([99]). That is based upon the fact that the $m - \mu$ principal integrals of equation (36) relative to $y_1 = 0$ will go directly to the principal integrals of the *Jacobi* system (25) with respect to $x_1 = x_1^0, \ldots, x_\mu = x_\mu^0$ after one eliminates the y using (35). The integration of the complete system (23) or (25) is thus reduced to that of a single equation (36) or the adjoint simultaneous system of $m - \mu + 1$ ordinary differential equations ([100]).

([95]) *Mayer*, Math. Ann. **5**, pp. 448; *C. Bouquet*, Darb. Bull. (1872), pp. 265; *Méray*, *loc cit.*; *C. Bourlet*, Ann. éc. norm. sup. (1891), Suppl., pp. 6, *et seq.*; *Delassus*, *ibidem* (1897), pp. 109, esp., pp. 129.

([96]) *Bourlet*, *loc. cit.*, pp. 15; *Lie*, *Transform.* 1, Chap. 10.

([97]) Christ. Forh. (1872), pp. 28.

([98]) Math. Ann. **5** (1872), pp. 458, *et seq.*, where a somewhat-more-general transformation was applied to the adjoint system (29), instead of (25).

([99]) *Du Bois-Reymond* had already reduced integration of an exact total differential equation to that of a system of ordinary differential equations *Beiträge*, § **1**; J. f. Math. **70** (1869), pp. 299; Math. Ann. **12**, pp. 123.

([100]) *F. Schur* reduced the integration of the *unsolved* form (23) to that of a system of *m* ordinary differential equations, Leipziger Ber. (1892), pp. 177.

By means of (35), any solution of (25) will imply one for (36), and any integral of the latter will imply *at least one* solution of (25) by algebraic processes ([101]). The integration of an m-parameter complete system with q known integrals will require $m - \mu - q$, $m - \mu - q - 1, \ldots, 2, 1$ operations.

III. – The Pfaff problem.

18. Historical remarks. Pfaff's method of reduction. – Whereas *L. Euler* ([102]) considered a total differential equation:

$$(37) \qquad 0 = \sum_{i=1}^{m} a_i\,(x_1, x_2, \ldots, x_m)\,dx_i \equiv \Delta$$

that is not exact, so it cannot be fulfilled by *one* relation of the form f = const., to be inadmissible, *G. Monge* ([103]) remarked that equation (37) can always be satisfied by $m - 1$ (and possibly fewer) equations in the x. From the proof that when $m = 2n$ or $2n - 1$, Δ can be put into a form $F_1\,df_1 + \ldots + F_n\,df_n$ with only n differential elements, so it can already be made to vanish by means of n equations of the form:

$$\psi(f_1, \ldots, f_n) = 0\,, \qquad F_1 : F_2 : \ldots : F_n = \frac{\partial\psi}{\partial f_1} : \frac{\partial\psi}{\partial f_2} : \cdots : \frac{\partial\psi}{\partial f_n}\,,$$

J. F. Pfaff ([104]) laid the foundations for the theory of the equation (37). It will then be referred to as the *Paff equation*, while its right-hand side will be referred to as the *Pfaff expression*. The problem of integrating equation (37) by means of the *least-possible* number of relations between the x (or also the problem of bringing Δ into a form with the least-possible number of differential elements) is called the *Pfaff problem*.

In the case $m = 2n$, *Pfaff* introduced new variables $t, y_1, \ldots, y_{m-1}$, in place of the x, such that one will have:

$$(38) \qquad \Delta \equiv M\,\Delta' \equiv M\,(t, y_1, y_2, \ldots, y_{m-1}) \sum_{i=1}^{m-1} b_i\,(y_1, y_2, \ldots, y_{m-1})\,dy$$

identically. When considered to be functions of $t, y_1, \ldots, y_{m-1}$, the x will then satisfy a system of differential equations ([105]) of the form:

$$(39) \qquad \sum_{k=1}^{m} a_{ik} \frac{\partial x_k}{\partial t} + \rho\, a_i = 0 \qquad (i = 1, \ldots, m),$$

([101]) *Mayer*, Math. Ann. **5**, pp. 463; cf., *Goursat A*, art. 32: *Delassus*, art. 28.
([102]) *Inst. Calc. Int.* 3, pp. 7, *et seq.*
([103]) Paris Mém. (1784), pp. 502; esp., pp. 535.
([104]) Berliner Abh. (1814-15), pp. 76; cf., *C. F. Gauss*, Gött. Anz. (1815), pp. 1025.
([105]) Cf., *Jacobi*, J. f. Math. **2** (1827), pp. 347 = *Werke* 4, pp. 19; *Mayer*, Math. Ann. **17**, pp. 523.

$$(40) \qquad \sum_{k=1}^{m} a_k \frac{\partial x_k}{\partial t} = 0 \qquad \left(a_{ik} \equiv -a_{ki} \equiv \frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i} \right).$$

If, as *Pfaff* assumed, the determinant $| a_{ik} |$ is not identically zero then the y, when considered to be functions of the x, will be integrals of the differential equation:

$$(41) \qquad \eta_1 \frac{\partial f}{\partial x_1} + \eta_2 \frac{\partial f}{\partial x_2} + \cdots + \eta_m \frac{\partial f}{\partial x_m} = 0$$

or the adjoint simultaneous system:

$$(42) \qquad dx_1 : dx_2 : \ldots : dx_m = \eta_1, : \eta_2 : \ldots : \eta_m ,$$

in which ρ, η_1, $\eta_2 : \ldots$, η_m mean the solutions of the system of equations:

$$(43) \qquad \sum_{k=1}^{m} a_{ik} \eta_k + \rho \, a_i = 0 \qquad (i = 1, \ldots, m).$$

In order to solve those equations, if one understands $\alpha_1, \ldots, \alpha_{2\nu}$ to mean any 2ν indices and $\beta_1, \ldots, \beta_{2\nu}$ to mean a permutation of them that arises by T transpositions (I A, 2 no. **3**), and one defines the *Pfaffian aggregate* $(\alpha_1, \ldots, \alpha_{2\nu})$ ([106]) by means of the equations:

$$(\alpha_1, \ldots, \alpha_{2\nu}) \equiv (-1)^T (\beta_1, \beta_2, \ldots, \beta_{2\nu}),$$
$$(\alpha_1, \ldots, \alpha_{2\nu}) \equiv \sum (\alpha_1, \alpha_2) \; (\alpha_3, \alpha_4, \ldots, \alpha_{2\nu}),$$

in which $\sum$ denotes a sum of $2\nu - 1$ terms that arise from the ones that are written down by zero-fold, one-fold, …, $(2\nu - 2)$-fold cyclic permutations of the indices $\alpha_2, \ldots, \alpha_{2\nu}$. If one sets:

$$(i\,k) \equiv -(k\,i) \equiv a_{ik}, \qquad (o\,i) \equiv -(i\,o) \equiv a_i \qquad (i, k = 1, 2, \ldots, m),$$

$$P \equiv (1, 2, \ldots, 2n), \qquad \Pi_k \equiv (1, 2, \ldots, k-1, 0, k+1, \ldots, 2n)$$

then the functions η_k, ρ will be proportional to the expressions Π_k, $-P$, resp. If one chooses the y_i to be principal integrals ([107]) of equation (41) with respect to $x_m = 0$ then the b_i on the right-hand side of (38) will be proportional to the functions $a_i (y_1, y_2, \ldots, y_{m-1}, 0)$.

([106]) Those expressions (which are called "Pfaffians" in England) were already found in *Pfaff*. The symbol $(\alpha_1, \ldots, \alpha_{2\nu})$ goes back to *Jacobi*, *loc. cit.*; cf., *A. Cayley*, J. f. Math. **38** (1849), pp. 93 = *Papers* 1, pp. 410; J. f. Math. **57** (1860), pp. 273 = *Papers* 4, pp. 359; Quart. J. Math. **26** (1893), pp. 195 = *Papers* 13, p. 405.

([107]) *Jacobi*, J. f. Math. **17**, pp. 156 = *Werke* 4, pp. 120, *et seq.*; *J. Binet*, C. R. Acad. Sci. Paris **15** (1842), pp. 74.

In the case of $m = 2n - 1$, one applies the previous reduction to the first $m - 1$ terms in Δ, in which x_m are treated as constants, and then modifies ([108]) the coefficients of dx_m . Repeated application of both reductions will yield a form with n differential elements for Δ in all cases.

19. Grassmann's method. The fundamental theorem. – *H. Grassmann* ([109]) had adapted *Pfaff*'s reduction (38) to the case in which the linear equations (43) are not independent. If κ, κ_1, κ_2 mean the ranks ([110]) of the three matrices:

$$\text{(A)}\quad \left\|\begin{matrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \\ a_1 & \cdots & a_m \end{matrix}\right\|, \qquad \text{(B)}\quad \left\|\begin{matrix} a_{11} & \cdots & a_{1m} & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} & a_m \\ a_1 & \cdots & a_m & 0 \end{matrix}\right\|, \qquad \text{(C)}\quad \| a_{ik} \|,$$

resp., and if κ is an even number 2λ then one will also have $\kappa_1 = \kappa_2 = 2\lambda$. If $\kappa = 2\lambda - 1$ then $\kappa_1 = 2\lambda$ and $\kappa_2 = 2\lambda - 2$ ([111]). $k \leq m$ and *Frobenius* ([112]) called it the *class of the Pfaffian expression* Δ. The case $\kappa = 2\lambda$ can then be characterized by the conditions:

$$\text{(44)}\qquad \begin{array}{ll} (1,2,\ldots,2\lambda) \not\equiv 0 & (1,2,\ldots,2\lambda-1,0) \not\equiv 0, \\ (1,2,\ldots,2\lambda,\rho,\sigma) \equiv 0 & (\rho,\sigma = 0, 2\lambda+1,\ldots,m), \end{array}$$

while the case $\kappa = 2\lambda - 1$ is characterized by the conditions:

$$\text{(45)}\qquad \begin{array}{ll} (1,2,\ldots,2\lambda-2) \not\equiv 0 & (0,1,2,\ldots,2\lambda-1) \not\equiv 0, \\ (1,2,\ldots,2\lambda-2,\rho,\sigma) \equiv 0 & (\rho,\sigma = 2\lambda+1, 2\lambda,\ldots,m). \end{array}$$

An expression Δ with m variables and $\kappa = m$ is called *condition-free*. If $\kappa = 1$ then Δ will be an exact differential, while in the case $\kappa = 2$, the equation $\Delta = 0$ will be exact.

There are $m - \kappa + 1$ linearly-independent systems of solutions $\eta_1, \ldots, \eta_m, \rho$ to equations (43), so there will be just as many equations of the form:

$$\text{(46)}\qquad X_0 f = 0, \qquad X_1 f = 0, \qquad \ldots, \qquad X_{m-\kappa} f = 0 .$$

$m - k$ of them, say the following ones:

([108]) *Gauss*, *loc. cit.*, pp. 1028 (*Ges. Werke* 3, pp. 233).
([109]) *Ausdehnungslehre*, 1862 = *Werke* 12; pp. 345-379; cf., *Engel*'s note, *ibidem*, pp. 482, *et seq.*; see also *Gauss*, *loc. cit.*; *Darboux*, Darb. Bull. **1** (1882), pp. 14, 19; *Forsyth A*, Chap. 4.
([110]) That is, the orders of the highest minors that do not vanish identically; I A 2, no. 24.
([111]) *Frobenius*, J. f. Math. **82** (1877), pp. 230. Those theorems are already found implicitly in *Grassmann*.
([112]) *Loc. cit.*, pp. 291.

(47) $$X_1 f = 0 , \qquad X_2 f = 0 , \qquad \ldots, \qquad X_{m-\kappa} f = 0 ,$$

correspond to the $m - \kappa$ systems of solutions of the equations:

(48) $$\sum a_k \eta_k = 0 , \qquad \sum a_{ik} \eta_k = 0 \qquad (i = 1, 2, \ldots, n).$$

According to *Grassmann*, in order for the identity (38) to exist, the y, when regarded as functions of the x, must satisfy a partial differential equation that is included in the system (46), but not (47), for even κ and an equation of the system (47) for odd κ. The use of the *principal integral* also allows one to give the forms of the ratios of the b_i in (38) from the outset (see the prev. no.). Repeated application of that process, which is inapplicable only when $\kappa = 2\lambda - 1 = m$, yields a form $\sigma\Delta_1$ for Δ, where Δ_1 means a condition-free expression in $2\lambda - 1$ variables, so from the previous no., it can be reduced to a form with λ differential elements. Thus, if $\kappa = 2\lambda$ or $2\lambda - 1$ then Δ will possess a reduced form:

(49) $$M\,(d\lambda_1 - \varphi_1\, df_1 - \varphi_2\, df_2 - \ldots - \varphi_{\lambda-1}\, df_{\lambda-1}) ,$$

in which the f_i, φ_κ are independent, and conversely ([113]). M is or is not representable as a function of the φ, f according to whether κ is odd or even, resp.

If one understands $y_1, y_2, \ldots, y_{m-1}$ to mean the integrals of a linear homogeneous first-order partial differential equations that is a linear combination of equations (46), but not equations (47), then for odd κ, Δ will take the form ([114]):

$$d\psi\,(t, y_1, y_2, \ldots, y_{m-1}) + \sum_{i=1}^{m-1} \psi_i(y_1, y_2, \ldots, y_{m-1})\, dy_i$$

after one introduces new variables $t, y_1, y_2, \ldots, y_{m-1}$. That theorem, in conjunction with *Grassmann*'s process, will yield the *fundamental theorem* ([115]): According to whether the rank κ of the matrix (A) is equal to 2λ or $2\lambda - 1$, the *Pfaffian* expression Δ can take on one or the other normal form:

(50) $$\pi_1\, d\xi_1 + \pi_2\, d\xi_2 + \ldots + \pi_\lambda\, d\xi_\lambda ,$$

(51) $$d\xi_\lambda - \pi_1\, d\xi_1 - \pi_2\, d\xi_2 - \ldots - \pi_{\lambda-1}\, d\xi_{\lambda-1} ,$$

([113]) *Grassmann*, *Werke* 12, pp. 355, *et seq.*, pp. 483; cf., *Jacobi*, J. f. Math. **29**, pp. 242 = *Werke* 4, pp. 426, *et seq.*; *Natani*, J. f. Math. **58** (1860), pp. 314.

([114]) For $m = \kappa = 2\lambda - 1$, cf., *Jacobi*, *loc. cit.* = *Werke* 4, pp. 424; *Darboux* (footnote 109). In my book (Chap. IV, § **3**), I referred to that reduction of Δ as the "Jacobi reduction."

([115]) It was only *postulated* by *Clebsch* [J. f. Math. **60** (1862), pp. 193]. In the year 1876, it was simultaneously proved by *Lie* (Norw. Arch. 2, pp. 338) by using two lemmas that were taken from *Clebsch*'s theory and directly by *Frobenius* (footnote 111; cf., no. **23**), and later by *Darboux* (footnote 109) and *Engel* (footnote 155).

resp., in which the π, ξ both mean κ *independent* functions of the x ([116]).

If one expresses the a_i, a_{ik} with the aid of the π_i, ξ_i by identifying the expression Δ with one or the other of those normal forms ([117]) then the converse of that theorem will follow, as well as the theorem:

For even κ, $\xi_1, \xi_2, \ldots, \xi_\lambda, \pi_2 / \pi_1, \ldots, \pi_\lambda / \pi_1$ ([118]) *are integrals of the system* (46), *while for odd* κ, *the functions* $\xi_1, \ldots, \xi_{\lambda-1}, \pi_1, \ldots, \pi_{\lambda-1}$ *are* ([119]). *In both cases, the* κ *functions* π, ξ, *and likewise the functions* M, f, φ *in the reduced form* (49), *are integrals of* (47). *The two systems of equations* (46) *and* (47) *are complete then* (no. **13**).

If one introduces κ arbitrary solutions $y_1, \ldots, y_\kappa$ of the complete system (47) as new variables in Δ in place of just as many x then that will produce a condition-free expression in the variables $y_1, \ldots, y_\kappa$. If one introduces $\kappa - 1$ integrals $y_1, \ldots, y_{\kappa-1}$ of (46) then Δ will take the form $\rho\,\Delta_1$ for even κ, while it will take the form $d\varphi + \Delta_1$ for odd κ, where Δ_1 means a condition-free expression in $y_1, \ldots, y_{\kappa-1}$, and φ is found by quadrature ([120]).

20. Integral equivalent. The most general normal form. – If $\kappa = 2\lambda$ or $2\lambda - 1$ then the *Pfaff* equation $\Delta = 0$ will be fulfilled by λ and no fewer relations between the κ functions π, ξ of the normal form ([121]). The most general integral equivalent of that type follows from:

$$\varphi_s(\xi_1, \xi_2, \ldots, \xi_\lambda) = 0\,, \qquad \sum_{k=1}^{q} \rho_k \frac{\partial \varphi_k}{\partial \xi_i} = \pi_i \qquad (s = 1, \ldots, q\,;\ i = 1, \ldots, \lambda)$$

upon eliminating the ρ_i . q is an arbitrary number $\leq \lambda$. The φ_s are arbitrary. For odd κ, one sets $\pi_\lambda \equiv -1$. For odd κ, it is possible that $\Delta = 0$ will also be fulfilled as a result of the relations $\pi_1 = 0, \ldots, \pi_\lambda = 0$. In addition, there might also possibly exist *singular integral equivalents* that make either all a_i or all κ-rowed determinants in the matrix (A) vanish ([122]).

One will get the most general normal form (50) [(51), resp.]:

([116]) A sharper formulation from function-theoretic standpoint is in my book, Chap. 6, § **3**.

([117]) *Clebsch*, *loc. cit.*, pp. 205, 222.

([118]) *Jacobi*, J. f. Math. **17**, pp. 155; *ibid.*, **29**, pp. 252 = *Werke* 4, pp. 120, 437.

([119]) For $m = k = 3$, cf., *Jacobi*, J. f. Math. **29**, pp. 241 = *Werke* 4, pp. 425; *Lie-Scheffers*, *Berührungs.*, pp. 198, *et seq.*

([120]) *Frobenius*, *loc. cit.*, pp. 314.

([121]) *Lie*, *Transform.*, II, Chap. 4.

([122]) *Engel*, in Bd. 1^2 of *Grassmann*'s *Werken*, pp. 472, *et seq.*; cf., my book, Chap. 7, § **2**; *É. Cartan*, Ann. éc. norm. sup. (1899), pp. 239. For the necessary and sufficient conditions for r given relations in the x to annul the expression Δ (reduce to an expression with $m - r$ variables and prescribed classes, resp.), and the problem that is connected with them of extending r given relations to an integral equivalent of $\Delta = 0$ by the fewest-possible number of equations, see my book, Chap. IX, § **4** and *Cartan*, *loc. cit.* (cf., also no. **26**). The latter problem also admits possible singular solutions (*Cartan*, pp. 285).

(52) $$\pi_1' \, d\xi_1' + \pi_2' \, d\xi_2' + \cdots + \pi_\lambda' \, d\xi_\lambda' ,$$

(53) $$d\xi_\lambda' - \pi_1' \, d\xi_1' - \pi_2' \, d\xi_2' - \cdots - \pi_{\lambda-1}' \, d\xi_{\lambda-1}' ,$$

resp., from a special one by means of a homogeneous contact transformation ([123]) of the 2λ variables π_i, ξ_i, or a contact transformation of the form ([124]):

(54) $$\xi_\lambda' = \xi_\lambda + \Omega, \quad \pi_i' = \Pi_i : \xi_i' = \Xi_i \qquad (i = 1, 2, \ldots, \lambda - 1),$$

in which the Ω, Π_i, Ξ_i mean functions of $\xi_1, \ldots, \xi_{\lambda-1}, \pi_1, \ldots \pi_{\lambda-1}$. Likewise, the most-general reduced form (49):

$$M'(df_\lambda' - \varphi_1' \, df_1' - \cdots - \varphi_{\lambda-1}' \, df_{\lambda-1}')$$

follows from a special one by an ordinary contact transformation in the $2\lambda - 1$ variables $f_\lambda, f_1, \ldots, f_{\lambda-1}, \varphi_1, \ldots, \varphi_{\lambda-1}$. Therefore, ξ_1' (likewise for f_1' in the case of even κ) can be taken to be an arbitrary integral of the system (46), and for odd κ, f_1' can be taken to be a solution of (47), so it can be taken to be an arbitrary function of x when $\kappa = 2\lambda - 1 = m$.

21. Transformation of a Pfaff expression ([125]). – The class κ is the *only invariant* of the *Pfaffian* expression Δ under an arbitrary transformation of the x. In other words: If one calls an expression Δ' in the variables $x_1', \ldots, x_m'$ *equivalent to* Δ when Δ can be converted into Δ' by a transformation of variables, then the necessary and sufficient conditions for the equivalence of Δ and Δ' is that Δ' must likewise possess the class κ. If, under that assumption, the expression (52) [(53), resp.] (in which the π', ξ' mean functions of the x_i') is the most-general normal form of Δ', and (50) [(51), resp.] is a special form of Δ, moreover, then one will get the most-general transformation of Δ' into Δ when one solves the κ relations $\xi' = \xi$, $\pi' = \pi$, and $m - \kappa$ arbitrary equations in the x, x' for the x'. The only invariant of the *Pfaff equation* $\Delta = 0$ is the rank 2λ of the matrix (B) (no. **19**). That is, if that number is the same for Δ and Δ', and $\sum F_i \, df_i$ is a reduced form for Δ with λ terms (no. **19**), while $\sum F_i' \, df_i'$ is the most-general form for Δ', then $\Delta' \equiv \rho \, \Delta$, under any transformation of variables that subsumes the equations $f_i' = f_i$; $F_1' : \ldots : F_\lambda' = F_1 : \ldots : F_\lambda$. If Δ' means Δ, when written in terms of the x', then one will get the most-general transformation of the *Pfaffian* expression Δ [the *Pfaff* equation $\Delta = 0$, resp.] *into itself* from the foregoing.

([123]) *Clebsch*, J. f. Math. **60**, pp. 196, 220. The fact that one cannot set two normal forms of different classes equal to each other was shown by *Lie* (Norw. Arch. **2**, pp. 343, *et seq.*) independently of the fundamental theorem.

([124]) *Lie*, *Transform.* 2, pp. 135; *ibidem*, pp. 125, *et seq.*

([125]) *Frobenius*, *loc. cit.*, §§ **25-27**.

22. Reduction methods of Clebsch and Lie. – The order and number of the operations that are necessary for one to exhibit the normal form of Δ can be reduced significantly. *L. Natani* ([126]) and *A. Clebsch* ([115]), following an idea of *Jacobi* ([127]), chose the ξ_1 in the expression (50) [(51), resp.] to be arbitrary integrals of the complete system (46) and removed one of the x by means of the relation $\xi_1 = c_1$, which made ξ_i, π_i, Δ go to $\xi_i^{(1)}$, $\pi_i^{(1)}$, $\Delta^{(1)}$, resp. $\Delta^{(1)}$ will then be an expression of class $k - 2$ with $m - 1$ variables ([128]) and will possess the normal form:

$$\sum_{i=2}^{\lambda} \pi_i^{(1)}\, d\xi_i^{(1)} \quad \text{or} \quad d\xi_\lambda^{(1)} - \sum_{i=2}^{\lambda-1} \pi_i^{(1)}\, d\xi_i^{(1)}, \text{ resp.}$$

$\xi_2^{(1)}$ is an arbitrary integral of the $m - \kappa + 2$-term complete system in $m - 1$ independent variables that belongs to $\Delta^{(1)}$ and is analogous to the system (46), and it will go to ξ_2 when one replaces the constant c_1 with ξ_1. The removal of one of the x from $\Delta^{(1)}$ by means of the equation $\xi_1^{(1)} = c_2$ likewise leads to an expression $\Delta^{(\kappa)}$ of class $\kappa - 4$ in $m - 2$ variables, etc. For an even κ, one will get reductions of the $\xi_1, \ldots, \xi_\lambda$ that will imply $\pi_1, \ldots, \pi_\lambda$ when one compares Δ with (50) for $\lambda = \kappa / 2$. For odd κ, $\lambda - 1$ reductions will yield the functions $\xi_1, \xi_2, \ldots, \xi_{\lambda-1}$, and an expression $\Delta^{(\lambda-1)}$ of class 1, i.e., an exact differential, from which ξ_λ will follow by a quadrature, and the π_i will follow as before.

Clebsch determined an integral ξ_1, $\xi_2^{(1)}$, $\xi_3^{(1)}$, … of each successive complete system with the help of *Jacobi*'s method (no. **15**). According to the method of no. **17**, for $\kappa = 2\lambda$, each determination requires $\kappa - 1, \kappa - 3, \ldots, 3, 1$ operations, resp., and for $\kappa = 2\lambda - 1$, they will require $\kappa - 1, \kappa - 3, \ldots, 4, 2$, resp., and a quadrature ([129]). If $x_1^0, \ldots, x_m^0$ is a location at which all coefficients a_i of Δ are regular, and for even κ, the *Pfaffian* aggregate (44) does not vanish, while for odd κ, it is the aggregate (45) that does not vanish, then the complete system (46) can be solved for the derivatives $\frac{\partial f}{\partial x_\kappa}, \frac{\partial f}{\partial x_{\kappa+1}}, \ldots, \frac{\partial f}{\partial x_m}$, and the coefficients of the solved form will be regular at the location $x_1^0, \ldots, x_m^0$. The transformation of no. **17** then reads:

$$x_{\kappa+1} = x_{\kappa+s}^0 + (x_\kappa - x_\kappa^0)\, y_{\kappa+s} \qquad (s = 1, 2, \ldots, m - \kappa). \tag{55}$$

Lie ([130]) performed that transformation on Δ directly, instead of on the complete system (46), under which the $y_{\kappa+s}$ were regarded as constants, and in that way Δ was converted into a condition-free

([126]) J. f. Math. **58**, pp. 318.

([127]) *Ibidem*, **27**, pp. 253 – *Werke* 4, pp. 438; cf., *Cayley*, J. f. Math. **57**, pp. 273 = *Papers* 4, pp. 359.

([128]) For a direct proof of this theorem, see my book, Chap. 6, § **1**; cf., also *Engel* (footnote 155).

([129]) If one chooses ξ_1 for odd κ to a function that fulfills the system (47), but not (46), so it will be an arbitrary function in the case $\kappa = 2\lambda - 1 = m$, then $\Delta^{(1)}$ will have class $\kappa - 1$, and the method above will lead to *Grassmann*'s form with $(\kappa + 1) / 2$ terms by means of $\kappa, \kappa - 2, \ldots, 3, 1$ operations. Cf., *Clebsch*, J. f. Math. **60**, pp. 224, *et seq.*

([130]) *Lie*, Christ. Forh. (1873), pp. 320; Norw. Arch. **2**, pp. 368, *et seq.*

expression Δ' with the κ variables $x_1, x_2, \ldots, x_\kappa$, from whose normal form Δ would be obtained by mere eliminations (for odd κ, by means a quadrature, in addition), and applied the analogous process repeatedly to the *Pfaffian* expressions that emerged from Δ' by the successive reductions of the *Clebsch* method.

23. Method of Frobenius ([131])**.** – The differential expression $\sum\sum a_{ik}\, dx_i\, \delta x_k$ is called the *bilinear covariant of the Pfaffian* expression Δ. Namely, if one subjects the x_i to any transformation $x_i' = \varphi_i\,(x_1, \ldots, x_m)$ and correspondingly subjects the two systems of differentials dx_i and δx_i to the transformation:

$$dx_i' = \sum_k \frac{\partial \varphi_i}{\partial x_k}\, dx_k\,, \qquad \delta x_i' = \sum_k \frac{\partial \varphi_i}{\partial x_k}\, \delta x_k \qquad (i = 1, 2, \ldots, m)$$

then the three differential expressions:

$$\sum a_i\, dx_i\,, \qquad \sum a_i\, \delta x_i\,, \qquad \sum\sum a_{ik}\, dx_i\, \delta x_k \tag{56}$$

will be simultaneously converted into the following ones:

$$\sum a_i'\, dx_i'\,, \qquad \sum a_i'\, \delta x_i'\,, \qquad \sum\sum a_{ik}'\, dx_i'\, \delta x_k' \qquad \left(a_{ik}' = \frac{\partial a_i'}{\partial x_k'} - \frac{\partial a_k'}{\partial x_i'}\right).$$

Furthermore, if Δ can be represented in the form:

$$\sum_{i=1}^{r} F_i\,(x_i, x_2, \ldots, x_m)\, df_i\,(x_i, \ldots, x_m) \tag{57}$$

then the three expressions (56) will vanish identically by means of the relations:

$$\sum f_{ik}\, dx_k = 0\,, \qquad \sum f_{ik}\, \delta x_k = 0 \qquad \left(i = 1, \ldots, r\,;\; f_{ik} = \frac{\partial f_i}{\partial x_k}\right),$$

and conversely. The problem of finding all of the properties of Δ that are invariant under an arbitrary transformation of the x [the problem of bringing the expression Δ into a form (57) with the least-possible number r of differential terms, resp.] then leads to the *purely-algebraic problem* (I B 2, no. **2**) of giving the invariants of the system of forms:

([131]) J. f. Math. **82**, pp. 230. Cf., also *G. Morera*, Mem.Torino **18** (1882), pp. 521. For the introduction of two different systems of differentials, cf., *Binet* (footnote 107); *Natani*, J. f. Math. **58**, pp. 307. A presentation of Frobenius's theory by means of symbolic methods was given by *É. Cartan* (footnote 122).

$$\sum a_i\, u_i\,, \qquad \sum a_i\, v_i\,, \qquad \sum\sum a_{ik}\, u_i\, v_k \qquad (a_{ik} = -\, a_{ki}), \tag{58}$$

under arbitrary congruent linear transformations of the two groups of variables u and v ([131.a]) [making the three forms (58) vanish by the least-possible number r of pairs of relations:

$$\sum f_{ik}\, u_k = 0\,, \qquad \sum f_{ik}\, v_k = 0 \qquad (i = 1, 2, \ldots, r)],$$

in which the a_i, a_{ik}, f_{ik} mean constants. The rank κ of the matrix (A) (no. **19**) proves to be the single invariant of the system of forms (58). If one further lets 2λ denote the rank of the matrix (B) (no. **19**) and lets (B_μ) denote the matrix:

$$(\mathrm{B}_\mu) \qquad \begin{Vmatrix} 0 & a_{12} & \cdots & a_{1m} & a_1 & f_{11} & \cdots & f_{\mu 1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & 0 & a_m & f_{1m} & \cdots & f_{\mu m} \\ a_1 & a_2 & \cdots & a_m & 0 & 0 & \cdots & 0 \\ f_{11} & f_{12} & \cdots & f_{1m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ f_{\mu 1} & f_{\mu 2} & \cdots & f_{\mu m} & 0 & 0 & \cdots & 0 \end{Vmatrix}$$

then $r = \lambda$, and the constants f_{ik} must be determined in such a way that (B_λ) also has rank 2λ. If one has determined the f_{hi} ($h < \mu$) in such a way that the matrices (B_1), (B_2), …, ($B_{\mu-1}$) all have rank 2λ then after one sets all $2l + 1$-rowed determinants of (B_μ) equal to zero, one will get a system of $m - 2l + \mu$ linear homogeneous equations for the unknowns $f_{\mu 1}$, $f_{\mu 2}$, …, $f_{\mu m}$. When the a_i, a_{ik}, f_{ik} keep their previous meanings, that system will be converted into an $m - 2\lambda + \mu$-parameter complete ([132]) system with the unknowns f_μ and the independent variables x_1, …, x_m that will possess $2\lambda - 2\mu + 1$ integrals in addition to f_1, …, f_μ as long as $\mu \leq \lambda$. That will once more imply *Grassmann*'s theorem that Δ can be reduced to λ and no fewer differential elements. The complete system f_λ is also produced by annulling all $\lambda + 1$-rowed determinants in the matrix:

$$\begin{Vmatrix} a_1 & a_2 & \cdots & a_m \\ f_{11} & f_{12} & \cdots & f_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\lambda 1} & f_{\lambda 2} & \cdots & f_{\lambda,} \end{Vmatrix}.$$

([131.a]) Cf., also *G. Morera*, Atti Torino **18** (1882), pp. 383.
([132]) *Frobenius*, *loc. cit.*, §§ **19** and **22**. Cf., *M. Hamburger*, Arch. Math. **60** (1877), pp. 203.

Conversely, if one follows *J. Zantschevski* ([133]) and expresses the idea that an $m - \lambda$-parameter *complete* system for f_λ will be obtained in that way then one will come back to the process above for determining $f_1, \ldots, f_{\lambda-1}$. One will get that process even more simply when one writes out the conditions for the system $\Delta = 0$, $df_1 = 0, \ldots, df_{\lambda-1} = 0$ to be unrestricted integrable ([134]), as in no. **14**.

If P is the *Pfaffian* aggregate $(1, 2, \ldots, 2\lambda)$ (no. **19**) and one defines the symbol P_{ik} by the equations:

$$(-1)^{i+k+1} P_{ik} \equiv (1, 2, \ldots, i-1, i+1, \ldots, k-1, k+1, \ldots, 2\lambda) \qquad (i < k)$$

$$P_{ik} \equiv -P_{ki}, \qquad P_{ii} \equiv 0 \qquad (i, k = 1, 2, \ldots, 2\lambda),$$

and one further sets:

$$(59) \qquad (\varphi f) \equiv \sum_{i=1}^{2\lambda}\sum_{k=1}^{2\lambda} \frac{P_{ik}}{P} \frac{\partial \varphi}{\partial x_k} \frac{\partial f}{\partial x_i}, \qquad (f) \equiv \sum_{i=1}^{2\lambda}\sum_{k=1}^{2\lambda} a_i \frac{P_{ik}}{P} \frac{\partial f}{\partial x_k}$$

then one can write the complete system that the f_μ have to satisfy in the case of $k = 2\lambda$ as ([135]):

$$(60) \qquad \begin{cases} 0 = (f) \equiv X_0 f; \quad X_1 f = 0, \quad \cdots \quad X_{m-\kappa} f = 0 \quad (\text{cf., no.}\,\mathbf{19}) \\ (f_1\, f) = 0, \quad (f_2\, f) = 0, \quad \cdots, (f_{\mu-1}\, f) = 0. \end{cases}$$

If one has determined $f_1, \ldots, f_\lambda$ in such a way that Δ has a normal form:

$$(61) \qquad F_1\, df_1 + F_2\, df_2 + \ldots + F_\lambda\, df_\lambda,$$

in which the F_i are determined by comparing that expression with Δ, then the complete system for f_1 will be identical to (46). One successively determines the f_i using the method of no. **22** when one removes $\mu - 1$ of the x_i from the complete system (60) by means of ([136]):

$$f_1 = c_1, \qquad \ldots, \qquad f_{\mu-1} = c_{\mu-1}.$$

In the case of $\kappa = 2\lambda - 1$, the complete system that f_μ has to satisfy reads as follows ([137]):

$$\begin{aligned} &X_1 f = 0, \quad &&X_2 f = 0, \quad &&\ldots, \quad &&X_{m-\kappa} f = 0, \\ &[f_1\, f] = 0, \quad &&[f_2\, f] = 0, \quad &&\ldots, \quad &&[f_{\mu-1}\, f] = 0, \end{aligned}$$

when one sets:

$$(62) \qquad [\varphi f] \equiv -\sum_{i=1}^{\kappa}\sum_{k=1}^{\kappa} \frac{Q_{ik}}{Q} \frac{\partial \varphi}{\partial x_k} \frac{\partial f}{\partial x_i},$$

([133]) Ann. éc. norm. sup. (1896), pp. 257.
([134]) See my book, Chap. 9, § **1**.
([135]) For $\kappa = 2\lambda = m$, cf., *Natani*, J. f. Math. **58**, pp. 321; *Clebsch*, *ibidem* **60**, pp. 243; *ibid.* **61** (1862), pp. 146.
([136]) The opposite path was shown by *Clebsch*, J. f. Math. **60**, pp. 232-242 for the case $m = \kappa$.
([137]) *Natani*, *loc. cit.*, pp. 312, *et seq. Hamburger*, *loc. cit.*, pp. 203, *et seq.*

and Q means the aggregate $(0, 1, \ldots, 2\lambda - 1)$, while the Q_{ik} are defined similarly to the P_{ik} . One will get a reduced form (61) by κ, $\kappa - 2, \ldots, 3, 1$ operations at each step.

In order to get a *normal form* in the case of $\kappa = 2\lambda - 1$, as well ([138]), one can either determine f_λ in such a way that the expression $\Delta' = \Delta - df_\lambda$ has the class $2\lambda - 2$ ([139]), and Δ' is put into normal form, from the above, or one determines $\lambda - 1$ functions $f_1, \ldots, f_{\lambda-1}$ in such a way that the matrix $(C_{\lambda-1})$ that arises from $(B_{\lambda-1})$ by deleting the $(m + 1)^{\text{th}}$ row and column will have rank $2\lambda - 2$. f_1 is then an arbitrary integral of (46), and the complete system that f_μ has to satisfy will arise from (46) by appending the relations:

$$\sum_{i=1}^{\kappa-1}\sum_{k=1}^{\kappa-1} \frac{P'_{ik}}{P'} \frac{\partial f_h}{\partial x_k} \frac{\partial f}{\partial x_i} = 0 \qquad (h = 1, \ldots, \mu - 1),$$

in which P' means the aggregate $(1, 2, \ldots, 2\lambda - 2)$, and the P'_{ik} are defined similarly to the P_{ik} . If $f_1, \ldots, f_{\lambda-1}$ are well-defined then one will find a normal form for Δ :

$$df_1 + F_1\, df_1 + \ldots + F_{\lambda-1}\, df_{\lambda-1} ,$$

in which f_λ can be ascertained by a quadrature, and the F_i are then obtained by comparing that expression to Δ. The method will then be reduced to the one in no. **22**.

24. The theory of contact transformations as a special case of the theory of the Pfaff problem. – If one introduces the k functions π, ξ into the normal form (50) [(51), resp.] in place of just as many x as new independent variables ([140]) then one will get the following expressions for (f), (φf), $[\varphi f]$:

$$\sum_{s=1}^{\lambda} \pi_s \frac{\partial f}{\partial \pi_s}, \qquad \sum_{s=1}^{\lambda} \frac{\partial \varphi}{\partial \pi_s}\frac{\partial f}{\partial \xi_s} - \frac{\partial \varphi}{\partial \xi_s}\frac{\partial f}{\partial \pi_s}, \qquad \sum_{s=1}^{\lambda-1} \frac{\partial \varphi}{\partial \pi_s}\left(\frac{\partial f}{\partial \xi_s} + \pi_s \frac{\partial f}{\partial \xi_\lambda}\right) - \frac{\partial f}{\partial \pi_s}\left(\frac{\partial \varphi}{\partial \xi_s} + \pi_s \frac{\partial \varphi}{\partial \xi_\lambda}\right),$$

resp.

The symbol (59) will then have the following meaning for the *Pfaffian* expression $p_1\, dx_1 + \ldots + p_m\, dx_m$ with the $2m$ variables x_i, p_i ([141]):

(63)
$$(f) - \sum_{s=1}^{m} p_s \frac{\partial f}{\partial p_s}; \qquad (\varphi f), \qquad \sum_{s=1}^{\lambda} \frac{\partial \varphi}{\partial \pi_s}\frac{\partial f}{\partial \xi_s} - \frac{\partial \varphi}{\partial \xi_s}\frac{\partial f}{\partial \pi_s}.$$

For the *Pfaffian* expression $dz - p_1\, dx_1 - \ldots - p_m\, dx_m$, one has:

([138]) *Frobenius, loc. cit.*, § **26**.

([139]) For the change in class that occurs upon subtracting a differential or multiplying it by a function of the x (for homogeneous a_i, resp.), cf., *Frobenius*, J. f. Math. **86** (1879), pp. 1 and my book, Chap. 3, § **2**; Chap. 5, § **3**.

([140]) Cf., my book, Chap. 10, **§ 3**; for $k = 2l = m$, see *Clebsch* (footnote 135).

([141]) The symbol (φf) is already found with this meaning in *S. D. Poisson,* J. éc. polyt. **8**, cah. 15 (1809), pp. 266.

(64)
$$[\varphi f] \equiv \sum_{s=1}^{m} \frac{\partial\varphi}{\partial p_s}\left(\frac{\partial f}{\partial x_s} + \pi_s \frac{\partial f}{\partial z}\right) - \frac{\partial f}{\partial p_s}\left(\frac{\partial\varphi}{\partial x_s} + p_s \frac{\partial\varphi}{\partial z}\right) .$$

From the previous no., one will now easily get the following theorems, which go back to *Lie* ([142]):

In order for the $2m$ independent functions X_i, P_i of the variables x, p to fulfill the identity:

(65)
$$P_1\, dX_1 + \ldots + P_m\, dX_m \equiv p_1\, dx_1 + \ldots + p_m\, dx_m ,$$

so for the right-hand sides to define a *homogeneous contact transformation:*

$$x_i' = X_i ; \qquad p_i' = P_i \qquad (i = 1, \ldots, m),$$

it is necessary and sufficient that they should satisfy the identities:

(66)
$$(X_i\, X_k) \equiv (P_i\, P_k) \equiv (P_i\, X_k) \equiv 0 ; \qquad (P_i\, X_k) \equiv 1 \qquad (i, k = 1, \ldots, m;\ i \neq k),$$

and that the X_i should be homogeneous of order zero in the p, while the P_i should be homogeneous of order one. If X_1 is an arbitrary function of x, p, and is homogeneous of order zero in the p_i then one can determine $m - 1$ other functions $X_2, \ldots, X_m$ in such a way that an identity of the form (65) will exist, and indeed for each index μ in the sequence $1, \ldots, m - 1$, the functions $X_{\mu+1}$ will be an integral of the $\mu + 1$-parameter complete system:

(67)
$$\sum p_s \frac{\partial f}{\partial p_s} = 0 , \qquad (X_1, f) = 0 , \qquad \ldots, \qquad (X_\mu, f) = 0 ,$$

with the known solutions $X_1, \ldots, X_m$, so $2m - 2\mu - 1$ of them will be found by one operation.

In order for $2m + 2$ functions ρ, Z, X_i, P_i of the variables z, x_i, p_i to fulfill the identity:

(68)
$$dZ - \sum_{i=1}^{m} P_i\, dX_i \equiv \rho\left(dz - \sum_{i=1}^{m} p_i\, dx_i\right) \qquad (\rho \not\equiv 0) ,$$

so in order for the equations:

$$z' = Z ; \qquad x_i' = X_i , \qquad p_i' = P_i \qquad (i = 1, 2, \ldots, m)$$

to represent a contact transformation, it is necessary and sufficient that the identities should exist ([143]):

([142]) Cf. III D 7 and *Lie*, *Transform.*, **2**, Chap. 5. *A. Mayer*, Math. Ann. **8** (1875), pp. 304. My book, Chap. 11.

([143]) These last three identities follow from (69) by means of the *Mayer* identity (next no. **25**).

(69) $$\begin{cases} [X_i\, X_k] \equiv [X_i\, Z] \equiv [P_i\, X_k] \equiv [P_i\, P_k] \equiv 0 \quad (i, k = 1, 2, \ldots, m; i \neq k) \\ [P_i\, X_i] \equiv \rho; \quad [P_i\, Z] \equiv \rho\, P_i\,, \end{cases}$$

(70) $$[\rho\, X_i] + \rho \frac{\partial X_i}{\partial z} \equiv [\rho\, P_i] + \rho \frac{\partial P_i}{\partial z} \equiv [\rho\, Z] + \rho \frac{\partial Z}{\partial z} - \rho^2 \equiv 0\,.$$

If X_1 is given arbitrarily then one can determine the functions $X_2, \ldots, X_m$, Z in such a way that an identity (68) will exist, and indeed $X_{\mu+1}$ will be a solution of the complete system:

(71) $$[X_1\,, f] = 0\,, \qquad \ldots, \qquad [X_\mu\,, f] = 0\,,$$

of which the solutions $X_1, \ldots, X_\mu$ are known already. If one has thus found each X_i by an operation $2m - 1$, $2m - 3, \ldots, 3$ then one will get Z by an operation 1, and the functions P_i and ρ by solving linear equations.

In order for $2m$ independent functions X_i, P_i of the $2m$ variables $x_1, \ldots, x_m, p_1, \ldots, p_m$ to fulfill an identity of the form:

(72) $$d\Omega\,(x_1, \ldots, x_m, p_1, \ldots, p_m) + \sum_{i=1}^{m} P_i\, dX_i \equiv \sum_{i=1}^{m} p_i\, dx_i\,,$$

it is necessary and sufficient that they should fulfill the conditions (66). If X_1 is given arbitrarily then one can determine $X_2, \ldots, X_m$ such that an identity (72) exists, and indeed such that $X_{\mu+1}$ is a solution of the *Jacobi* system:

(73) $$(X_1\, f) = \,, \qquad \ldots, \qquad (X_\mu\, f) = 0\,,$$

which possesses the integrals $X_1, \ldots, X_\mu$, so it will be found by an operation $2m - 2\mu$. One will get Ω by a quadrature, and one will get the P_i from (72) by solving linear equations.

25. The Jacobi and Mayer identities. – The bracket symbol (63) satisfies the identities:

(74) $$(f\,(\varphi\, \psi)) + (\varphi\,(\psi\, f)) + (\psi\,(f\, \varphi)) \equiv 0\,;$$

(75) $$((f)\, \varphi) + (f\,(\varphi)) + (\varphi\, f) = ((f\, \varphi))\,.$$

The former is called the ***Jacobi** identity* ([144]); the latter is a special case of the ***Mayer** identity* ([145]):

$$[f\,[\varphi\, \psi]] + [\varphi\,[\psi\, f]] + [\psi\,[f\, \varphi]] \equiv \frac{\partial f}{\partial z}[\varphi \psi] + \frac{\partial \varphi}{\partial z}[\psi\; f] + \frac{\partial \psi}{\partial z}[f\; \varphi]\,.$$

([144]) J. f. Math. **60**, pp. 42 [*Werke* **5**, pp. 46]. For a conceptual interpretation, see *Lie, Transform.* **2**, pp. 278-280.
([145]) Math. Ann. **9**, pp. 370. Cf., also *Goursat A*, art. 142.

Poisson*'s theorem* ([146]) follows from (74), which says that two known solutions ψ and χ of the linear homogeneous partial differential equation $(\varphi f) = 0$ will imply a third one in the form of $(\psi \chi)$ ([147]). The identities (74), (75) are also true when the bracket symbols have the meaning (59). According to *Clebsch* ([148]), in the case of $\kappa = 2\lambda = m$, they will then serve to derive new integrals of the partial differential equation $(f) = 0$ from known ones, and also to prove *Jacobi*'s ([149]) theorems on the multiplier of that equation.

26. Generalization of Frobenius's theory ([150]). – If the functions $f_1, \ldots, f_\mu$ are given arbitrarily and one eliminates μ of the variables x from Δ by means of the relations:

$$f_1 = c_1\,,\ f_2 = c_2\,,\ \ldots,\ f_\mu = c_\mu$$

then Δ will be converted into a *Pfaffian* expression with $m - \mu$ variables and class $\sigma + \sigma' - 2\mu$ when 2σ, $2\sigma'$ denote the ranks of the matrices (B_μ), (C_μ), resp. ([151]), and σ is the smallest number such that Δ can be represented in the form:

$$F_1\, df_1 + \ldots + F_\mu\, df_\mu + F_{\mu+1}\, df_{\mu+1} + \ldots + F_\sigma\, df_\sigma .$$

σ' is equal to σ or $\sigma - 1$. In the latter case, it can happen that $F_\sigma \equiv 1$. By means of a theorem on determinants that is due to *J. J. Sylvester* ([152]), the arrays (B_μ), (C_μ) can be replaced with matrices whose elements are certain bracket symbols that are constructed from $f_1, \ldots, f_\mu$. For example, one will then get the theorem:

By means of the relations:

$$f_i\,(z, x_1, x_2, \ldots, x_m, p_1, \ldots, p_m) = 0 \qquad (i = 1, 2, \ldots, \mu), \tag{76}$$

the *Pfaffian* expression $dz - p_1\, dx_1 - \ldots - p_m\, dx_m$ will reduce to an expression in $2m + 1 - \mu$ variables and class:

$$2\,(\rho + m - \mu) + 1 \qquad \text{or} \qquad 2\,(\rho + m - \mu) + 2\,,$$

when the matrix:

$$\|\,[f_i\, f_k]\,\| \qquad (i, k = 1, 2, \ldots, \mu) \tag{77}$$

([146]) J. éc. polyt. **8**, cah. 15 (1809), pp. 281, art. 7. *Jacobi*'s *Werke* **5**, pp. 47, cf., no. **41** (footnote 234.a).
([147]) For the profundity of this theorem, esp., for dynamics, cf., *J. Bertrand*, J. de math. (1) **17** (1852), pp. 393.
([148]) J. f. Math. **60**, pp. 246; **61**, pp. 160.
([149]) J. f. Math. **29**, pp. 236 [*Werke* **4**, pp. 420]. Cf., II A **4** b, no. **12**.
([150]) My book, Chap. 9, **§ 4**. *É. Cartan*, footnote 122.
([151]) (C_μ) arises from (B_μ) upon deleting the $(m + 1)^{\text{th}}$ row and column.
([152]) Phil. Mag. (1851), pp. 279. Cf., *Frobenius*, J. f. Math. **86**, pp. 54.

has rank 2ρ due to (76), and the symbol [ ] has the meaning that it had in (64). Therefore, in order for equations (76) to define an element-$M_{2m+1-\mu}$ (no. **9**), it is necessary and sufficient that the rank of the matrix (77) should be $2\mu - 2m - 2$ ([153]). In order for the $\mu = m + 1$ relations (76) to represent an integral-M_m, it is necessary and sufficient that all expressions $[f_i\ f_k]$ should vanish because of (76). In other words: If the $m + 1$ defining equations of an element-M_m include the relation $\varphi(z, x_1, \ldots, p_m) = 0$ then it will admit the infinitesimal transformation $[\varphi f]$ ([154]).

27. Relations between Pfaff expressions and infinitesimal transformations ([155]). – The most general infinitesimal transformation $X f \equiv \sum \xi_i \frac{\partial f}{\partial x_i}$ that satisfies the conditions that:

$$0 \equiv \sum a_i\, \xi_i \equiv \Lambda\ ; \qquad X\,\Delta \equiv 0 \qquad \text{(cf., no. \textbf{14})}$$

has the form:

$$(78) \qquad \rho_1\, X_1 f + \rho_2\, X_2 f + \ldots + \rho_{m-\kappa}\, X_{m-\kappa} f \qquad \text{(cf., no. \textbf{19})},$$

in which the ρ_i mean arbitrary functions. The most general infinitesimal transformation that satisfies the conditions ([156]):

$$0 \equiv (?)\ \Lambda\ ; \qquad X\,\Delta = \rho\,\Delta$$

for even κ and the condition ([157]):

$$X\,\Delta \equiv d\,\Lambda$$

for odd κ has the form:

$$(79) \qquad \rho_0\, X_0 f + \rho_1\, X_1 f + \ldots + \rho_{m-\kappa}\, X_{m-\kappa} f\,.$$

One then gets the theorem that the two differential systems (46) and (47) are complete and invariantly coupled with Δ, and that will define a simple invariant-theoretic basis for the theory of the *Pfaff* problem ([158]). Thus, e.g., the *Pfaff-Grassmann* reduction method (nos. **18**, **19**) will be deduced immediately from the fact ([159]) that in all of the cases where $\kappa = 2\lambda$ [$2\lambda - 1$, resp.], the

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([153]) *A. V. Bäcklund*, Math. Ann. **11** (1877), pp. 412.

([154]) *Lie*, *Transform.* **2**, Chap. 4; for the analogous theorems in regard to the *Pfaffian* expression $p_1\, dx_1 + \ldots + p_m\, dx_m$, cf., *Lie*, *ibidem* and my book, *loc. cit.*

([155]) *Lie*, Norw. Arch. **2** (1877), pp. 156; Leipz. Ber. (1896), pp. 405. *Engel*, *ibidem*, pp. 412. Cf., my book, Chap. 10.

([156]) The set of infinitesimal transformations that satisfy the equation $X\,\Delta \equiv \rho\,\Delta$ was given in some special cases by *G. Vivanti* [Rend. Palermo **12** (1898), pp. 1] and by myself (*ibidem*, pp. 133). Cf., my book, *loc. cit. F. Engel*, Leipz. Ber. (1899), pp. 296. For infinitesimal *contact transformations*, cf., III D 7 and *Lie*, *Transform.* **2**, Sec. **3**.

([157]) For the most general infinitesimal transformation that satisfies an equation of the form $X\,\Delta \equiv d\,\Omega$, cf., my book, *loc. cit.*

([158]) *Engel*, *loc. cit.* My book, Chap. 10, **§ 2**. Cf., also, the analogous formulation of *G. Darboux* (footnote 109).

([159]) *W. de Tannenberg*, C. R. Acad. Sci. Paris **120** (1895), pp. 674.

Pfaff equation $\Delta = 0$ will go to itself under all transformations of the one-parameter group (II A 6) that is generated by an infinitesimal transformation of the form (79) [(78), resp.] (no. **21**).

IV. – Nonlinear first-order partial differential equations with one unknown

28. Methods of Lagrange and Pfaff. – After *L. Euler* ([160]) had integrated various special categories of nonlinear partial differential equations of the form:

$$f(x, y, z, p, q) = 0 \qquad \left(p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}\right), \tag{80}$$

J. L. Lagrange ([161]) was the first to reduce the general equation (80) to a system of ordinary differential equations. To that end, he sought to find a second relation $\varphi(x, y, z, p, q) = c$ such that after eliminating p and q, the total differential $dz = p\,dx + q\,dy$ would be exact (no. **14**). The integration of the latter would then yield a complete integral $V(x, y, z, c) = c'$ of equation (80). The function φ is an integral of the simultaneous systems:

$$\begin{aligned} & dx : dy : dz : -dp : -dq \\ & = \frac{\partial f}{\partial p} : \frac{\partial f}{\partial q} : p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} : \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} : \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} \end{aligned} \tag{81}$$

that is independent of f.

The first-order partial differential equation with m independent variables:

$$F(z, x_1, \ldots, x_m, p_1, \ldots, p_m) = 0 \qquad \left(p_i \equiv \frac{\partial z}{\partial x_i}\right) \tag{82}$$

was first reduced to ordinary differential equations by *Pfaff* ([104]). In order to find the most general integral function z of (82), one must put that equation into the form:

$$0 = p_m + \psi(z, x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_{m-1}), \tag{83}$$

such that the total differential equation:

$$0 = dz - p_1\,dx_1 - p_2\,dx_2 - \ldots - p_{m-1}\,dx_{m-1} + \psi\,dx_m \quad (\equiv \nabla) \tag{84}$$

([160]) *Institutiones calculi integralis*, **3**, Petersburg, 1770. *G. Elliot* investigated the equations (80) that were free of z and quadratic in p, q in Ann. éc. norm. sup. (1892), pp. 329.

([161]) Berl. Nouv. Mém. (1772) [*Oeuvres* **3**, pp. 546]. Historical remarks in *Jacobi*, J. f. Math. **23**, pp. 3 [*Werke* **4**, pp. 151]. *Lie* and *Scheffers*, *Berührungs.*, pp. 514, 516, *et seq.*

will reduce to a form with m differential elements (no. **18**), and fulfill m relations (no. **20**) in the most general way from which $z, p_1, \ldots, p_{m-1}$ can be calculated as functions of x. The first *Pfaffian* auxiliary system (no. **18**) will then be written as:

$$\text{(85)} \qquad \frac{dx_i}{dx_m} = \frac{\partial \psi}{\partial p_i}\;; \qquad \frac{dp_i}{dx_m} = -\frac{\partial \psi}{\partial p_i} - p_i \frac{\partial \psi}{\partial z}\;; \qquad \frac{dz}{dx_m} = \sum_{h=1}^{m-1} p_h \frac{\partial \psi}{\partial p_h} - \psi$$

$$(i = 1, \ldots, m-1)$$

here. Its integration is equivalent to that of the system:

$$\text{(86)} \qquad \frac{dx_k}{dt} = \frac{\partial F}{\partial p_k}\;; \qquad \frac{dp_k}{dt} = -\frac{\partial F}{\partial x_k} - p_k \frac{\partial F}{\partial z}\;; \qquad \frac{dz}{dt} = \sum_{h=1}^{m-1} p_h \frac{\partial F}{\partial p_h}$$

$$(k = 1, \ldots, m).$$

Namely, that system possesses the integral F, and eliminating p_m by means of (83) and dt will lead to equations (85).

Jacobi ([162]) arrived at the system (86), *inter alia*, from the remark that every integral z of (82) and its derivatives p_i will satisfy the differential system:

$$\sum \frac{\partial F}{\partial p_h}\frac{\partial z}{\partial x_h} = \sum p_h \frac{\partial F}{\partial p_h}\;; \qquad \sum \frac{\partial F}{\partial p_h}\frac{\partial p_i}{\partial x_h} = -\frac{\partial F}{\partial x_i} - p_i \frac{\partial F}{\partial z} \quad (i = 1, \ldots, m),$$

so (no. **11**, footnote 63) the linear homogeneous differential equation:

$$\sum \frac{\partial F}{\partial p_h}\left(\frac{\partial f}{\partial x_h} + p_h \frac{\partial f}{\partial z}\right) - \left(\frac{\partial F}{\partial x_h} + p_h \frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial p_h} = 0$$

will be defined by (82) and m other relations ([163]).

29. Cauchy's method. – *A. Cauchy* ([164]) showed that in order to integrate (82), it is sufficient to integrate the simultaneous system (85) or (86). If an integral of (82) is defined by the equations:

([162]) J. f. Math. **2** (1827), pp. 317 [*Werke* **4**, pp. 1].

([163]) How that is determined in order for the p_i to be derivatives of z was shown by *L. Boltzmann*, Wiener Ber. **2**, 2 Abt. (1875), pp. 471.

([164]) Bull. soc. philomath. (1819), pp. 10; C. R. Acad. Sci. Paris **14** (1842), pp. 740, 769, 881, 952, 1026 [*Oeuvres* (1) **6**, pp. 423, 431, 444, 459, 467].

$$(87) \qquad z = f(x_1, x_2, \ldots, x_m)\,, \qquad p_i = f_i(x_1, \ldots, x_m) \qquad \left(f_i \equiv \frac{\partial f}{\partial x_i}\right),$$

and one introduces the new variables ([165]) t, u_2, u_3, …, u_m in place of x such that the first m of equations (86) will be satisfied for constant u then the functions z, p_i of the variable t that are defined by (87) will fulfill the remaining equations (86) automatically. Therefore, if x_i^0, z^0, p_i^0 are those functions of the parameter u to which x_i, f, f_i, resp., reduce for $t = 0$, and furthermore if:

$$(88) \qquad x_i = \xi_i(t, x_1^0, \ldots, x_m^0, z^0, p_1^0, \ldots, p_m^0)\,, \qquad z = \zeta(t, x_1^0, \ldots)\,, \qquad p_i = \pi_i(t, x_1^0, \ldots)$$

are those integral functions of the system (86) that go to x_i^0, z^0, p_i^0 when $t = 0$ then eliminating t, u from (88) will once more yield the relations (87).

An identity of the form ([166]):

$$(89) \qquad \delta\zeta - \sum \pi_i\, \delta\xi_i = -\,(?)\ \rho(\delta z^0 - \sum p_i^0\, \delta x_i^0) + \sigma\, \delta F(z^0, x_1^0, \ldots, p_m^0)$$

exists for arbitrary increments of t, u. If the x^0, z^0, p^0 fulfill the equations:

$$(90) \qquad F(z^0, x_1^0, \ldots, x_m^0, p_1^0, \ldots, p_m^0) = 0\,,$$

$$(91) \qquad \delta z^0 - p_1^0\, x_1^0 - \ldots - p_1^0\, \delta x_m^0 = 0$$

then the functions of the m variables t, u_i that are defined by (88) will satisfy the condition (82) and:

$$(92) \qquad dz - p_1\, dx_1 - \ldots - p_m\, dx_m = 0$$

identically. If the elimination of t, u from (88) leads to $m + 1$ equations of the form (87) then the f_i will be the derivatives of f, and f will be an integral of (82).

The integration of (82) is then reduced to determining the quantities x_i^0, z^0, p_i^0 as functions of $m - 1$ parameters u in such a way that the relations (90), (91) will be fulfilled, i.e., the *Pfaff* equation (91) will be satisfied in the most general way by equation (90) and $m + 1$ further relations between the x_i^0, z^0, p_i^0, which is possible with no integration (no. **9**). Examples of $m + 1$ such relations are, e.g., the following ones:

([165]) Cf., also *Ampère*, J. éc. polyt., **10**, cah. 17, and esp. **11**, cah. 18, pp. 1-34 and 43, *et seq.* (1820). His presentation of the theory (for $m = 2$) was based upon a consideration of the arbitrary functions that enter into the general integral (see nos. **4** and **48**). It was adapted to the case of an arbitrary m by *E. Padova* (*Collectanea matematica*, Mailand, 1881, pp. 105).

([166]) Cf., also *Binet*, C. R. Acad. Sci. Paris **14** (1842), pp. 654.

$$x_m^0 = \text{const.}, \qquad z^0 = \varphi(x_1^0, \dots, x_{n-1}^0), \qquad p_i^0 = \frac{\partial\varphi}{\partial x_i^0} \qquad (i = 1, \dots, m-1)\,. \tag{93}$$

Eliminating t, x_i^0, z^0, p_i^0 from (88), (90), (93) will give the integral z that will go to $\varphi(x_1, \dots, x_{m-1})$ when $x_m = x_m^0$ (no. **1**).

One will also get that integral when one determines the principal integral:

$$\mathfrak{x}_k(z, x_1, \dots, x_m, p_1, \dots, p_{m-1})\,, \quad \mathfrak{z}(z, x_1, \dots)\,; \quad \mathfrak{x}_k(z, \dots) \qquad (k = 1, \dots, m-1) \tag{94}$$

of the system (85) with respect to $x_m = x_m^0$, and eliminates the x_i^0, z^0, p_i^0 from (90), (93), and the equations:

$$\mathfrak{x}_k = x_k^0\,; \qquad \mathfrak{z} = z^0\,; \qquad \mathfrak{p}_k = p_k^0 \qquad (k = 1, 2, \dots, m-1). \tag{95}$$

30. Jacobi's first method ([167]) is a special case of *Cauchy*'s method ([168]) and is based upon the fact that the *Pfaff* expression ∇ on the right-hand side of (84) will already be obtained in the normal form:

$$\nabla \equiv \rho\left(d\mathfrak{z} - \sum_{h=1}^{m-1} \mathfrak{p}_h\, d\mathfrak{x}_h\right) \tag{96}$$

when one uses the principal integral (94) of *Pfaff*'s auxiliary system (85) (no. **18**, footnote 107). In so doing, one must distinguish three cases:

α) The function ψ is not independent of z.
β) ψ is free of z ([169]), but not homogeneous of first order in the p_i .
γ) ψ is free of z and homogeneous of first order in the p_i .

In case β), the factor in (96) is $\rho \equiv 1$, the $\mathfrak{x}$, $\mathfrak{p}$ are the principal integrals of the "canonical" ([170]) system:

([167]) J. f. Math. **17** (1837), pp. 97. [*Werke* **4**, pp. 57, esp., pp. 104].

([168]) Cf., *Cauchy*'s remarks in C. R. Acad. Sci. Paris **14** (1842), pp. 881 [*Werke* (1) **6**, pp. 444] and *Exercices d'anal. et de phys.*, Paris, 1841, pp. 239.

([169]) *Jacobi* gave two methods for replacing every equation (82) with one that does not include the unknowns explicitly. The first one consists of defining z by means of a relation $V(z, x_1, \dots, x_m) = 0$, which will make (82) go to an equation with the unknown V and the independent variables z, x_i (*Jacobi*, J. f. Math. **23**, pp. 18 [*Werke* **4**, pp. 166]; cf., *Goursat A*, art. 16, *Delassus*, art. 23. A generalization to higher differential problems was given by *L. Königsberger*, J. f. Math. **109**, pp. 338, *et seq.*). For the second method, see (J. f. Math. **60**, pp. 1 [*Werke* **5**, pp. 1]; 31. *Vorlesung über Dynamik* [*Werke* Suppl., pp. 237]). Cf., *A. Mayer*, Math. Ann. **9**, pp. 366; *Mansion*, **§ 1.**

([170]) *W. R. Hamilton*, Trans. Lond. Math. Soc. (1834), pp. 247; (1835), pp. 95. Cf., *Imschenetzky*, Arch. Math. **50**, pp. 428 and the following no.

$$(97) \qquad dx_k = \frac{\partial \psi}{\partial p_k}\, dx_m\,, \qquad dp_k = -\frac{\partial \psi}{\partial x_k}\, dx_m \qquad (k = 1, 2, \ldots, m-1)$$

with respect to $x_m = x_m^0$, and $\mathfrak{z}$ has the form $z - U$ ([171]), where U emerges from the integral ([172]):

$$(98) \qquad \int_{x_m^0}^{x_m} \left(p_1 \frac{\partial \psi}{\partial p_1} + \cdots + p_{m-1} \frac{\partial \psi}{\partial p_{m-1}} - \psi \right) dx_m$$

when one expresses the x_k and p_k by means of the equations:

$$\mathfrak{x}_k = a_k\,; \qquad \mathfrak{p}_k = b_k \qquad (k = 1, \ldots, m-1)$$

and the arbitrary constants a_i, b_i before the integration and after replacing a_i, b_i with $\mathfrak{x}_k$, $\mathfrak{p}_k$, resp. In the case γ), one has $\mathfrak{z} \equiv z$, while the $\mathfrak{p}_k$ and $\mathfrak{x}_k$ are homogeneous of order one (zero, resp.) in the p, and the *Pfaffian* expression:

$$\nabla_1 \equiv p_1\, dx_1 + \ldots + p_{m-1}\, dx_{m-1} - \psi\, dx_m$$

possesses the normal form $\mathfrak{p}_1\, d\mathfrak{x}_1 + \ldots + \mathfrak{p}_{m-1}\, d\mathfrak{x}_{m-1}$. Those theorems agree with the fact ([173]) that the expression ∇ possesses class $2m$ in the case α), class $2m$ in the case β), and class $2m - 1$ in case γ), while the class of ∇_1 is equal to $2m - 1$ in case β), and equal to $2m - 2$ in case γ) (no. **19**).

Upon eliminating $p_1, \ldots, p_{m-1}$, the m relations $\mathfrak{x}_k = a_k$, $\mathfrak{z} = a_m$ will yield, *inter alia*, a complete integral (no. **7**):

$$(99) \qquad z = \Phi\,(x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_m)$$

of the given equation (83), that will have the form:

$$(100) \qquad z = \Psi\,(x_1, x_2, \ldots, x_m, a_1, \ldots, a_{m-1}) + a_m$$

in case β). However, that elimination can also yield several ([174]) relations in $z, x_1, \ldots, x_m$, e.g., that is always true in case γ). However, since:

([171]) $\sum_{i=1}^{m} p_i\, dx_i$ will then be an exact differential when one replaces the p_i with their values that follow from (82) and $\mathfrak{x}_k = a_k$. Cf., *A. Cayley*, Math. Ann. **11**, pp. 194 [*Papers* **10**, pp. 134].

([172]) That quadrature can be performed immediately when a complete integral of (82) is already known (*Mayer*, Math. **6**, pp. 166; *Goursat A*, art. 56, remark).

([173]) See *G. Morera*, Rend. Lombardo (2) **16** (1883), pp. 637, 691 and my book, Chap. 12, **§ 2.**

([174]) *J. A. Serret*, Ann. éc. norm. sup. (1866), pp. 153, *et seq.*

$$\nabla \equiv \rho[d(\mathfrak{z} - \sum \mathfrak{p}_h \mathfrak{x}_h) + \sum \mathfrak{x}_h \mathfrak{p}_h] ,$$

due to (96), one will always get a complete integral of the form (99) when one eliminates the p_i from the equation:

$$\mathfrak{z} - \mathfrak{p}_1 \mathfrak{x}_1 - \ldots - \mathfrak{p}_{m-1} \mathfrak{x}_{m-1} = a_m$$

by means of the equations $\mathfrak{x}_k = a_k$ ([175]), which is always possible.

In case γ), every complete integral can be written in the form:

(101) $$z = a_{m-1} \cdot V(x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_{m-2}) + a_m .$$

In order for equation (99) to represent a complete integral of (83), it is necessary and sufficient that the equations:

(102) $$z = \Phi , \qquad p_1 = \frac{\partial \Phi}{\partial x_1}, \ldots, p_{m-1} = \frac{\partial \Phi}{\partial x_{m-1}}$$

can be solved for $a_1, \ldots, a_m$, and the relation (83) will arise by substituting the expressions obtained in $p_m = \partial\Phi / \partial x_m$. That substitution will then convert the expression:

(103) $$\frac{\partial \Phi}{\partial a_1} da_1 + \frac{\partial \Phi}{\partial a_2} da_2 + \cdots + \frac{\partial \Phi}{\partial a_m} da_m$$

into a normal form of the *Pfaffian* expression ∇, from which, from a remark in no. **19** (footnote 119), it will follow that when the relations:

(104) $$z = \Phi , \qquad p_i = \frac{\partial \Phi}{\partial x_i} ; \qquad \frac{\partial \Phi}{\partial a_i} + b_i \frac{\partial \Phi}{\partial a_m} = 0 \qquad (i = 1, \ldots, m-1)$$

are solved for the arbitrary a, b, they will yield the general integral equations of the simultaneous system (85). In particular, in case β), one will get the general integral equations of the canonical system in the form ([176]):

$$p_i = \frac{\partial \Psi}{\partial x_i}, \qquad b_i = \frac{\partial \Psi}{\partial a_i} \qquad (i = 1, \ldots, m-1)$$

from a complete integral (100).

([175]) *A. Mayer*, Math. Ann. **3** (1871), pp. 435. The process will be identical to the one in the previous no. when φ is replaced with $a_m + a_1 x_1^0 + \cdots + a_{m-1} x_{m-1}^0$ in (93). Other methods were given by *G. Darboux*, C. R. Acad. Sci. Paris **79** (1875), pp. 1488, 160 [Darboux Bull. (1) **8**, pp. 249]. *J. Bertrand*, C. R. Acad. Sci. Paris **82** (1876), pp. 641. *J. Farkas*, *ibid*., **98**, pp. 352.

31. The Hamilton-Jacobi theory ([176]). – The case of the differential equations of dynamics is included in the results of the previous no. as a special case, and that defines the starting point for *Jacobi*'s investigations.

Let $q_1, \ldots, q_m$ be functions of the variable t, and let $q_1', \ldots, q_m'$ be their derivatives. In order for the variation of the integral:

$$S = \int_\tau^t V(t, q_1, \ldots, q_m, q_1', \ldots, q_m')\, dt$$

to vanish under the assumption that the variations of t, q_i, q_i' are zero at the limits of integration, it is necessary and sufficient that the q_i should satisfy the system of simultaneous second-order differential equations:

$$\frac{d}{dt}\frac{\partial V}{\partial q_s'} - \frac{\partial V}{\partial q_s} = 0 \qquad (s = 1, 2, \ldots, m). \tag{105}$$

If $H(t, q_1, \ldots, q_m, p_1, \ldots, p_m)$ is the function that arises from:

$$p_1 q_1' + \cdots + p_m q_m' - V$$

when one eliminates the q_m' from it by means of the equations:

$$p_i = \frac{\partial V}{\partial q_i'} \qquad (i = 1, 2, \ldots, m) \tag{106}$$

then the following relation will exist between the *canonical* system:

$$\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \qquad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s} \qquad (s = 1, 2, \ldots, m) \tag{107}$$

of the system (105) and the partial differential equation:

$$\frac{\partial z}{\partial t} + H\left(t, q_1, \ldots, q_m, \frac{\partial z}{\partial q_1}, \ldots, \frac{\partial z}{\partial q_m}\right) = 0, \tag{108}$$

namely:

([176]) *Hamilton*, Trans. Lond. Math. Soc. (1834), pp. 247; (1835), pp. 95. *Jacobi*, 19 and 20 *Vorlesungen über Dynamik*, and the posthumous treatise in *Werke* **5**, pp. 217-395. Cf., *A. Cayley*'s "Report on Theoretical Dynamics," Rep. Brit. Ass. (1857), pp. 1-42, and the textbooks on mechanics, e.g., *F. Tisserand, Mécanique celeste*, I. Introduction (Paris, 1889). *P. Painlevé*, *Leçons sur l'intégration des équ. diff. de la mécanique* (Paris, 1895).

One will get the general integral equations of (105) from those of (107) when one eliminates p_i by means of (106), and conversely, one will get the general integral equations of (107) from those of (105) by eliminating the q_i'. If one has integrated the system (107) in the form:

$$q_i = \kappa_i (t, \tau, q_1^0, \ldots, q_m^0, p_1^0, \ldots, p_m^0) \qquad (i = 1, \ldots, m)\,, \tag{109.1}$$

$$p_i = \pi_i (t, \tau, q_1^0, \ldots, q_m^0, p_1^0, \ldots, p_m^0)\,, \tag{109.2}$$

in which the functions κ_i, π_i reduce to q_i^0, p_i^0, resp., for $t = \tau$, then one will get a complete integral:

$$z = \Omega(t, q_1, \ldots, q_m, q_1^0, \ldots, q_m^0) + c$$

with the arbitrary constants q_i^0, c for the partial differential equation (108). One eliminates the q_i' from V by means of (106), which will make the integral S assume the form:

$$S \equiv \int_\tau^t \left(p_1 \frac{\partial H}{\partial p_1} + \cdots + p_m \frac{\partial H}{\partial p_m} - H \right) .$$

One then replaces the q_i, p_i under the ∫ sign with their expressions (109), eliminates the p_i, p_i^0 by means of (109) ([177]) after performing the quadrature, and denotes the function that thus arises from S by Ω.

Equations (109.1), (109.2) can then be written in the equivalent form ([178]):

$$\frac{\partial \Omega}{\partial q_i} = p_i\,, \qquad \frac{\partial \Omega}{\partial q_i^0} = -\,p_i^0 \qquad (i = 1, 2, \ldots, m).$$

More generally, an arbitrary complete integral:

$$z = \Phi\,(t, q_1, \ldots, q_m, c_1, \ldots c_m) + c$$

of equation (108) will yield the general integral equations of the system (107) in the form:

([177]) For the function-theoretic sense of those eliminations, cf., my book, art. 381, 382. Ω satisfies a partial differential equation with respect to the independent variables τ, q_i^0 that is entirely similar to equation (108); cf., *Hamilton* and *Jacobi*, *loc. cit.*

([178]) *Hamilton* proved those theorems (in part, that will follow from the previous no.) by means of the expression for the variation δS; cf., also *Jacobi*, *loc. cit.*, and *E. J. Routh*, *Dynamics of Systems of Rigid bodies* (Ger. ed., Leipzig, 1898, **2**, Chap. 10).

$$\frac{\partial \Phi}{\partial q_i} = p_i\,, \qquad \frac{\partial \Phi}{\partial c_i} = \gamma_i \qquad (i = 1, 2, \ldots, m). \tag{110}$$

The three integration problems (105), (107), (108) are therefore completely equivalent ([179]).

If t is the time, and $q_1, \ldots, q_m$ are the *Lagrangian* coordinates of a dynamical problem with m degrees of freedom, and furthermore:

$$T\,(t, q_1, \ldots, q_m, q'_1, \ldots, q'_m)$$

is the *vis viva* of the system, while $U\,(t, q_1, \ldots, q_m)$ is the force function, and finally $V \equiv T + U$ then equations (105) will be identical to the *Lagrangian* differential equations of motion, and the solution of the dynamical problem will then be reduced to the problem of finding a complete integral to ***Hamilton's*** *partial differential equation* (108). Under the assumption that the constraint equations of the problem do not depend upon t, so T will be homogeneous in the q'_i, H will be identical to $T - U$ by means of (106) and will be called the ***Hamiltonian*** *function* (*Jacobi*), while Ω is the "principal function" (*Hamilton*) of the dynamical problem.

If one considers two dynamical problems at the same time, namely, the *unperturbed* problem with the *Hamiltonian* function H, whose general integral equations are given by (110), and the *perturbed* problem with the *Hamiltonian* function $H + H_1$, and one introduces the new variables c_i, γ_i in place of p_i, q_i, resp., by means of (110), which will make H go to $H'_1(t, c_1, \ldots, c_m, \gamma_1, \ldots, \gamma_m)$, then the differential equations of the perturbed problem:

$$\frac{dq_s}{dt} = \frac{\partial (H + H_1)}{\partial p_s}\,; \qquad \frac{dp_s}{dt} = -\frac{\partial (H + H_1)}{\partial q_s}$$

will, in turn, be converted into a canonical system:

$$\frac{dc_s}{dt} = -\frac{\partial H'_1}{\partial \gamma_s}\,; \qquad \frac{d\gamma_s}{dt} = \frac{\partial H'_1}{\partial c_s} \qquad (s = 1, \ldots, m). \tag{111}$$

Those equations define the changes that the quantities c_i, γ_i (which are constant in the unperturbed problem) will experience in the course of time by the addition of the *perturbation function* H_1. The c_i, γ_i are called the canonical elements of the unperturbed problem ([179.a]). The principal integrals of the simultaneous system (107) for $t = 0$ are examples of such canonical elements ([179.b]). The relationship between the c, γ and the p, q that is mediated by (110) is a *contact transformation*

([179]) For the application of this theory to the isoperimetric problems, see *Jacobi*, *Werke* **5**, pp. 465.

([179.a]) *Hamilton, Jacobi* (footnote 176). For the older theories of perturbations (Lagrange, Laplace, Poisson, etc.), cf., *Cayley*'s Report.

([179.b]) *Lagrange* had already considered those elements and the associated perturbation equations (111) in *Mécan. anal.*, **1**, 2nd partie, sect. 5, § **2**.

([179.c]), and the partial derivatives of the c, γ with respect to the p, q give relations that are characteristic of the right-hand side of such a thing ([179.d]). If one imagines that the p, q are expressed as functions of the c, γ then some simple relations will exist between the derivatives of the c, γ and those of the p, q ([179.e]).

The most general *canonical substitution* ([179.f]), i.e., the transformation of variables:

$$\bar{q}_i = Q_i(t, q_1, \ldots, q_m, p_1, \ldots, p_m), \qquad \bar{p}_i = P_i(t, q_1, \ldots, q_m, p_1, \ldots, p_m),$$

that will convert any canonical system (107) into another one, was given by *Lie* ([179.g]). In the event that the Q, P are free of t, it is a contact transformation, so it will satisfy an identity:

$$dW(x_1, \ldots, x_m, p_1, \ldots, p_m) + \sum P_i \, dQ_i \equiv \sum p_i \, dq_i .$$

32. Variation of constants. Characteristic curves. – According to no. **30**, one will get the most general integral from an arbitrary complete integral (99) of equation (83) when one annuls the expression (103) in the most general way by means of m relations between the a_i, x_i, z, i.e., r ($< m$) arbitrary equations:

(112) $$\varphi_s(a_1, a_2, \ldots, a_m) = 0 \qquad (s = 1, 2, \ldots, r),$$

as well as (99), and eliminates the a, λ from the relations:

(113) $$\frac{\partial \Phi}{\partial a_i} = \lambda_1 \frac{\partial \varphi_1}{\partial a_i} + \lambda_2 \frac{\partial \varphi_2}{\partial a_i} + \cdots + \lambda_r \frac{\partial \varphi_r}{\partial a_i} \qquad (i = 1, 2, \ldots, m)$$

[*J. L. Lagrange*'s "Variation of constants" ([180])]. The integral surface that one obtains then takes the form of the envelope of the ∞^{m-r} surfaces that are selected from the family (99) by the r equations (112), and corresponding to the values $r = 1, 2, \ldots, m$, there will be m (not essentially different) categories of integral surfaces. If:

$$\Phi(x, y, z, a, b) = 0$$

([179.c]) *Lie*, Norw. Arch. **2** (1877), pp. 129.

([179.d]) *Hamilton, Jacobi, loc. cit. Donkin, Liouville, Bour* (footnote 212). *E. Schering*, Gött. Abh. **18** (1873); **19** (1874); see also no. **36**.

([179.e]) *Jacobi*, J. f. Math. **30**, pp. 117 [*Werke* **4**, pp. 137; *Werke* **5**, pp. 317].

([179.f]) *Jacobi*, *Werke* **4**, pp. 136; **5**, pp. 369, *et seq. Schering, loc. cit.*

([179.g]) *Loc. cit.*; the most general transformation that will convert a *particular* canonical system (107) into another one was also given there.

([180]) See footnote 161 and Berl. Hist. 1774 [*Werke* **4**, pp. 5, esp. pp. 62, *et seq.*]; *Leçons sur le calcul des fonctions* (Paris, 1806), pp. 353. For exceptional cases in which the elimination of the a, λ yields more than one relation in the z, x, see *Darboux*, "Sol. sing.," pp. 98.

is a complete integral of equation (80) in no. **28** then from the foregoing, the most general integral surface V will be the envelope ([181]) of any ∞^1 surfaces that are selected from the family above by a relation $b = \varphi(a)$, each of which will cut out a *characteristic curve* ([182]) from the neighboring one, so it will be generated by ∞^1 characteristic curves, which will envelope a curve in their own right, namely, the *edge of regression* of the surface V ([183]). There are ∞^3 [and merely ∞^2 for a linear equation (80)] characteristic curves. They are defined by the equations:

$$\Phi = 0\,, \qquad \frac{\partial \Phi}{\partial a} + c\frac{\partial \Phi}{\partial b} = 0\,. \tag{114}$$

The condition for two of those curves that neighbor each other to intersect is $db - c\,da = 0$ ([184]). An arbitrary ∞^1-family of characteristic curves that each intersect its neighbor will generate an integral surface, and in particular, all of the ∞^1 curves that contain the same point P (x, y, z) will generate the *integral conoid* ([185]) with the vertex P. The tangents to those ∞^1 curves at the point P are the generators of the *elementary cone* with its vertex at P ([186]). That cone is defined by a certain "Monge equation" ([187]):

$$\varphi(x, y, z, dx, dy, dz) = 0\,, \tag{115}$$

and is the envelope ∞^1 planes whose equations have the form:

$$\zeta - z = p\,(\xi - x) + q\,(\eta - y)\,, \tag{116}$$

whose coefficients p, q fulfill equation (80). It is only for a *linear* equation that the elementary cone will degenerate into a pencil of planes. The edges of regression of the integral surfaces fulfill the *Monge* equation (115). Conversely, one and only one integral surface will go through any curve that is not a characteristic curve and fulfills (115) whose edge of regression is that curve ([188]). Any such curve is called an "integral curve" [*Lie*, ([189])] of the partial differential equation (80).

([181]) The geometric interpretation of the variation of constants goes back to *G. Monge*, *Application de l'analyse à la géométrie*, 5th ed., Paris, 1850, pp. 421, 432. Cf., *Du Bois-Reymond*, *Beiträge*, Sects. 1 and 3. *Lie-Scheffers*, *Berührungs*, Chap. 11.

([182]) *Monge* said: "Characteristic," *loc. cit.*, pp. 432. Cf., no. **34** in the text.

([183]) *Monge*, *loc. cit.*; cf., *Darboux*, "Sol. sing."

([184]) For the relationships between that equation and the *Monge* equation (115), and above all, between an arbitrary non-integrable *Pfaff* equation and a *Monge* equation, cf., *Lie*, Leipz. Ber. (1897), pp. 704, *et seq*.

([185]) *Du Bois-Reymond*, *Beiträge*, pp. 62. The integral conoid was already considered by *O. Bonnet*, C. R. Acad. Sci. Paris **45** (1857), pp. 581.

([186]) *O. Bonnet*, *loc. cit.*; *Lie-Scheffers*, *Berührungs.*, Part III. Historical remarks, *ibidem*, pp. 518, *et seq.*

([187]) See footnote 103; *Lie-Scheffers*, *Berührungs.*, Chap. 7; *Du Bois-Reymond*, *Beiträge*, pp. 22. *Lie*, Leipz. Ber. (1897), pp. 687.

([188]) There are, *inter alia*, ∞^2 integral surfaces with one bicuspidal edge, *Darboux*, "Sol. sing.," pp. 47.

([189]) Math. Ann. **5**, pp. 151, *et seq.* Cf., *Du Bois-Reymond*, *Beiträge*, Chap. 7, 8, 11. *Darboux*, "Sol. sing.," pp. 36-58.

An integral surface is characterized by the fact that it will contact the associated elementary cone at each of its points *P*, and indeed along the tangent direction to the characteristic curve that goes through *P* and lies on *V*.

If an integral curve contacts an integral surface of (80) then there will be at least two points where it contacts it. Therefore, if the ∞^3 curves of a complex are found among the integral curves then the ∞^1 characteristic curves will be, at the same time, principal tangent curves on any integral surface ([190]).

The concepts of characteristic curve, elementary cone, integral conoid can be adapted to an equation (82) with *m* independent variables with no further considerations.

33. Singular integrals. – The expression (103) can also be made to vanish by the assumption that:

$$\frac{\partial \Phi}{\partial a_1} = 0, \qquad \frac{\partial \Phi}{\partial a_2} = 0, \qquad \ldots, \qquad \frac{\partial \Phi}{\partial a_m} = 0. \tag{117}$$

If the family of surfaces (99) is given arbitrarily, and (82) is the associated differential equation, which follows upon eliminating the *a* from (99) and the equations $p_i = \partial\Phi / \partial a_i$, then the envelope *W* of the family (99) that is obtained by eliminating the *a* from (99), (117) will fulfill the partial differential equations:

$$\frac{\partial F}{\partial p_i} = 0, \tag{118.1}$$

$$\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial z} = 0 \qquad (i = 1, 2, \ldots, m), \tag{118.2}$$

so it will be a singular integral ([191]) (no. **5**) of (82). The surface *W* is not generated by characteristic curves ([192]). It will contact ∞^{m-1} characteristic curves at each point *P*, and they will generate a surface of the family (99) that contacts *P*. The integral surface, whose equation will arise upon eliminating the *a* and λ from (99), (112), (113), contacts *W* along an $(m - r)$-fold extended point manifold, and it will be determined when the latter is given ([193]).

An *arbitrarily*-given equation (82) possesses no singular integral ([194]). The elimination of the *p* from (82) and (118.1) [(82) and (118.2), resp.] will generally yield two different surfaces that are loci of point [tangent, resp.] singularities of the ∞^m surfaces (99), and the first of those surfaces

([190]) *Lie*, Math. Ann. **5**, pp. 154, 189; *ibidem*, on pp. 192, 196, first-order partial differential equations on whose integral surfaces the characteristic curves were lines of curvature or geodetic lines were considered. Cf., *Lie-Scheffers*, *Berührungs.*, Chap. 9 and 14. *Du Bois-Reymond*, *Beiträge*, pp. 127, *et seq.*

([191]) *J. L. Lagrange*, Berl. nouv. mém. (1774) [*Werke* **4**, pp. 5]. Cf., *Darboux*, "Sol. sing.," *F. Casorati*, Rend. Lombardo (2) **9** (1876), pp. 522.

([192]) The fact that there can be only *one* such integral was shown by *Lie*, Math. Ann. **9**, pp. 264.

([193]) Cf., *H. Weber*, J. f. Math. **66**, pp. 227, *et seq.*

([194]) *Darboux*, "Sol. sing.," pp. 113.

is the locus of vertices of the characteristic curves ([195]). If both of them coincide in *one* surface W then in the event that it does not satisfy the partial differential equation $\partial F / \partial z = 0$, it will be a singular integral ([196]) of (82) and the envelope of the ∞^m surfaces of a complete integral, which can, however, also possess singularities at its contact points with W. There can be infinitely many singular integrals for which $\partial F / \partial z = 0$ ([197]). In all cases, the question of the possible existence of singular integrals is resolved by the search for all common integrals of the differential system (82), (118) (cf., no. **48**).

34. Characteristic strips. Mapping and classification of first-order partial differential equations. – If one, with *Lie* (no. **9**), understands an integral or an integral-M_m of equation (82) to mean any m + 1-parameter system of relations in z, x_i, p_i that subsumes (82) and fulfills the **Pfaff** equation (92), and one calls a system of values z, x_i, p_i that satisfies (82) a *singular* (*non-singular*, resp.) surface element according to whether it does (does not, resp.) satisfy all relations (118.1), (118.2) then that will imply the following interpretation and generalization of *Cauchy*'s theory (no. **29**) ([198]):

The ∞^{2m} non-singular surface elements of (82) arrange themselves into ∞^{2m-1} first-order *characteristic strips* or *characteristics* that are defined by the simultaneous system (86) or its integral equations (88). One and only one such strip (88) goes through each non-singular surface element with the coordinates z^0, x_i^0, p_i^0. The point manifold that belongs to a characteristic is a characteristic *curve*. If an integral-M_m does not include the non-singular element E then it will subsume the entire characteristic that goes through E. Every non-singular integral-M_m is then generated by ∞^{m-1} characteristics, i.e., its m + 1 defining equations admit the infinitesimal transformation:

$$[F, f] \equiv \sum_{k=1}^{m} \frac{\partial F}{\partial p_h}\left(\frac{\partial f}{\partial x_h} + p_h \frac{\partial f}{\partial z}\right) - \left(\frac{\partial F}{\partial x_h} + p_h \frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial p_h} \tag{119}$$

(cf., the conclusion of no. **26**, footnote 154), or what amounts to the same thing, the infinitesimal contact transformation: $[F, f] - F\,(\partial f / \partial z)$ (see III D 7). If two neighboring surface elements of (82) are in united position then due to (89), the characteristics that emanate from them will be found to be in united position along their entire extent. The $\infty^{\nu-1}$ characteristics that start from the respective elements of an integral-$M_{\nu-1}$ then define an integral-M_ν that will be called a *ν-dimensional characteristic* or *characteristic* M_ν for $\nu < m$. The integration of (82) will then be achieved when one knows the characteristics or also the finite equations of the one-parameter group of contact transformations $[F\, f] - F\,(\partial f / \partial z)$. The most general integral-M_m will be obtained by applying the ∞^1 transformations of that group to an arbitrary non-singular integral M_{m-1}. Whereas an integral-M_m is established uniquely in that way from a non-singular and non-

([195]) *Darboux*, *loc. cit.*, pp. 146, *et seq.*; 172, *et seq.*

([196]) *H. Weber*, *loc. cit.*, pp. 216, *et seq. Darboux*, *loc. cit.*, pp. 177, 185.

([197]) *Darboux*, *loc. cit.*, pp. 193, *et seq.*

([198]) Developed by *Lie* since 1871; cf., *Transform.*, **2**, Abt. 1; *Lie-Scheffers*, *Berührungs.*, Chap. 12; Historical remarks, *ibidem*, pp. 518 and 564, rem.

characteristic integral-M_{m-1}, there are integral-M_m that include the same characteristic M_{m-1}. Thus, e.g., in the case of equation (80), the condition for the relations:

$$\frac{\partial f}{\partial x}+p\,\frac{\partial f}{\partial z}+r\,\frac{\partial f}{\partial p}+s\,\frac{\partial f}{\partial q}=0\,,\qquad \frac{\partial f}{\partial y}+q\,\frac{\partial f}{\partial z}+s\,\frac{\partial f}{\partial p}+t\,\frac{\partial f}{\partial q}=0\,,$$

$$dp = r\,dx + s\,dy\,;\qquad dq = s\,dx + t\,dy$$

to leave r, s, t undetermined will lead back to the defining equations (81) of the characteristics ([199]). Not only will ∞^1 planes go through every point x, y, z, but also every plane (116) will be assigned ∞^1 points x, y, z that lie on it ([200]) and define a curve κ. The defining equations (81) of the characteristics then express the idea that in every element E of an integral surface of (80), the tangent directions to the characteristic curve and the tangent directions to the curve κ that lies in the plane of E are conjugate ([201]).

The relations (95) of no. **29**, along with (83), represent the characteristic that starts from the surface element z^0, x_i^0, p_i^0 in any event. Similarly, the ∞^{2m-1} characteristic strips will be defined by equations (83), (104) with the help of a complete integral (99) ([202]).

From no. **9**, one must consider every m + 1-parameter system of equations of the form:

$$(120)\qquad p_m + \psi = 0\,,\quad \Xi_i\,(z, x_1, \ldots, x_m, p_1, \ldots, p_{m-1}) = c_i\qquad (i = 1, 2, \ldots, m)$$

that defines an element-M_m for arbitrary constants c_i to be a *complete integral* of equation (83). In order for that to exist, it is necessary and sufficient that the *Pfaff* equation (84) can be put into the form:

$$(121)\qquad d\,\Xi_m - \Pi_1\,d\,\Xi_m - \Pi_1\,d\,\Xi_m - \ldots - \Pi_{m-1}\,d\,\Xi_{m-1} = 0\,.$$

The Π_i will be obtained by solving linear equations when one knows the Ξ, and the ∞^{2m-1} characteristics will be defined by the relations:

$$p_m + \psi = 0\,,\quad \Xi_i = c_i\,,\quad \Pi_k = \gamma_k\qquad (i = 1, \ldots, m\,;\ k = 1, \ldots, m-1).$$

([199]) See also *Goursat*, Acta math. **19**, pp. 285.

([200]) This dualistic character (*Darboux*, “Sol. sing.,” pp. 17) will emerge especially when one uses ***Clebsch* *connection coordinates***, i.e., homogeneous point and plane coordinates x_1, x_2, x_3, x_4, and u_1, u_2, u_3, u_4, resp., that are coupled by the relation $\sum u_i\,x_i = 0$, as coordinates of the elements and regards the first-order partial differential equation accordingly as the *principal coincidence* of a space connection. Moreover, all surface elements that lie at finite or infinite points will then appear to be equivalent. Cf., *Clebsch-Lindemann*, *Geometrie* I 2, Abt. 7; “Sol. sing.,” pp. 238.

([201]) *Monge*, *Applications*, pp. 430; *Darboux*, “Sol. sing.,” pp. 23.

([202]) The solution of the *Cauchy* problem (no. **2**) in terms of a complete integral was given by *A. Mayer*, Math. Ann. **3**, pp. 452.

From no. **30**, a complete integral of the form (99) can be exhibited from an *arbitrary* complete integral by means of certain eliminations. One will get the $m + 1$ defining equations of the most general non-singular integral-M_m when one fulfills the *Pfaff* equation (121) as in no. **9** by means of m relations between the Ξ, Π and adds (83). *Lagrange*'s variation of constants is a special case of that method.

The complete integral $\mathfrak{z} = z^0$, $\mathfrak{x}_1 = x_1^0$, …, $\mathfrak{x}_{m-1} = x_{m-1}^0$ that was found in no. **30** consists of all integral conoids whose vertices lie in the plane $x_m - x_m^0 = 0$. From no. **30** (cf., footnote 174), the integral conoids do not need to be surfaces, but rather they can degenerate into element-M_m^ρ ($\rho < m$, cf., **9**), e.g., for every equation (83) of type γ) (no. **30**), as well as for equations that are *linear* in p. In the latter case (and only in that one), there are not ∞^{m+1}, but merely ∞^m integral conoids, which consist of the element-M_m^1 that are connected with the ∞^m respective characteristic curves (no. **11**).

Every representation (121) of the *Pfaff* equation (84) implies a *map of the partial differential* (83)([203]) that is based upon the interpretation of the $2m - 1$ quantities Ξ, Π as coordinates of the surface elements in a space R_m with the point coordinates Ξ_m, Ξ_1, …, Ξ_{m-1}. Every surface element and every element-$M_{\nu-1}$ in the space R_m then corresponds to a characteristic (an integral-M_ν, resp.) of equation (83) and conversely. In particular, the quantities $\mathfrak{z}$, $\mathfrak{x}_i$, $\mathfrak{p}_i$ in no. **30** can be interpreted as the element coordinates of the space R_m ($\mathfrak{z}$, $\mathfrak{x}_1$, …, $\mathfrak{x}_{m-1}$) that is selected from R_{m+1} (z, x_1, …, x_m) by the equation $x_m = x_m^0$. Every characteristic of (83) then corresponds to the surface element in R_m that it selects from R_m, and every integral M_ν corresponds to the element $M_{\nu-1}$ in R_m that it selects from the latter ([204]).

The transition from one certain complete integral (120) to another ([205]) [so from one particular type of mapping of equation (83) to another, as well ([206])] is completed by an arbitrary contact transformation of the $2m - 1$ variables Ξ, Π ([207]).

According to *Lie* ([208]), the first-order partial differential equations can be *classified* by their complete integrals as follows: A first-order equation belongs to the ν^{th} class when one of its complete integrals consists of nothing but element-$M_m^{m-\nu+1}$, and ν is the greatest such number. A special case of this is defined by the equations whose integral conoids all degenerate into element-$M_m^{m-\nu+1}$ ([209]). In that case (and only in it), any $\infty^{\nu-1}$ characteristics strips will contain the same characteristic curve. The number ν is an invariant of the partial differential equation with respect

([203]) *Lie-Scheffers*, *Berührungs.*, pp. 535, *et seq. A. V. Bäcklund*, Math. Ann. **9**, pp. 313.

([204]) *Lie-Scheffers*, *Berührungs.*, pp. 544, *et seq.*

([205]) *Jacobi*, *Werke*, **5**, pp. 420, 431. *H. Weber*, J. f. Math. **66**, pp. 210. Cf., also *Jacobi*, *Werke* **5**, pp. 369-377, where the most general system of canonical elements for a dynamical problem is derived from a particular one (no. **31**).

([206]) *Lie-Scheffers*, *Berührungs.*, pp. 548.

([207]) For the behavior of the singular integrals that might exist under that transform, cf., *H. Weber*, *loc. cit.*, pp. 231. *Darboux*, "Sol. sing.," pp. 102-108.

([208]) Gött. Nachr. (1872), pp. 473.

([209]) Cf., *S. Lie*, Leipz. Ber. (1895), pp. 85.

to all (extended) point transformation of the space R_{m+1} $(z, x_1, \ldots, x_m)$. The m^{th} class consists of the equations that are linear in the p (cf., *supra*), and the $(m + 1)^{\text{th}}$ class consists of the equations of the form $F(z, x_1, \ldots, x_m) = 0$ whose integral-M_m can be found without integration. Equations of the 2nd, 3rd, …, $(m - 1)^{\text{th}}$ class are called *semi-linear* ([210]). The equations that are homogeneous in the p_i belong to the 2nd class, in general.

35. Homogeneous element coordinates. – In the case of an equation of type (γ):

$$(122) \qquad F\left(x_1, x_2, \ldots x_m, \frac{p_2}{p_1}, \frac{p_3}{p_1}, \ldots, \frac{p_m}{p_1}\right) = 0 \qquad \text{or} \qquad p_m + \psi = 0 \, ,$$

one can interpret the x, p as homogeneous element coordinates ([211]) in the space of $x_1, \ldots, x_m$. A *surface element* is the set that consists of a point $x_1, \ldots, x_m$ and a plane $\sum p_i \, (\xi_i - x_i) = 0$ that goes through it. Two neighboring elements are *united* when they fulfill the equation:

$$(123) \qquad p_1 \, dx_1 + p_2 \, dx_2 + \ldots + p_m \, dx_m = 0 \, ,$$

and an *element*-M_ν is defined by $2m - \nu - 1$ relations in the x, p that fulfill equation (123) and are homogeneous in the p. Equation (122) is said to be *integrated* when all of its *integral*-M_{m-1} are found, i.e., all element-M_{m-1} whose defining equations include the relation (122). Any *complete integral:*

$$(124) \qquad p_m + \psi = 0 \, , \qquad \Xi_i\left(x_1, \ldots x_m, \frac{p_2}{p_1}, \frac{p_3}{p_1}, \ldots, \frac{p_{m-1}}{p_1}\right) = c_i \qquad (i = 1, \ldots, m - 1)$$

will yield a normal form $\sum \Pi_i \, d\Xi_i$ for the expression ∇_1 in no. **30**, and conversely. The ∞^{2m-3} *characteristics* are now represented by the relations:

$$\Xi_1 = c_1 \, , \ldots, \quad \Xi_{m-1} = c_{m-1} \, ; \qquad \Pi_1 : \Pi_2 : \ldots : \Pi_{m-1} = \gamma_1 : \ldots : \gamma_{m-1} \, .$$

Statements analogous to the previous ones are true in regard to generating the integration by characteristics. The transition from one well-defined complete integral (124) to another takes place by way of a *homogeneous contact transformation* of the Π, Σ.

After adding $z = c$, the complete integral (124) will yield a complete integral in the previous sense. If (101) is a complete integral in the sense of no. **30** then $V(x_1, \ldots, x_m, a_1, \ldots, a_{m-2}) = a_{m-1}$ will be a complete integral, with the present meaning, that consists of ∞^{m-1} *surfaces*, and conversely.

([210]) The cases $m = 3$, $\nu = 2$, and $m = 4$, $\nu = 2, 3$ were studied by *A. V. Bäcklund*, Math. Ann. **17** (1880) pp. 285.

([211]) *Lie, Transform.*, **2**, pp. 108, *et seq.*

The equation (82) of type (*a*) or (*b*) will be reduced to the case considered here by the substitution $p_i = - q_i / q_{m+1}$, $z = x_{m+1}$. That conversion has the advantage that integral-M_m of (82) whose associated point manifolds are determined by relations $x_1, \ldots, x_m$ will now come under consideration, so ones whose $m + 1$ defining equations will subsume the equation $q_{n+1} = 0$ when they are written in terms of x_i, q_i .

36. Jacobi's second method ([212]) for integrating a partial differential equation:

$$X_1 (x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m) = a_1 \tag{125}$$

is based upon the following theorem (cf., no. **24**): If m functions $X_1, \ldots, X_m$ of the variables x_i, p_i satisfy the identities ([213]):

$$0 \equiv (X_i\, X_k) \qquad (i, k = 1, \ldots, m) \tag{126}$$

then the values of the p that are inferred from the equations $X_i = a_i$ will convert the expression $\sum p_k\, dx_k$ into an exact differential $dV (x_1, \ldots, x_m, a_1, \ldots, a_m)$, and $z = V + a$ will be a complete integral of (125) with the arbitrary constants $a_1, a_2, \ldots, a_m$. If μ ($< m$) functions $X_1, \ldots, X_m$ that are independent of $p_1, \ldots, p_m$ and satisfy the conditions (126) have been determined already, and one has solved the equations:

$$X_1 (x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m) = a_1 \qquad (i = 1, 2, \ldots, \mu) \tag{127}$$

in the form $p_1 = h_1, \ldots, p_\mu = h_\mu$ ([214]) then one will have $(p_i - h_i, p_k - h_k) \equiv 0$, and the m-parameter *Jacobi* system $(p_i - h_i, f) = 0$ will be equivalent to the system (73) in no. **24**. It will possess $2m - 2\mu$ integrals that do not depend upon $p_1, \ldots, p_\mu$ and fulfill the *Jacobi* system:

$$\frac{\partial f}{\partial x_i} + \sum_{k=\mu+1}^{m} \left(\frac{\partial h_i}{\partial x_k} \frac{\partial f}{\partial p_k} - \frac{\partial h_i}{\partial p_k} \frac{\partial f}{\partial x_k} \right) = 0 \qquad (i = 1, 2, \ldots, \mu). \tag{128}$$

One will get $X_{\mu+1}$ ([215]) from an arbitrary integral of that system that does not depend upon $p_{\mu+1}$:

([212]) This method was already given by *Jacobi* in 1836 (letter to *Encke*, J. f. Math. **17**, pp. 68; *Werke* **4**, pp. 41, esp. pp. 52, *et seq.* Cf., Lect. 30-34 in *Vorl. über Dynamik* and J. f. Math. **60**, pp. 1 [unpublished work] = *Werke* **5**, pp. 1; for $m = 3$, cf., *Werke* 5, pp. 439), and found later, in dependently, by *W. F. Donkin* [Trans. London Math. Soc. (1854), pp. 71; *ibid*. (1855), pp. 299]. *J. Liouville* [J. de math. **20** (1855), pp. 137] and *É. Bour* [*ibidem*, pp. 185 = Paris sav. [étr.] **14**, pp. 792; J. éc. polyt. **22**, cah. 39 (1862), pp. 149]. Cf., *P. Gilbert*, Brux. Ann. 5[2] (1881), pp. 1. *G. Boole*, Trans. London Math. Soc. (1863), pp. 485. *J. Collet*, Ann. éc. norm. sup. (1870), pp. 7; *ibid*. (1876), pp. 49. *H. Laurent*, J. de math. (3) **5** (1879), pp. 249.

([213]) In what follows, the symbols $(\varphi\ \psi)$ and $[\varphi\ \psi]$ will always have the meaning that they were given in no. **24**.

([214]) *A. Mayer* (Math. Ann. **6**, pp. 165) referred to such a system as a "Jacobi" system.

([215]) Cf., the presentation of *J. König*, Math. Ann. **23**, pp. 504, in which the adjoint system of total differential equations (no. **14**) was used instead of the system (128). *Jacobi* employed the process in no. **15** in order to determine

$$\varphi(x_1, \ldots, x_m, p_{\mu+1}, \ldots, p_m, a_1, \ldots, a_\mu)$$

when one replaces the a_i with the X_i . If one has thus determined the functions $X_1, \ldots, X_m$ then they will satisfy an identity of the form (72), and indeed Ω will arise from the function V that was defined above when one replaces the a with the X.

With that, the integration of the canonical system (107), or what amounts to the same thing, from no. **31**, the *Hamilton* partial differential equation (108) of a dynamical problem, will each require $2m, 2m - 2, \ldots, 4, 2$ operations, and one quadrature. One will arrive at a system of canonical elements (no. **31**) directly by that method. If the *vis viva* T and the force function U, and therefore the *Hamiltonian* function H, are free of t then the integration of (108) will reduce to the integration of the equation ([216]):

$$H(q_1, \ldots, q_m, p_1, \ldots, p_m) = h \qquad \left(p_i = \frac{\partial z}{\partial q_i};\ h = \text{arb. const.}\right),$$

since a complete integral $z = \Psi - h\,t + c$ of equation (108) will be obtained from any complete integral $z = \Psi(q_1, \ldots, q_m, c_1, \ldots, c_{m-1}, h) + c$ of that equation, so each integration will require $2m - 2, 2m - 4, \ldots, 2$ operations, and one quadrature.

In order to integrate the partial differential equation:

$$X_1(q_1, \ldots, q_m, p_1, \ldots, p_m) = a_1 , \tag{129}$$

one must determine the m functions $X_2, \ldots, X_m, Z$ of the variables z, x_i, p_i in such a way that the identities $[Z\,X_i] \equiv [X_i\,X_k] \equiv 0$ will exist, and the equations:

$$X_1 = a_1 , \qquad X_2 = a_2 , \qquad \ldots, \qquad X_m = a_m , \qquad Z = a_{m+1} \tag{130}$$

will be soluble for $p_1, \ldots, p_m, z$ ([217]). The expression for z that follows from that will then be a complete integral of (129). One can arrange the integration such that the first m equations (130) can be solved for $p_1, \ldots, p_m$. After replacing the p_i, the *Pfaff* equation (92) will become exact, and its integration will yield a complete integral of (129); for $m = 2$, that is *Lagrange*'s method (no. 28).

37. Lie's generalization of Jacobi's second method ([218]). – On the basis of *Lie*'s more general definition of the concept of a "complete integral," when a differential equation (129) is given, one can overlook the condition that equations (130) are soluble for $p_1, \ldots, p_m, z$ in the derivation of a system of functions X_i, Z, P_i that satisfy the identity (68). Those equations will then define a complete integral in a more general sense. That method, like the one in the previous no., will

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a solution for each successive *Jacobi* system (128) or (73); other methods were given by *A. Weiler* (footnote 92) and *A. Clebsch* (J. f. Math. **65**, pp. 263).

([216]) *Jacobi*, Lect. 21, *Vorl. über Dynamik*.

([217]) Cf., *V. G. Imschenetzky*, Arch. Math. **50**, pp. 304.

([218]) Math. Ann. **8**, pp. 240.

require $2m - 1$, $2m - 3$, …, 3, 1 operations. In the case of the partial differential equation (125), the derivation of the identity (72) will require $2m - 2$, $2m - 4$, …, 2, and one quadrature. If X_1 is homogeneous of order zero in the p_i then one can impose the same condition on the functions X_2, …, X_m . Ascertaining it will then take place by way of $2m - 3$, $2m - 5$, …, 1 operations. They will satisfy the identity (65), and the equations:

$$X_1 = a_1, \qquad X_2 = a_2, \qquad \ldots, \qquad X_m = a_m \tag{131}$$

will define a complete integral in the sense of no. **35**.

From the fact that when X_1 means an arbitrary function of the $2m$ variables x_i, p_i, the expression (X_1, f) will represent the most general infinitesimal transformation of the infinite group of all contact transformations of the $2m$ variables x_i, p_i, *Lie* ([219]) concluded that a further simplification of the process of integration would be impossible.

38. Systems in involution. – A common integral-M_m of several equations:

$$F_i\,(z, x_1, x_2, \ldots, x_m, p_1, \ldots, p_m) = 0 \qquad (i = 1, 2\ldots) \tag{132}$$

will also fulfill all relations $[F_i\,F_k] = 0$ ([220]). Repeated application of that theorem will lead to either more than $m + 1$ independent relations, and there will then be no common integral-M_m of the differential system (132), or to a *system in involution*, i.e., to m ($\leq m + 1$) equations $F_i = 0$, by means of which all $[F_i\,F_k]$ will vanish ([221]). One can always succeed in making those expressions vanish *identically*. One then says: The functions F_i are *involutory* (*in involution*). In the case of $\mu = m + 1$, there is only *one* common integral-M_m that is represented by just the equations $F_i = 0$. If $\mu \leq m$ then one can write the system in involution in the form (127), in which the a_i mean *arbitrary* constants. The integral of (127) will be obtained when one determines $m - \mu + 1$ functions:

$$X_{\mu+1}, X_{\mu+2}, \ldots, X_m, Z \tag{133}$$

according to no. **24** in such a way that an identity (68) will exist [an identity (72), resp., when the $X_1, \ldots, X_m$ do not depend upon z], and then adds any $m - \mu + 1$ relations that satisfy the *Pfaff* equation:

$$dZ - P_{\mu+1}\,dX_{\mu+1} - \ldots - P_m\,dX_m = 0 \tag{134}$$

to equations (127). In particular, one will then find a *complete integral* (130) of the system in involution (127) that depends upon $m - \mu + 1$ arbitrary constants $a_{\mu+1}, \ldots, a_{m+1}$.

([219]) Math. Ann. **11**, pp. 529, *et seq.*; Leipziger Ber. (1895), pp. 265; cf., the conclusion of no. **11**.

([220]) It arises by eliminating the second derivatives of z from the first derivatives of the system (132). A generalization of the bracket operation for two differential equations of arbitrary order was given by *É. Combescure*, C. R. Acad. Sci. Paris **78** (1874), pp. 1212.

([221]) *É. Bour*, J. éc. polyt. **22**, cah. **39**, pp. 171, *et seq. A. Mayer*, Math. Ann. **4** (1871), pp. 88.

One understands the *characteristics* of the system in involution (127) ([222]) to mean any element-M_m that is defined by equations of the form:

$$X_1 = a_1\,, \qquad \ldots, \qquad X_m = a_m\,, \qquad Z = a_{m+1}\,; \qquad P_{\mu+1} = b_{\mu+1}\,, \quad \ldots, \qquad P = b_m\,,$$

in which the $a_{\mu+i}$, $b_{\mu+i}$ mean arbitrary constants. The left-hand side of those equations are the integrals of the μ-parameter complete system (71). The characteristics of (127) will also be represented by the equations:

$$X_1 = a_1\,, \qquad \ldots, \qquad X_\mu = a_\mu\,, \qquad \Phi_i = c_i \qquad (i = 1, 2, \ldots, 2m - 2\mu + 1)$$

when the Φ_i mean any solutions of the complete system (71) that are independent of each other and of $X_1, \ldots, X_\mu$. The characteristics are the characteristic M_ν for each individual partial differential equation in (127) (see no. **34**).

A surface element z, x_i, p_i that satisfies equations (127) for a certain system of constants $a_1, \ldots, a_\mu$ might be called singular or nonsingular, respectively, according to whether all or not all equations that arise from annulling the μ-rowed subdeterminants of the matrix:

$$\text{(135)} \qquad \left\| \begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial X_i}{\partial p_1} & \cdots & \dfrac{\partial X_i}{\partial p_m} & \dfrac{\partial X_i}{\partial x_1} + p_1 \dfrac{\partial X_i}{\partial z} & \cdots & \dfrac{\partial X_i}{\partial x_m} + p_m \dfrac{\partial X_i}{\partial z} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \right\| \qquad (i = 1, 2, \ldots, \mu)$$

are satisfied. One and only one characteristic of the system in involution (127) goes through any nonsingular surface element. An integral-M_m that includes a nonsingular surface element E will encompass the entire characteristic that starts from E. If two neighboring surface elements of (127) are united then that will also be true of the characteristics that go through them. The $\infty^{\nu-\mu}$ characteristics that go through the elements of a common integral $M_{\nu-\mu}$ of equations (127), resp., generate an integral M_ν, since the common integral-$M_{m-\mu}$ will be found without integration, so the integration of the system in involution (127) will be achieved when one knows all solutions of the complete system (71) (***Lie****’s generalized* ***Cauchy*** *method*).

The *Pfaff* equation (134) mediates a *map* (cf. no. **34**) of the system in involution (127) in the space $R_{m-\mu+1}$ with the point coordinates (133) under which every characteristic (integral-M_m, resp.) of the system in involution corresponds to a surface element (element-$M_{m-\mu}$, resp.) in the space $R_{m-\mu+1}$, and conversely. The transition from a certain type of map to another arbitrary one, or what amounts to the same thing, from one complete integral (130) to another, takes place by way of a contact transformation of the $2m - 2\mu + 1$ variables Z, $X_{\mu+k}$, $P_{\mu+k}$.

A somewhat-different way of looking at the same state of affairs ensures from the remark that for an arbitrary contact transformation of the variables z, x_i, p_i, every system in involution J will

([222]) *Lie*, Gött. Nachr. (1872), pp. 321; Math. Ann. **9** (1876), pp. 245.

correspond to a system in involution J' (no. **10**) ([223]), so the characteristics (integral-M_m, resp.) of the system in involution (127) will emerge by way of the contact transformation:

$$(136)\qquad z' = Z, \qquad x_i' = X_i\,; \qquad p_i' = P \qquad (i = 1, 2, \ldots, m)$$

from the characteristics (integral-M_m, resp.) of the special system in involution:

$$(137)\qquad x_1' = a_1\,, \qquad x_2' = a_2\,, \qquad \ldots, \qquad x_\mu' = a_\mu\,.$$

On the same grounds, when any contact transformation (136) is known, one can find the integral of all partial differential equations of the form $\varphi(Z, X_1, \ldots, X_m) = 0$ ([224]) and all systems in involution that consist of several such equations without integration.

Since every μ-parameter system in involution can be put into the form (137) ([225]), a system in involution will possess no invariant besides the number μ under all contact transformations of the variables z, x_i, p_i . The same thing will be true for the system of involution that are independent of z with respect to all contact transformations of the form:

$$z' = z + U(x, p)\,; \qquad x_i' = X_i(x, p)\,; \qquad p_i' = P_i(x, p)\,.$$

The first equation of the system (127) be put into the form $x_1' = a_1$ by a contact transformation when any of its complete integrals is known, which will convert the remaining equations (127) into a (μ – 1)-parameter system in involution with the variables z, $x_2', \ldots, x_m'$, $p_2', \ldots, p_m'$. *A. Korkin*'s ([226]) method for integrating a system involution is obtained by repeating that process.

The possible *singular* ([227]) integrals for the system in involution (127) are the common integral-M_m of equations (127) and the relations that emerge from the matrix (135) by annulling all μ-rowed determinants.

If a system in involution (127) is given whose left-hand sides are free of z and are homogeneous of order zero in the p_i then one can determine $m - \mu$ further functions $X_{\mu+1}, \ldots, X_m$ with the same behavior, as in no. **37**, in such a way that the identity (65) exists. Based upon the conception of an integral in no. **35**, one will get a theory that is entirely analogous to the one before when the concepts of "characteristic" and "complete integral" are modified correspondingly. A system in

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([223]) The contact transformations are the only transformation of that type. However, there are always other transformations besides the contact transformations that take two *well-defined* μ-parameter systems in involution to each other (or a well-defined system in involution to itself) in such a way that every characteristic or integral-M_m, resp., will again correspond to such a thing. *Lie*, Christ. Forh. (873), pp. 242. *Lie-Scheffers*, *Berührungs.*, pp. 581, *et seq. A. V. Bäcklund*, Math. Ann. **9**, pp. 313.

([224]) That theorem was already given for $m = 2$ by *G. Monge*, Paris Hist. (1784), pp. 174, *et seq.* Cf., *A. de Morgan*, Trans. Camb. Phil. Soc. **9** (1854), pp. [136]. A special case of that theory is defined by the *generalized Clairaut equation*: *Lie-Scheffers, Berührungs.*, pp. 265, *et seq.*, rem. 518. *Darboux*, "Sol. sing.," pp. 205, *et seq.*

([225]) *Lie*, Math. Ann. **8**, pp. 215.

([226]) C. R. Acad. Sci. Paris **68** (1869), pp. 1460. See *A. Mayer*, Math. Ann. **6** (1873), pp. 173.

([227]) *Goursat A*, art. 117. *Delassus*, art. 22.

involution (127) of that type will possess only m for an invariant under all homogeneous contact transformations of the $2m$ variables x, p.

39. Special systems in involution. – For a system in involution of the form:

$$p_i = \psi_i\,(z, x_1, \ldots, x_m, p_{\mu+1}, \ldots, p_m) \qquad (i = 1, 2, \ldots, \mu), \tag{138}$$

the generalized *Cauchy* method (see the prev. no.) will assume the following form: If the ψ_i are regular at the location z^0, x_1^0, …, p_m^0 then the complete system:

$$(p_i - \psi_i\,, f) = 0 \qquad (i = 1, 2, \ldots, \mu)$$

will possess $2m - 2\mu + 1$ solutions ζ, $\xi_{\mu+h}$, $\pi_{\mu+k}$ that are independent of $p_1, \ldots, p_\mu$ and reduce to z, $x_{\mu+h}$, $p_{\mu+k}$, resp., by means of:

$$x_1 = x_1^0, \qquad \ldots, \qquad x_\mu = x_\mu^0, \tag{139}$$

and the identity will exist ([228]):

$$dz - \psi_1\, dx_1 - \ldots - \psi_\mu\, dx_\mu - \sum_{h=\mu+1}^{m} p_h\, dx_h \equiv \rho\Big(d\zeta - \sum_{h=\mu+1}^{m} \pi_h\, d\xi_h\Big)\,.$$

If the function $\varphi\,(x_{\mu+1}, \ldots, x_m)$ is regular at the location $x_{\mu+1}^0$, …, x_m^0 and one has:

$$z^0 = \varphi\left(x_{\mu+1}^0, \ldots, x_m^0\right), \qquad p_{\mu+k}^0 = \frac{\partial\varphi\left(x_{\mu+1}^0, \ldots, x_m^0\right)}{\partial x_{\mu+k}^0}$$

then one will obtain the integral z of the differential system (138) that is regular at the location x_1^0, …, x_m^0, and goes to $\varphi\,(x_{\mu+1}, \ldots, x_m)$ by means of (139) by eliminating $p_{\mu+1}, \ldots, p_m$ from the equations ([229]):

$$\zeta = \varphi\,(\xi_{\mu+1}, \ldots, \xi_m)\,, \quad \pi_{\mu+k} = \frac{\partial\varphi\left(\xi_{\mu+1}, \ldots, \xi_m\right)}{\partial\xi_{\mu+k}} \qquad (k = 1, \ldots, m - \mu)\,.$$

([228]) For the classification of this *Pfaff* expression, cf., *G. Morera* (footnote 173) and my book, Chap. 13, art. 363. The consideration of the identity above leads to obvious generalizations of the theorems that were given in no. **20** [*N. Saltykov*, J. de math. (5) **3** (1897), pp. 423; C. R. Acad. Sci. Paris **128** (1899), pp. 166, 225, 274, 1550.]

([229]) Its existence follows from the passivity of the system (138). According to *É. Delassus* [Ann. éc. norm. sup. (1897), pp. 117], it can also be obtained by successive integrations of μ first-order partial differential equations with $m - \mu + 1$ independent variables. For $m = 2$, $\mu = 2$, cf., *Darboux, Surfaces*, **2**, pp. 258.

The integration of *any* system of first-order partial differential equations in involution can be reduced to a system in involution of the form ([230]):

$$p_i = h_i\,(x_1, x_2, \ldots, x_m, p_{\mu+1}, \ldots, p_m) \qquad (i = 1, 2, \ldots, \mu). \tag{140}$$

If one introduces the new variables y_i into it in place of the $x_1, \ldots, x_\mu$ by means of formulas (35), no. **17**, which will make h_i go to h_i', then one of the transformed equations will be:

$$\frac{\partial z}{\partial y_1} = h_1' + y_2\,h_2' + \cdots + y_\mu\,h_\mu' \,, \tag{141}$$

and the integral of it z, which will be converted into $\varphi\,(x_{\mu+1}, \ldots, x_m)$ by means of $y_1 = 0$, will give the integral of equations (140) after eliminating y_i by means of (35), which reduces to φ by means of (139) [*Lie*'s ([231]) *fundamental theorem*]. In that way, the integration of the μ-parameter system of involution (140) is reduced to that of a single first-order partial differential equation with $m - \mu + 1$ independent variables ([232]). The process is based upon the same geometric facts as the method in no. **17** and will go to the latter when the h_i are homogeneous and linear in the p. In order to integrate, e.g., the equation $p_1 = \varphi\,(x_1, \ldots, x_m, p_2, \ldots, p_m)$ afterwards, one determines an equation $f\,(x, p) = a$ that is in involution with it and reduces that system in involution to *one* first-order partial differential equation with $m - 1$ independent variables. By repeating that process, one will ultimately arrive at a first-order equation with one independent variable whose complete integral is obtained by quadrature and will yield a complete integral for each of the foregoing equations with the help of mere eliminations. The method requires the same operations as the second *Jacobi* method. If φ is homogeneous of degree one in the p then one can choose f to be a function that is homogeneous of degree zero in the p, and analogously in the reductions that follow, and in that way, the number of necessary operations will be reduced by *one* unit. One can also truncate the process after the k^{th} step and integrate the first-order partial differential equation with $m - k$ independent variables that one obtains using *Cauchy*'s method.

40. Function groups ([233]). – If s independent functions $f_1, f_2, \ldots, f_s$ of the $2m$ variables $x_1, \ldots, x_m, p_1, \ldots, p_m$ have the property that all bracket expressions $(f_i\,f_k)$ can be represented as functions of $f_1, \ldots, f_s$ then one calls the set of all functions $\varphi\,(f_1, \ldots, f_s)$ an s-parameter *function group G*. Then (and only then) does the system of linear partial differential equations:

([230]) *A. Mayer*, Math. Ann. **8** (1875), pp. 313. *Lie*, Math. Ann. **9**, pp. 275, *et seq.*

([231]) Christ. Forh. (1872), pp. 29. Gött. Nachr. (1872), pp. 321. *Mayer*, Math. Ann. **6** (1873), pp. 162.

([232]) Naturally, that method is also applicable to a system of the form (138) [*Delassus*, Ann. éc. norm. (1897), pp. 118. His book, pp. 38]. *F. Schur* reduced the integration of the *unsolved* form of the system in involution (127) to a system of ordinary differential equations, Leipz. Ber. (1892), pp. 182; *ibid.* (1894), pp. 38.

([233]) *S. Lie*, Christ. Forh. (1873), pp. 16. Math. Ann. **8**, pp. 215, *ibidem*, **11**, pp. 464. "*Transform.*," **2**, Abt. **2**. Cf., *Goursat A*, Chap. 12. My book, Chap. 14.

(142) $$(f_1\, f) = 0\,, \qquad (f_2\, f) = 0\,, \qquad \ldots, \qquad (f_s\, f) = 0\,,$$

and its solutions define a $2m - s$-parameter function group G', namely, the *polar group* of G; the polar group of G' is once more G. Any function of G that is also contained in G', so it is in involution with all functions of G (no. **38**), is called a *distinguished function* of G. If the rank of the skew-symmetric matrix:

(143) $$\| (f_i\, f_k) \| \qquad (i, k = 1, 2, \ldots, s)$$

is 2ν then $s - 2\nu$ independent distinguished functions $u_1, \ldots, u_{s-2}$ will be included in G. The linear relations:

$$\lambda_1\, (f_1\, f_i) + \lambda_2\, (f_2\, f_i) + \ldots + \lambda_s\, (f_s\, f_i) = 0 \qquad (i = 1, 2, \ldots, s)$$

possess $s - 2\nu$ independent systems of solutions $\lambda_1^{(k)}, \ldots, \lambda_s^{(k)}$, and the equations:

(144) $$\lambda_1^{(k)}(f_1\, f) + \lambda_2^{(k)}(f_2\, f) + \cdots + \lambda_s^{(k)}(f_s\, f) = 0 \qquad (k = 1, \ldots, s - 2\nu)$$

will define an $s - 2\nu$-parameter complete system that is equivalent to the system $(u_i\, f) = 0$ and possesses the solutions $f_1, \ldots, f_s$, *inter alia*.

For $s > m$, one will have $\nu \geq s - m$. An identity ([234]):

(145) $$p_1\, dx_1 + \ldots + p_m\, dx_m \equiv dV\,(x_1, \ldots, x_m, p_1, \ldots, p_m) + \sum_{k=1}^{s} \Phi_k\, df_k$$

will exist if and only if $\nu = s - m$, in which V is found by a quadrature, and the Φ_k are found by solving linear equations. In all cases, there are $2m$ independent functions P_i, X_i such that the identities (66), no. **24** will exist, and the groups G, G' will take the *canonical forms:*

(G) $$P_1, X_1\,; P_2, X_2\,, \ldots, P_\nu, X_\nu\,; X_{\nu+1}, X_{\nu+2}, \ldots, X_{s-\nu}\,;$$
(G') $$P_{s-\nu+1}\,, X_{s-\nu+1}\,; \ldots, P_m, X_m\,; X_{\nu+1}, X_{\nu+2}, \ldots, X_{s-\nu}\,,$$

resp.; $X_{\nu+1}, \ldots, X_{s-\nu}$ are the distinguished functions. Hence, G possesses only the two invariants s and 2ν with respect to arbitrary contact transformations of the variables x, p.

In the case of $\nu = s - m$, and only in that case, G will contain an m-parameter system in involution, e.g., the following one: $X_1, X_2, \ldots, X_{s-\nu}$.

The group G is called *homogeneous* when the s expressions:

([234]) This theorem is included in the generalized *Frobenius* theory (no. **26**) as a special case; cf., my book, Chap. 14.

(146) $$\sum p_h \frac{\partial f_1}{\partial p_h}, \qquad \sum p_h \frac{\partial f_2}{\partial p_h}, \ldots, \sum p_h \frac{\partial f_s}{\partial p_h}$$

are contained in G. If all f_i are homogeneous of degree zero in the p then they will define an s-parameter system in involution. In the other case, G can be written in the form $N_1, N_2, \ldots, N_{s-1}, H$, where the N_i are homogeneous of degree zero in the p, and H is homogeneous of degree one in them. The $s - 2\nu$ distinguished functions of G are all or not all homogeneous of degree zero according to whether the rank $2\nu'$ of the matrix (143) that is bordered by the elements (146) is equal to 2ν or $2\nu + 2$, resp. If $2\nu' = 2\nu$ then the linear equations:

$$\lambda_1 (f_1\, f_i) + \ldots + \lambda_s (f_s\, f_i) + \lambda_{s+1} \cdot \sum p_k \frac{\partial f_i}{\partial p_h} = 0 \qquad (i = 1, \ldots, s)$$

will possess $s - 2\nu + 1$ independent solutions $\lambda_1^{(k)}, \ldots, \lambda_{s+1}^{(k)}$, and the partial differential equations:

(147) $$\lambda_1^{(k)} (f_1\; f) + \cdots + \lambda_s^{(k)} (f_s\; f) + \lambda_{s+1}^{(k)} \cdot \sum p_k \frac{\partial f}{\partial p_k} \qquad (k = 1, \ldots, s - 2\nu + 1)$$

will define an $s - 2\nu + 1$-parameter complete system that the functions $f_1, \ldots, f_s$ also satisfy, *inter alia*. If G is homogeneous then there will always be $2m$ independent functions P_i, X_i such that the equations (66) will exist, the P_i will be homogeneous of degree one in the p, the X_i will be homogeneous of degree zero, and G can take one or the other *canonical form:*

$$P_1, X_1\,; \ldots, P_\nu, X_\nu\,; X_{\nu+1}, \ldots, X_{s-\nu}\,;$$
$$P_1, X_1\,; \ldots, P_\nu, X_\nu\,; P_{\nu+1}, \ldots, P_{s-\nu}\,,$$

resp., according to whether $2\nu'$ equals 2ν or $2\nu + 2$, resp. Thus, a homogeneous group G possesses only the invariants s, 2ν, $2\nu'$ with respect to all *homogeneous* contact transformations of the x, p.

An identity ([234]):

(148) $$p_1\, dx_1 + \ldots + p_m\, dx_m \equiv \Psi_1\, df_1 + \Psi_2\, df_2 + \ldots + \Psi_s\, df_s$$

exists if and only if $s \geq m$ and $n = \nu' = s - m$. In that case (and only in that case), G will include an m-parameter system in involution of order zero, e.g., the following one: $X_1, X_2, \ldots, X_{s-\nu}$.

41. Continuation. – If the problem is to integrate a q-parameter system in involution of the form:

(149) $$f_i (x_1, x_2, \ldots, x_m, p_1, p_2, \ldots, p_m) = c_i \qquad (i = 1, 2, \ldots, q),$$

and if φ, ψ are solutions of the *Jacobi* system:

$$(f_1\, f) = 0\,,\qquad (f_2\, f) = 0\,,\ \ldots,\ (f_q\, f) = 0\,, \tag{150}$$

then from *Poisson*'s theorem (no. **25**), $(\varphi\ \psi)$ will also be a solution ([234.a]). Thus, if f_{q+1}, f_{q+2}, …, f_r are known integrals (150) then the same thing will be true of all functions f_1, …, f_r, …, f_s of the smallest group G in which the f_1, …, f_r are included. If G subsumes m-parameter systems in involution, i.e., if $s > m$ and $2\nu = 2s - 2m$ is the rank of the matrix (143), then the identity (145) will be true, and the functions f_1, …, f_s, Φ_{q+1}, …, Φ_s, $z - V$ will collectively give all solutions to the complete system $[f_1\, f] = 0$, …, $[f_q\, f] = 0$, with which, the integration of the system in involution (149) by the generalized *Cauchy* method is achieved (no. **38**). If G includes no m-parameter system in involution then, in addition to f_1, …, f_s, the complete system (144) will include other solutions that are independent of them. Ascertaining such a solution f_{s+1} will require $2\rho = 2m - 2s + 2\nu$ operations. The application of the analogous process to the group that is generated by f_1, …, f_{s+1} will require $2\rho - 2l$ ($l \geq 1$), etc. If one has found enough functions f_{s+1}, …, $f_{s'}$ that the complete system, which has the same relationship to the group G' that is generated by f_1, …, $f_{s'}$ that (144) has to G, possesses no integrals besides f_1, …, $f_{s'}$ then G' will include m-parameter systems in involution, and the integration ([235]) of (149) will require yet another quadrature ([236]).

If f_1, …, f_q are homogeneous of degree zero in p then along with φ, $\sum p_i \dfrac{\partial\varphi}{\partial p_i}$ will also be a solution of (150). Thus, if f_{q+1}, …, f_r are known solutions then the same thing will be true of all functions f_1, …, f_s of the smallest *homogeneous* group G that includes the functions f_1, …, f_r. If one has $s \geq m$ and $\nu = \nu' = s - m$, i.e., G includes an m-parameter system in involution of order zero, then an identity (148) will exist, and the functions f_1, …, f_s, Ψ_{q+1}, …, Ψ_s will yield all solutions of (150), with which the integration of (149) is achieved. Otherwise, let $2\nu' = 2\nu + 2$. One can then derive no advantage from the fact that G is *homogeneous*, and the previous process will be applied. However, if $2\nu' = 2\nu$ then all distinguished functions of G will have degree zero, so one will determine an integral f_{s+1} of the complete system (147) that is independent of f_1, …, f_s by $2m - 2s + 2\nu - 1$ operations. All distinguished functions of the homogeneous group that is generated by f_1, …, f_{s+1} will again be homogeneous of degree zero then, etc. If one has determined the functions f_{s+1}, …, $f_{s'}$ in such a way that the complete system that has the same relationship to the homogeneous group G' that is generated by f_1, …, $f_{s'}$ that (147) has to G possesses no solutions

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([234.a]) A generalization of this is found in *H. Laurent*, J. de math. (2) **17** (1872), pp. 422. Cf., *Goursat A*, Note II.

([235]) *Jacobi*'s theorems on the multipliers of equations of the form $[F\, f] = 0$ or $(f_1\, f) = 0$ [J. f. Math. **29**, § 19 = *Werke* **4**, pp. 413; Lect. 18 in *Vorl. über Dynamik*] are included in this theory as a corollary; see *Lie*, Math. Ann. **11,** pp. 519. *Jacobi* treated other special cases in J. f. Math. **60**, pp. 143 = *Werke* 5, pp. 151, along with *G. Boole* in Trans. London Math. Soc (1863), pp. 495-501.

([236]) *Lie*'s original method (Math. Ann. **8**) requires the determination of the distinguished functions of G.

besides $f_1, \ldots, f_{s'}$ then G' will include an m-parameter system in involution of order zero, and the integration of (149) will be achieved ([237]).

Lie ([238]) showed that the foregoing theorems will ensure the best-possible utilization of the known solutions $f_{q+1}, \ldots, f_r$.

If φ is a solution of (150) then that complete system will admit the infinitesimal transformation $(\varphi\, f)$. *Lie* ([239]) also obtained the theorems of this subsection using that remark by means of his theorem on the integration of complete systems with known infinitesimal transformations.

If the aforementioned homogeneous group G includes not just functions that are homogeneous of degree zero in p then it can include the forms $\varphi_1, \varphi_2, \ldots, \varphi_s$, where all φ are homogeneous of degree one in the p. The homogeneous system in involution (149) will then *admit* the infinitesimal homogeneous contact transformations $(\varphi_1\, f), \ldots, (\varphi_s\, f)$, so all transformations of the infinite group of transformations that they generate ([240]). If one endows a system in involution:

$$F_i\,(z, x_1, \ldots, x_{m-1}, p_1, \ldots, p_{m-1}) = c_i \qquad (i = 1, \ldots, q) \tag{151}$$

with the homogeneous form (149), as in no. **35**, then any infinitesimal contact transformation of the variables $z, x_1, \ldots, x_{m-1}, p_1, \ldots, p_{m-1}$ that leaves the system (151) invariant will be converted into an infinitesimal homogeneous contact transformation that takes the system (149) to itself. The theory of homogeneous groups then shows how one can exploit known infinitesimal contact transformations ([241]) that leave a system in involution (151) invariant in order to integrate it. Thus, the simplification that the integration of an equation that does not include the unknown z explicitly will afford is based upon the fact that such an equation will admit the infinitesimal translation $\partial f / \partial z$. *Lie* ([242]) has treated the more general question of the advantage that this condition might yield for the integration of a first-order equation with known infinitesimal point transformations. *W. de Tannenberg* ([243]) has carried out a classification and theory of integration for all first-order equations with two independent variables that admit known finite groups of point transformations, following *Lie*'s suggestions ([244]).

42. Bäcklund's theory ([245]). – If μ ($\leq 2m$) independent equations:

$$F_i\,(z, x_1, \ldots, x_m, p_1, \ldots, p_m) = 0 \qquad (i = 1, 2, \ldots, \mu) \tag{152}$$

([237]) For the form that this theory assumes for systems in involution that include z explicitly, cf., my book, Chap. 14.

([238]) Math. Ann. **11**, pp. 540, 544, *et seq.*

([239]) *Ibidem*, pp. 521-529 (cf., no. **13**, footnote 80).

([240]) Cf., *Lie*, Math. Ann. **24**, pp. 554, rem. "*Transform.*" 2, Chap. 16. See also III D 7.

([241]) See also *Bäcklund*, Math. Ann. **15**, pp. 50-62. My book, Chap. 14, § 4.

([242]) Math. Ann. **5**, pp. 200. *Lie-Scheffers*, *Berührungs.*, Chap. 13.

([243]) Toul. Ann. **5** (1891).

([244]) "*Tr.*," 3, pp. 128, rem.

([245]) *A. V. Bäcklund*, Math. Ann. **11** (1877), pp. 412. My book, Chap. 14, § 5. The theory is a special case of the "generalized *Frobenius* theory" of the *Pfaff* problems, cf., no. **26**.

are given, and if $2k$ ([246]) is the order of the highest-order determinant in the matrix $\| [F_i\, F_k] \|$ that does not vanish identically because of (152) then equations (152) will possess common integral-M_{m-k} when $k > \mu - m$ that collectively subsume all surface elements that are defined by (152) and will be generated by "characteristic" manifolds $M_{\mu-2k}$, and they will possess a simple infinitude of integral-M_{m-k} in the case of $k = \mu - m$. By contrast, not all surface elements that satisfy equations (152) can belong to a common integral-M_{m-k+1} of those equations. If $\mu \geq m + 1$ and $k = \mu - m - 1$ then equations (152) will define an element-$M_{2m+1-\mu}$. The theory of systems in involution (no. **38**) is included in these theorems as a special case ($k = 0$).

V. Advanced differential problems.

1. *Differential systems with two independent variables.*

43. Classification of second-order partial differential equations with respect to their first-order characteristics. – The relations:

$$F(x, y, z, p, q, r, s, t) = 0 \qquad \left(r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x\, \partial y}, t = \frac{\partial^2 z}{\partial y^2} \right) \tag{153}$$

$$dp = r\, dx + s\, dy, \qquad dq = s\, dx + t\, dy \tag{154}$$

generally determine a discrete number of systems of values r, s, t. Therefore ([247]), in general, a discrete number of integral surfaces of the partial differential equation (153) will go through a strip S of first-order surface elements in the space $R_3 = (x, y, z)$. If the values r, s, t remain indeterminate for any two neighboring surface elements x, y, z, p, q and $x + dx$, …, $q + dq$ of the strip S by means of (153), (154) then the latter will be called a *first-order characteristic* of the partial differential equation (153). If one interprets x, …, q as parameters and r, s, t as the point coordinates of an R_3 then equation (153) can be interpreted as an ∞^5-family of surfaces F and the relations (154) as the equations of a line in the complex (κ) that consists of lines parallel to the generators of the cone:

$$r\, t - s^2 = 0 \tag{155}$$

then that will imply the following classification ([248]) of the equations (153):

a) Equation (153) has the *Ampére* form ([249]):

([246]) One always has $\mu - m - 1 \leq k \leq \frac{1}{2}\mu$.

([247]) *É. Goursat*, Acta math. **19**, pp. 291 – *Goursat B*, 1, art. 16.

([248]) *Goursat*, *loc. cit.*, pp. 297 = *Goursat B*, 1, art. 87.

([249]) J. éc. polyt. **11**, cah. **18** (1820), pp. 34; cf., *de Morgan*, Trans. Camb. Math. Soc. **9** (1854), pp. 515, esp., art. 20 and 23; *L. Natani*, *Analysis*, pp. 265-380; *É. Bour*, J. éc. polyt. **22**, cah. **39** (1862), pp. 186. *G. Boole*, J. f. Math. **61** (1863), pp. 309 = *Boole*, Suppl. Vol., Chaps. 28, 29; *Imschenetsky*, Arch. Math. **54** (1872), pp. 209 = *Mansion*, Ger. ed., Appendix 2; *Goursat B* 1, Chap. 2; *A Cayley*, Quart. J. Math. **26** (1893), pp. 1 = *Papers* 13, pp. 358.

(156) $$Hr + 2Ks + Lt + M + N(r\,t - s^2) = 0$$
$$(H, K, \dots \text{ are functions of } x, y, z, p, q).$$

Any surface F includes two different families of ∞^1 lines of the complex (κ). Accordingly, there are two different systems of first-order characteristics. One of them is defined by:

(157.1) $$dz = p\,dx + q\,dy\,, \qquad N\,dp + L\,dx + \lambda_1\,dy = 0\,, \qquad N\,dq + \lambda_2\,dx + H\,dy = 0\,,$$

$$(\lambda_1, \lambda_2 = -K \pm \sqrt{K^2 - H\,L - M\,N})$$

while the other one is defined by equations (157.2), which one gets by switching λ_1, λ_2. In the case of the *Monge* equation ([250]), i.e., for $N \equiv 0$, the systems (157.1), (157.2) are replaced with the two systems ([250.a]):

(158.i) $$dy = \lambda_i\,dx\,, \quad H\,\lambda_i\,dp + L\,dq + M\,\lambda_i\,dx = 0\,, \quad dz = p\,dx + q\,dy \qquad (i = 1, 2)$$

b) Equation (153) represents the complex cone of (κ) or tangential planes to such a thing so it will have the form (156), in which $K^2 - H\,L - M\,N \equiv 0$. The two characteristic systems are then identical.

c) The surfaces F are developables whose generators belong to the complex (κ), but not a cone. Equation (153) arises by eliminating α from two equations of the form:

(159) $$r + 2\,\alpha\,s + \alpha^2\,t + 2\,\psi\,(x, y, z, p, q, \alpha) = 0\,, \qquad s + {}^t\alpha + \psi'_\alpha = 0\,.$$

There is a doubly-counted system of first-order characteristics that is defined by three equations:

(160) $$dz = p\,dx + q\,dy\,, \qquad \varphi_1\,(x, \dots, q, dx, \dots, dq) = 0\,, \quad \varphi_2\,(x, \dots, \alpha_2) = 0\,.$$

d) The surfaces F are ruled surfaces of the complex (κ) that are neither second-degree nor developable. There is a singly-counted first-order system of characteristics that is defined by three equations of the form (160).

e) Any surface F includes a discrete number of complex lines. There are ∞^4 first-order characteristics.

f) The general case.

([250]) Paris Hist. (1784), pp. 118; cf., *A. M. Legendre, ibidem* (1787), pp. 309.
([250.a]) For the case in which one or both functions *H, L* vanish, cf., *Goursat B,* 1, art. 25.

For the categories *b*) and *c*), and only for them, one has:

(161) $$4RT - S^2 \equiv 0 \qquad \left(R = \frac{\partial F}{\partial r}, \text{etc.}\right)$$

as a result of (153).

Under an arbitrary contact transformation of space $R_3 = (x, y, z)$, any first-order characteristic will again go to another such thing, so any second-order partial differential equation will go to an equation in the same category.

An element-M_2 (no. **9**) that consists of ∞^2 surface elements x, y, z, p, q, r, s, t but does not define a surface is linked with either a first-order strip or *one* first-order surface element. By contrast, a second-order element-M_2 whose first-order surface elements likewise define a ∞^2-family and are linked with a curve in R_3 or include the same point, cannot be represented in the coordinates $x, y, \ldots, t$ (cf., no. **9**, footnote 46.a). *F. Engel* ([251]) then used the homogeneous coordinates $-\rho : u, -\sigma : u, -\tau : u$, in place of r, s, t, and a fifth coordinate v that is coupled with the previous one by the relation $\rho\tau - \sigma^2 + uv = 0$. $u = 0$ will then be the second-order partial differential equation that has all curves in R_3 for its integrals. *Ampère*'s equation will be linear and homogeneous in the coordinates ρ, σ, τ, u, v. The *Monge* equation is linear and homogeneous in ρ, σ, τ, u, and characterized by the fact that it has the "points" of R_3 for its integrals. Since the coordinates $\rho, \ldots, v$ transform linearly and homogeneously under any extended contact transformation of R_3, a *Monge* equation will generally be converted into an *Ampère* equation under an arbitrary contact transformation. Conversely, equation (156) will take on the *Monge* form ([252]) under any contact transformation that transforms the ∞^3 of its integral surfaces into the points of an R_3'.

44. First integrals of a second-order partial differential equation. – Any integral surface of a second-order equation of the first four categories is generated by ∞^1 first-order characteristics, and indeed in two different ways in case *a*). Conversely, any M_2 of surface elements x, y, z, p, q that consists of ∞^1 characteristics will be an integral of (153). The characteristics of an intermediate integral $f(x, y, z, p, q) = 0$ (no. **6** and **34**) are included among those of equation (153). If one then replaces the $dx, \ldots, dq$ in the defining equations of the first-order characteristics of (153) with their values in (81) then that will yield a system Σ of first-order partial differential equations for f with the independent variables $x, \ldots, q$ ([253]). One finds all first integrals $f = 0$ by seeking all functions f that fulfill the system Σ *as a result of* $f = 0$. Any solution f of the system Σ will imply a first integral:

(162) $$f(x, y, z, p, q) = \text{const.}$$

([251]) Leipziger Ber. (1893), pp. 468.
([252]) *A. V. Imschenetsky*'s "Methode der Variation der Konstanten" [*loc. cit.*, Chap. 4; cf., *Lie*, Christ. Forh. (1872), pp. 24.] reduces to this simple remark.
([253]) *Goursat B* 1, art. 90.

of the partial differential equation (153), and conversely.

For the category *a*), one then obtains two systems Σ_1 and Σ_2 that each consist of two linear first-order partial differential equation that are adjoint to the system (157.1) [(157.2), resp.] (no. **14**). If (162) is a first integral of equation (156) then *df* will be an integrable combination of one of the systems (157.1) and (157.2), and conversely. If *df* is an integral combination of one system (157) then the characteristics of the partial differential equation (162) will belong to the characteristic system of (153), and conversely. If u_1, u_2 are solutions of Σ_1, Σ_2, resp., then the first-order partial differential equations $u_1 = c_1$ and $u_2 = c_2$ will be involutory ([254]).

Two solutions *u*, *v* to the same system Σ_i are involutory if and only if $\lambda_1 = \lambda_2$. One will then be dealing with the case *b*), and the systems Σ_i will coincide with a single *complete* system that will then possess a third solution $w(x, y, z, p, q)$ (that is involutory with *u* and *v*). Conversely, if the system Σ_1 is complete then it will be identical to Σ_2, its three solutions *u*, *v*, *w* will be involutory. Equation (156) will then have two different general first integrals $v = \varphi(u)$, $w = \psi(u)$ ([255]). Its most general integral-M_2 is obtained by determining *u*, *v*, *w*as the enveloping structure to any ∞^1 of the triple infinitude of element-M_2's:

$$u = a\,, \quad v = b\,, \quad w = c\,.$$

An equation of that type ([256]) is equivalent to $r = 0$, or $r\,t - s^2 = 0$, or to the equation whose general integral consists of all "curves" in $R_3 = (x, y, z)$ under a contact transformation (see the previous no.).

If equation (156) from category *b*) possesses *one* first integral (162) then it can be brought into a form that is free of *s* and *t* by a contact transformation whose determination is equivalent to the integral of (162) ([257]).

45. Continuation. – If two solutions *u*, *v* of the system Σ_1 exist for an equation (156) of the category *a*) then equation (156) will possess a general first integral:

$$u(x, y, z, p, q) = \varphi[v(x, y, z, p, q)] \tag{163}$$

that will be recovered by differentiating (footnote 2) that equation with respect to *x*, *y* and eliminating $\partial\varphi / \partial v$. There is one and only one equation of the form (163) that includes an arbitrarily-given first-order strip *S*, and to whose integration the solution of the *Cauchy* problem comes down, i.e., determining the integral surface that goes through *S*.

([254]) *Monge* [Par. Hist. (1784), pp. 168] used that relation in order to integrate the first-order partial differential equation.

([255]) *Boole*, Trans. London Math. Soc. (1862), pp. 451, *et seq.*; Suppl. Vol., pp. 123; J. f. Math. **61**, pp. 309; *Lie*, Math. Ann. **5**, § 19; *Goursat B*, Chap. 1.

([256]) The generalization to *m* independent variables was given by *J. Kürschak*, Math. u. naturw. Ber. aus Ungarn **14** (1898), pp. 285; see no. **57**.

([257]) *Ampère*, *loc. cit.* (footnote 249), pp. 126, *et seq.*

If Σ_2 also possesses two solutions u', v' then there will exist a second general intermediate integral $u' = \psi(v')$, and if one has determined u and u' then the search for v and the integration of the partial differential equation (163) will each require one quadrature ([258]). All equations (156) of that type are equivalent to $s = 0$ under contact transformations. If Σ_2 has *only one* solution then the integration of each equation of the form (163) will be accomplished by an operation 1.

Equation (156) can be freed of one or both of the derivatives r, t by a contact transformation according to whether only one or both of the systems Σ_1, Σ_2, possess a solution ([259]).

An equation of the category *e*) can possess *one* first integral (162), while an equation of the category *f*) can possess *at most one* such thing of the form $f = 0$.

If the two first-order partial differential equations of the system Σ (no. **44**), together with the relation that is derived from it by the bracket operation (no. **38**), define a three-parameter system in involution for an equation (153) of the category *c*) or *d*) then it will possess a solution with two arbitrary constants a, b, so equation (153) will possess a first integral of the form ([260]):

$$V(x, y, z, p, q, a, b) = 0 . \tag{164}$$

A general first integral will follow from this by variation of the constants, so the solution to the *Cauchy* problem for (153) will follow upon integrating a system of ordinary differential equations. Eliminating a and b from (164) and $\partial V / \partial a = 0$, $\partial V / \partial b = 0$ will produce a singular first integral under some circumstances. According to *A. V. Bäcklund* ([261]), one will get every equation (153) of that type from an equation in x', y', z', p', q' by a *surface transformation* (no. **10**):

$$x' = \xi,\ y' = \eta,\ \ldots,\ q' = \kappa \qquad (\xi, \eta, \ldots, \kappa \text{ are functions of } x, y, z, p, q, r, s, t). \tag{165}$$

Should the system Σ be involutory ([262]), so equation (153) would possess a first integral with three arbitrary constants, then (153) would belong to category *e*) and the function y in (159) would satisfy a certain second-order partial differential equation with the independent variables x, y, z, p, q, α. Every first integral of the form (164), together with the equations:

$$\frac{\partial V}{\partial a} = a', \quad \frac{\partial V}{\partial b} = b' \qquad (a, b, a', b' \text{ arbitrary constants}), \tag{166}$$

will then define a system in involution. If $W(x, y, z, a, b, a', b') = 0$ is the result of eliminating p and q from equations (164), (166) then the general integral of (153) will have the form ([263]):

([258]) *Lie*, Norw. Arch. **2** (1877), pp. 1; cf., Leipziger Ber. (1895), pp. 498.

([259]) *Ampère*, *loc. cit.*, pp. 122, 154, *et seq.*

([260]) *Lagrange*, Berl. Mém. (1774) = *Werke* 4, pp. 89; *Goursat*, C. R. Acad. Sci. Paris **112**, pp. 1117.

([261]) Math. Ann. **11**, pp. 213-226.

([262]) *N. J. Sonin*, Moscow Math. Soc. (1874), translated by *Engel* in Math. Ann. **49**, pp. 417, esp. § **7**; *H. A. Speckman*, Amsterdam Versl. (1892), pp. 465; *Goursat B* 1, art. 93-96.

([263]) How one must determine W in order for the relations (167) to represent the general integral of an equation of the form (153) is in *Goursat*, Acta math **19**, pp. 331.

(167)
$$\begin{cases} W[x, y, z, a, \varphi(a), \varphi'(a), \psi(a)] = 0, \\ \dfrac{\partial W}{\partial a} + \dfrac{\partial W}{\partial \varphi}\varphi' + \dfrac{\partial W}{\partial \varphi'}\varphi'' + \dfrac{\partial W}{\partial \psi}\psi' = 0. \end{cases}$$

46. Higher-order characteristics of a second-order partial differential equation. – By h-fold derivation of equation (153) with respect to x and y, one will get two relations of the form:

(168.h)
$$M_k^h + R\, z_k^{h+2} + S\, z_{k+1}^{h+2} + T\, z_{k+2}^{h+2} = 0 \qquad (k = 0, 1, \ldots, h)$$

$$\left(z_k^s \equiv \frac{\partial^s z}{\partial x^k\, \partial y^{s-k}} ;\, z_0^0 \equiv z ;\, R \equiv \frac{\partial F}{\partial r}, \text{etc.} \right)$$

A "strip of order ν of the partial differential equation (153)," i.e., a system of ∞^1 values $x, y, z, p, \ldots, z_\nu^\nu$ that satisfies the relations:

(169.k)
$$dz_i^k = z_i^{k+1} \cdot dx + z_{i+1}^{k+1} \cdot dy \qquad (i = 0, 1, \ldots, k)$$

for $k = 0, 1, \ldots, \nu - 1$, as well as equations (153) and (168.h) for $h = 1, 2, \ldots, \nu - 2$, will generally determine one and only one $(\nu + 1)^{\text{th}}$-order strip of (153) that subsumes it. However, if it is a ν^{th}-*order characteristic* ([264]), i.e., it satisfies one of the two systems of relations:

(170.j)
$$\begin{cases} a) \quad dy - \Lambda_j\, dx = 0; \quad b) \quad dz_i^k - \left(dz_i^{k+1} + \Lambda\, z_{i+1}^{k+1}\right) dx = 0, \\ c) \quad R\, dz_{h-1}^\nu + (S - R\,\Lambda_j)\, dz_h^\nu + M_{h-1}^{\nu-1}\, dx = 0 \end{cases} \qquad (j = 1, 2)$$

$$(i = 0, 1, \ldots, k ;\ k = 0, 1, \ldots, \nu - 1 ;\ h = 1, 2, \ldots, \nu),$$

in which Λ_1, Λ_2 mean the roots of the *characteristic equation:*

(171)
$$R\,\Lambda^2 - S\,\Lambda + T = 0\,,$$

then equations (168.ν – 1) and (169.ν) will leave the $(\nu + 1)^{\text{th}}$ derivatives of z undetermined. The relations (170.j), c) will reduce to *only one* independent one by means of (168.ν – 2) [by means of (153) in the case $\nu = 2$, resp.]. One of the differentials dz_0^ν, dz_1^ν, …, dz_ν^ν will then remain arbitrary by means of (170.j). Any characteristic of order ν includes each characteristic of order $\nu - 1$, ν –

([264]) *A. V. Bäcklund*, in the papers cited above (footnote 55); *Goursat B*, Chap. 4; *Lie*, Leipziger Ber. (1895), pp. 61; my note in Math. Ann. **44** (1894), pp. 466 and **47**, pp. 230. The second-order characteristics were considered already by *G. Monge*, Paris. Mém. (1784), pp. 190; *Du Bois-Reymond*'s *Beiträge*, Chap. 5 and 16.

2, …, 2, and if $\Lambda_1 \neq \Lambda_2$ then it will include a simple infinitude ([265]) of characteristics of order $\nu + 1$ ([266]). Any non-singular integral surface will be generated by ∞^1 characteristics of order ν from each of the two systems (170). The two associated systems of ∞^1 curves are called the *characteristic curves* ([267]) of the integral surface. Along each characteristic of order n, ∞^∞ integral surfaces have contact of order ν. One and only one integral surface of (153) is determined ([268]) through two arbitrary characteristics of order ν that belong to two different systems and have a surface element of order ν in common.

47. Continuation. – Any *first*-order characteristic of an equation (156) from category *a*) is included in a simple infinitude of second-order characteristics, and each of the latter includes one such thing of order 1. For the category *d*), the first-order characteristic systems have the same relationship to each other as the ones in one of the two second-order characteristic systems. For $\Lambda_1 \equiv \Lambda_2$, i.e., for an equation of category *b*) or *c*), any second-order strip that subsumes a first-order characteristic will indeed satisfy the defining equations of the characteristics. However, *just one* integral surface will go through a given first-order characteristic. The only exceptions ([269]) are defined by the *Ampère* equation with two intermediate integrals $v = \varphi(u)$, $w = \psi(u)$ (footnote 255) and *Sonin*'s equation (footnote 262). There are ∞^5 distinguished first-order characteristics for them that will generate all non-singular integral surfaces. The characteristics of order ν that lie in non-singular integral surfaces define a distinguished $\infty^{2\nu+3}$-family, and every distinguished first-order characteristic is included in ∞^2 distinguished second-order characteristics, each of the latter are included in ∞^2 distinguished third-order characteristics, etc.

48. Characteristics of n^{th}-order partial differential equations. – The concept of a characteristic can be adapted to an n^{th}-order equation directly ([270]):

$$F(x, y, z, p, q, r, \ldots, z_i^k, \ldots, z_0^n, \ldots, z_n^n) \qquad \left(z_i^k = \frac{\partial^k z}{\partial x^i\, \partial y^{k-i}}\right) \tag{172}$$

A *characteristic of order* $n + \nu$ is a strip that fulfills equation (172) and the derived equations up to order $n + \nu$ in such a way that for any two neighboring elements of the strip equations

([265]) See my paper in Math. Ann. **47**, pp. 234, *et seq.*

([266]) The latter will be ascertained with no integration when the characteristic system in question possesses an invariant of order $\nu + 1$ (no. **51**), *Goursat B*, **2**, art. 165.

([267]) *Monge* (*Applications*, pp. 471) called them *characteristics*. The second-order equations on whose integral surfaces those curves are the lines of curvature or asymptotic lines were considered by *Lie*, Math. Ann. **5**, pp. 209-233. See also *Du Bois-Reymond*'s *Beiträge*, pp. 129, *et seq. E. Stephan*, Ann. éc. norm. sup. (1855), pp. 1.

([268]) *Goursat B* 1, art. 82-85. A special case of the theorem was given by *Darboux*, *Surfaces* 2, pp. 92.

([269]) See my note, C. R. Acad. Sci. Paris **124**, pp. 1215; *Goursat*, *ibidem*, pp. 1294 (1897).

([270]) Cf., *Du Bois-Reymond*, *Beiträge*, pp. 198, *Bäcklund* (footnote 55) and my article (footnote 264).

(169.$n+\nu$) and the derived ones of order $n + \nu + 1$ for (172) have undetermined highest-order derivatives. Corresponding to the n roots Λ_i of the *characteristic equation*:

$$\frac{\partial F}{\partial z_0^n}\Lambda^n - \frac{\partial F}{\partial z_1^n}\Lambda^{n-1} + \cdots + (-1)^n \frac{\partial F}{\partial z_n^n} = 0\,, \tag{173}$$

there will be n generally-different characteristic systems of order $n + \nu$ ($\nu = 0, 1, 2, \ldots$) to which all of the theorems in no. **46**, with the exception of the last one, will find an application. The theory of first-order characteristics of a second-order equation also has its analogue here ([271]): A strip of order $n - 1$ is called a *characteristic of order* $n - 1$ when the same equation (172) will be fulfilled independently of z_n^n for any two neighboring elements of order $n - 1$ after one substitutes the values of $z_0^n, z_1^n, \ldots, z_{n-1}^n$ that follow from (169.$n - 1$). Of the categories of n^{th}-order equations that this implies, only *L. Natani*'s ([272]) generalization of *Ampère*'s equation:

$$A + \sum A_i\, z_i^n + \sum\sum B_{ik}\,(z_i^n\, z_{k+1}^n - z_k^n\, z_{i+1}^n) = 0 \tag{174}$$

$$(A, A_i, B_{ik} \text{ are functions of } x, y, z, z_0^n, \ldots, z_0^{n-1}, z_1^{n-1}, \ldots, z_{n-1}^{n-1})$$

has been examined in detail. The elimination of $z_0^n, \ldots, z_n^n$ from equations (169.$n - 1$) and (174) leads to two homogeneous nonlinear equations in dx, dy, $dz_0^{n-1}, \ldots, dz_{n-1}^{n-1}$ that will be fulfilled by one, or at most two ([273]), different pairs of *linear* homogeneous equations in dx, dy, dz_i^{n-1} in the event that not all $B_{ik} = 0$, under certain conditions that must be fulfilled by A, A_i, B_{ik} ([274]), while it will decompose into n different pairs of linear equations ([275]) if and only if $B_{ik} \equiv 0$. Any characteristic of order $n - 1$ is generally included in ∞^1 n^{th}-order characteristics, and conversely, any n^{th}-order characteristic subsumes one such thing of order $n - 1$. If the linear defining equations of a characteristic system of order $n - 1$ of equation (174) admit an integrable combination df (x, $\ldots$, $z_0^{n-1}, \ldots, z_{n-1}^{n-1}$) then $f = c$ will be a first integral of (174). If it possesses two integrable combinations df, df' then there will exist a first integral $f = \varphi(f')$, in general. An equation (174) that is nonlinear in the z_i^n can then possess at most two first integrals of the form above, while a linear equation will possess n different ones, in general ([276]).

If $\rho_1, \ldots, \rho_n$ are the arguments of the arbitrary functions φ_{ik} that appear in the general integral (no. **4**) of an n^{th}-order equation of the "first class" then the relations:

([271]) Cf., my article, Math. Ann. **47**, pp. 239.
([272]) *Analysis*, pp. 380-388.
([273]) *A. V. Bäcklund*, Math. Ann. **13**, esp., pp. 97.
([274]) *M. Hamburger*, J. f. Math. **81** (1876), pp. 272.
([275]) *G. Monge*, Paris Hist. (1784), pp. 155, *et seq.*
([276]) *Bäcklund*, *loc. cit.*; *M. Falk*, Ups. Nov. Act. (1872), pp. 1.

$$\rho_1 = \text{const.}, \ldots, \rho_n = \text{const.}$$

will define n systems of characteristic curves that belong to the roots $\Lambda_1, \ldots, \Lambda_n$, resp., of (173) on any integral surface. If the expressions for $p, q, r, \ldots, z_n^n$ that are to be calculated from the general integral equations include no higher derivatives of the arbitrary functions $\varphi_{1k}(\rho_1)$ than the integral equations themselves then one can determine ∞^∞ integral surfaces that contact a given integral surface V along the characteristic curves $\rho_1 = \rho_1^0$ n times by a suitable choice of the φ_{1k}. By contrast, there will be ∞^∞ integrals surfaces that contact V along the curve $\rho_1 = \rho_1^0$ only $n - 1$ times when the expressions for $p, \ldots, z_n^n$ include derivatives of the functions φ_{1k} that do not occur in the integral equations. *A. M. Ampère* (277) deduced the condition for the appearance of a characteristic system of order $n - 1$ and the defining equations of the latter from the latter assumption.

49. Relations between two second-order partial differential equations. – Should two second-order partial differential equations:

$$(175) \qquad F_1(x, y, z, p, q, r, s, t) = c_1, \qquad F_2(x, y, \ldots, t) = c_2,$$

define an unrestricted-integrable system, with *Lie*'s terminology (41), for arbitrary c_1, c_2, i.e., they have ∞^4 integral surfaces in common, then the six fourth-order equations that follow from (175) upon differentiating twice with respect to the fourth derivatives would have to reduce to only five independent ones (278), which will yield a linear second-order partial differential equation for F_2 for a given F_1 (279). The integration of (175) will then require that of a system of ordinary differential equations (nos. **3** and **16**, footnote 96) and deliver a complete integral with five arbitrary constants for $F_1 = c_1$. If an n^{th}-order equation $F(x, y, \ldots, z_n^n) = c$ is to have ∞^{2n+1} common integral surfaces with $F_1 = c_1$ then the function F must fulfill a certain linear second-order partial differential with the independent variables $x, y, \ldots, z_n^n$ (279).

Equations (175) will define a system in involution (280) for arbitrary values of c_1, c_2, or only two particular ones, according to whether they reduce to the four third-order equations that follow from them by a single derivation identically or to only *three* independent ones by means of (175). If the curve (175) in the space $R_3 = (r, s, t)$ (cf., no. **43**) represents a system in involution then the developable that is generated by its tangents will represent a second-order equation of the category

(277) J. éc. polyt. **10**, cah. 17, pp. 590, *et seq.*; cf., *Goursat B* 1, art. 98, 99.

(278) *Bäcklund*, Math. Ann. **15**, pp. 50.

(279) *Sonin* (footnote 262); *J. König*, Math. Ann. **24** (1884), pp. 501.

(280) Systems of that type were occasionally considered by *Du Bois-Reymond* (*Beiträge*, pp. 177), and then *G. Darboux*, Ann. éc. norm. sup. (1870), pp. 163 = C. R. Acad. Sci. Paris **70**, pp. 675, 746. Cf., *Sonin*, *König*, *loc. cit.*; *Bäcklund* (footnote 274); *L. Bianchi*, Rend. Lincei (4) **2**, pp. 218, 237, 307; *J. Beudon*, Ann. éc. norm. sup. (1896), Suppl. art. 14; my note in Münchener Ber. **25** (1895), pp. 101.

b) or *c*) that belongs to the type that was discussed at the end of no. **47**, and conversely ([281]). The integrals of the system in involution (175) are singular solutions of that second-order equation.

The characteristic equations (no. **46**) of the two involutory equations (175) have a root Λ_1 in common, while the equations (175) themselves have a family of ∞^5 second-order characteristics of the first system in common. If two common integral surfaces of (175) osculate at a point then they will do that along a common characteristic. If the system in involution (175) can be brought into a form that is linear in r, s, t ([282]) then it will include ∞^1 common second-order characteristics of the same first-order strip, and there will be ∞^1 common first-order characteristics that can be combined into ∞^∞ surfaces, and in particular, they can be identical to the characteristics of a common intermediate integral (162). One and only one of the ∞^1 integral surfaces that are generated by the common second-order characteristics goes through each common second-order strip of the involutory equations (175), and it is then found by a method ([283]) that is entirely analogous to that of *Cauchy* (no. **34**).

In order to integrate the system in involution (175), with *Lie* ([284]), one can also look for all equations $V(x, y, z, p, q) = \alpha$ that have ∞^2 integral surfaces in common with the system (175), which will lead to a *semi-linear* first-order equation (no. **34**, footnote 208) for the unknown V. One can also follow *N. Sonin* ([285]) and integrate a linear first-order partial differential equation with the independent variables x, y, z, p, q, r, s, t and determine a second-order equation of the form $F_3(x, \ldots, t) = c_3$ that has an integral with three constants c_4, c_5, c_6 in common with the system (175), and in that way obtain a complete integral of the equation $F_1 = c_1$ that depends upon the arbitrary constants $c_2, \ldots, c_6$ and whose surfaces can be combined into families of a simple infinitude of such things in such a way that their envelope will osculate along a curve ([286]).

There exists no corresponding process for an arbitrary complete integral of a second-order equation, and the variation of constants will generally lead to no simplification of the business of integration ([287]) in that case; the same thing will be true for higher differential problems *a fortiori* ([288]). The theory of singular solutions of a first-order partial differential equation (no. **33**) has no analogue for partial differential equation of higher order, either ([289]).

([281]) *Goursat*, C. R. Acad. Sci. Paris **122** (1896), pp. 1258; J. éc. polyt. (2), cah. 3 (1897), pp. 75.

([282]) See *Bäcklund*, Math. Ann. 13, pp. 73, *et seq.* and the thorough presentation in *Goursat*, J. éc. polyt., *loc. cit.*

([283]) According to *W. de Tannenberg* (C. R. Acad. Sci. Paris **120**, pp. 674), that fact can also be expressed by saying that the system of *Pfaff* equations (no. **8**) that is equivalent to the system in involution (175) admits a one-parameter group (no. **60**).

([284]) Leipziger Ber. (1895), pp. 70, *et seq.*

([285]) *Loc. cit.* (footnote 262), § **20**.

([286]) For the adaptation of *Monge*'s theory of envelopes to this case, see my article Math. Ann. 47, pp. 254; *Beudon*, *loc. cit.* (footnote 280), art 15, and C. R. Acad. Sci. Paris **128** (1899), pp. 1215.

([287]) *Lagrange*, *Oeuvres* **4**, pp. 101; *Goursat B* 1, art. 21.

([288]) The method of variation of constants that was developed by *V. Sersawy* [Wiener Denschr. **53** (1887)] and *L. Königsberger* (J. f. Math. **109**, pp. 313-319, 321-328) and presented in general can be applied successfully only in special cases that are more easily resolved by means of the theory of characteristics.

([289]) Cf., my article, Math. Ann. **46**, pp. 1.

50. Darboux systems. Systems in involution. – If one has a second-order equation and an n^{th}-order equation ([290]):

$$(176) \qquad F_1(x, \dots, t) = c_1 ; \qquad F_2(x, y, z, p, \dots, z_0^n, z_1^n, \dots, z_n^n) = c_2 \qquad (n \geq 2)$$

then the $n + 2$ equations of order $n + 1$ that follow from (176) by derivation will reduce to $n + 1$ independent ones if and only if the two characteristic equations of $F_1 = c_1$ and $F_2 = c_2$ have a common root Λ_1, while equations (176) themselves have ∞^{2n+1} n^{th}-order characteristics of the first system in common. One and only one of the integral surfaces that are generated by the common characteristics will then go through each common n^{th}-order strip of equations (176). The equations $F_2 = c_2$ will then be an intermediate integral of $F_1 = c_1$, i.e., it will have ∞^∞ integral surfaces in common with that equation. That will be true for any arbitrary c_1 if and only if dF_2 is an integrable combination of the total differential equation that is defined by the n^{th}-order characteristic system of the equation $F_1 = c_1$ that belongs to the root Λ_2 (no. **46**). If equations (175) define a system in involution for arbitrary c_1, c_2 then F_1 will have an analogous relationship to $F_2 = c_2$.

In order for a system of k ($\leq n + 1$) n^{th}-order partial differential equations to define a system in involution ([291]) such that an integral surface of the system goes through every common n^{th}-order strip, it is necessary and sufficient that the $2k$ derivatives of order $n + 1$ must represent only $k + 1$ independent relations. Any k different first integrals of a *Natani* equation of order $n + 1$ (no. **48**) that belong to k different n^{th}-order characteristic systems of the latter, resp., define a system in involution ([292]). For $k = n + 1$, the general integral of the system in involution depends upon a finite number of constants, and for $k \leq n$, it depends upon $n - k + 1$ arbitrary functions. In the latter case, the system in involution will possess $n - k + 1$ systems of characteristics of order n and higher that define a complete analogue of the characteristic systems of a partial differential equation of order $n - k + 1$.

Lie ([293]) called a system of partial differential equation of arbitrary order that has a common integral with r arbitrary functions a *Darboux system of class r*. One will obtain the conditions for such a thing when one expresses the idea that the derived equations of a certain order leave exactly r of the highest derivatives of z arbitrary; that is then true automatically for the higher derivatives. In the case $r = 1$, the general integral will always be found by a method that is similar to that of *Cauchy* by integrating systems of ordinary differential equations.

51. The Darboux-Lévy theory of integration and its generalizations. – If $du(x, y, z, p, \dots, z_i^k, \dots, z_n^n)$ is an integrable combination of the defining equations of one of the two n^{th}-order characteristic systems of the partial differential equation (153) then, according to *Goursat* ([294]), u

([290]) *Darboux* (footnote 280); *Bäcklund*, Math. Ann. **13**, pp. 76; *ibid*. **15**, pp. 45, *et seq.*; *Sonin*, *loc. cit*. (footnote 262), § **22**, *et seq*. An example for $n = 3$ was given by *J. Beudon*, C. R. Acad. Sci. Paris **120**, pp. 902.

([291]) See my article Münchener Ber. **25** (1895), pp. 423.

([292]) *Bäcklund*, Math. Ann. **11** and **13**, *loc. cit.*

([293]) Leipziger Ber. (1895), pp. 71.

([294]) *Goursat B* 2, Chap. 7; C. R. Acad. Sci. Paris **123** (1896), pp. 680; cf., also *Sonin*, *loc. cit.* (footnote 262).

will be called an n^{th}-*order invariant* of (153). If the roots Λ_1, Λ_2 are different then the following theorems ([294]) will be true for the invariants that belong to the same root Λ_1 :

An n^{th}-order invariant must satisfy a system $\Sigma^{(n)}$ of two homogeneous linear first-order partial differential equations into which x, y, z, p, q and those of the 2^{nd}, 3^{rd}, …, n^{th} derivatives of z that remain arbitrary as a result of (153) and its derivatives enter as independent variables; $\Sigma^{(n)}$ will also be satisfied by the possible 1^{st}, 2^{nd}, …, $(n-1)^{\text{th}}$-order invariants. For $n > 2$, there is *at most one* n^{th}-order invariant that belongs to the root Λ_1 . It can be brought into a form that is linear in the n^{th} derivatives. The former is also true for $n = 2$ in the case of the *Ampère* equation or when a first-order invariant is present. The total number of the invariant of order n and lower that belong to Λ_1 is at most $n + 1$. That number will be achieved if and only if equation (153) possesses a first integral with two arbitrary constants. Namely, the number of invariants of first and second order combined will then be three, and there will be an invariant of order 3, …, n. If two different invariants u, v ([295]) belong to Λ_1 then one will an unbounded series of them:

$$u, v\,;\; v_1 \equiv \frac{dv}{du}\,;\; v_2 \equiv \frac{dv_1}{du}, \ldots \tag{177}$$

when one replaces the differential dz, dz_i^k with their values in (170.1), *b*). Conversely, if there is *more than one* invariant that belongs to Λ_1 then one can choose two of them u, v in such a way that every invariant that belongs to Λ_1 can be represented in the form $\varphi(u, v, v_1, v_2, \ldots, v_k)$.

If (153) is free of r and t then the invariants x and y belong to Λ_1, Λ_2, resp., and any other invariant will have the form ([296]):

$$\varphi\left(x, y, z, \frac{\partial z}{\partial x}, \ldots, \frac{\partial^n z}{\partial x^n}\right) \qquad \text{or} \qquad \psi\left(x, y, z, \frac{\partial z}{\partial x}, \ldots, \frac{\partial^n z}{\partial x^n}\right), \text{ resp.}$$

If two invariants u, v (u', v', resp.) exist for each of the two systems of characteristics in (153), i.e., if equation (153) possesses two different general intermediate integrals (no. **6**):

$$u = \varphi(v)\,, \qquad u' = \psi(v')\,, \tag{178}$$

then they will collectively define an unrestricted-integrable system, in the *Lie* sense (footnote 41), for every form in the arbitrary functions φ and ψ, and the integration of (153) will follow from that of ordinary differential equations [*Darboux's method* ([280])]. However, according to *M. Lévy* ([297]), that will also be true when *only one* intermediate integral $u = \varphi(v)$ exists. Namely, if n is its order then one can determine φ in one and only one way such that the partial differential equation u =

([295]) If the invariants u, v belong to the same root Λ_1 then the equations $u = c$, $v = c'$ will have ∞^∞ common integral surfaces when the possess *one* of them, *Goursat B*, art. 144.

([296]) *Goursat B*, art. 145.

([297]) C. R. Acad. Sci. Paris **75** (1872), pp. 1094; cf., *Sonin, König, loc. cit.*; *Speckman* (footnote 262); *V. Sersawy*, Wiener Denkschr. 49^2 (1882), pp. 7-33.

$\varphi(v)$ is fulfilled by a given n^{th}-order strip of equation (153), so the integral surface V of equation (153) that it establishes has something in common with it ([298]), which implies that V itself will be found by integrating a system of ordinary differential equations that depends upon the choice of initial strip.

If a second-order equation with two different characteristic systems possesses a general integral of the form ([299]):

$$x = V_1(\alpha_1, \alpha_2, \varphi_1(\alpha_1), \varphi_1'(\alpha_1), \ldots, \varphi_1^{(p)}(\alpha_1), \varphi_2(\alpha_2), \varphi_2'(\alpha_2), \ldots, \varphi_2^{(q)}(\alpha_2), F_1, \ldots, F_l),$$

$$y = V_2(\alpha_1, \ldots, F_l), \qquad z = V_3\ \alpha_1, \ldots, F_l),$$

in which the α_1, α_2 mean independent parameters, while the φ_i are arbitrary functions, the φ_i' are their derivatives, and the F_i are defined to be integrals of system of total differential equations:

$$dF_i = \sum_{k=1,2} \Phi_{ik}(\alpha_1, \alpha_2, \varphi_1, \varphi_1', \ldots, \varphi_2, \varphi_2', \ldots, F_1, \ldots, F_l)\, d\alpha_k \qquad (i = 1, 2, \ldots, l)$$

that is unrestricted-integrable for any form of φ_1, φ_2, There will then exist two independent invariants for each root Λ_1, Λ_2 ([300]), and conversely. All equations (153) in *Ampère*'s first class (no. **4**) are of this type. If only α_2, but not α_1, enter into V_i, Φ_{ik} in the given way then only the first characteristic system will possess two invariants, and conversely.

The demand that a partial differential equation (153) should be integrable by ordinary differential equations will, under a certain auxiliary assumption, translate into the other one that every second-order strip of one of the two systems should be capable of admitting only a one-parameter manifold of positions under translation along an arbitrary integral surface that includes it ([301]).

In order for an n^{th}-order invariant to exist for a second-order equation with coincident roots Λ_1, Λ_2, according to *Goursat*, the system of equations $\Sigma^{(n)}$ (cf., *supra*) must be a complete system ([294]). The conditions for that are independent of the number n and express the idea that equation (153) belongs to one of the categories that were listed at the conclusion of no. **47**. They are the only second-order partial differential equations that belong to *Ampère*'s first class that have *only one* characteristic system.

If the function u of the variables $x, y, z, p, \ldots, z_i^k, \ldots, z_\nu^\nu$ is an "invariant" of the partial differential equation (172) that belongs to the root Λ_i of (173), i.e., du is an integrable combination of the defining equations for the i^{th} characteristic system of order ν (no. **48**), then u will satisfy a system of linear homogeneous first-order partial differential equations, in which those of the

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([298]) *Lie*, Leipziger Ber. (1895), pp. 65.

([299]) *Goursat B*, art. 184.

([300]) *De Boer* (Harl. Arch. **27**, pp. 355) determined all equations of the form $f(r, s, t) = 0$ that are integrable by the *Darboux* method. *Goursat* [C. R. Acad. Sci. Paris **127** (1898), pp. 654, Toul. Ann. (2) **1** (1899), pp. 31] determined all equations of the form $s = f(x, y, z, p, q)$ that are *Darboux*-integrable.

([301]) See my article, Münchener Ber. **26** (1896), pp. 26. *Goursat B* 2, art. 185.

quantities $x, \ldots, z_\nu^\nu$ that remain arbitrary as a result of (172) and the derived equations represent the independent variables. If $u_1, \ldots, u_k$ are the invariants of arbitrary order that belong to $\Lambda_1, \ldots, \Lambda_k$, resp., then the equations:

$$F(x_1, \ldots, z_n^n) = 0, \qquad u_1 = c_1, \ldots, \quad u_1 = c_1 \qquad (c_i \text{ are arbitrary constants})$$

will define a *Darboux* system (no. **50**) of class $n - k$ for $k < n$, and an unrestricted-integrable system with a finite-parameter family of integral surfaces for $k = n$. If two different invariants u_i, v_i belong to each of the roots $\Lambda_1, \ldots, \Lambda_k$, i.e., if there are k different general intermediate integrals:

(179) $$u_1 = \varphi_1(v_1), u_2 = \varphi_2(v_2), \ldots, u_k = \varphi_k(v_k) \qquad (\varphi_1, \ldots, \varphi_k \text{ are arbitrary functions}),$$

then the solution of the *Cauchy* problem for equation (172) will come down to the integration of a $k + 1$-parameter *Darboux* system of the form (172), (179), so to a system of ordinary differential equations for $k = n$ or $k = n - 1$ ([302]).

That method can be adapted to systems of partial differential equation in involution ([302]), and above all, to *Darboux* systems.

52. First-order differential systems with several unknowns. – If ν ($< 2n$, but $\geq n$) equations:

(180) $$F_i(x, y, z_1, \ldots, z_n, p_1, q_1, \ldots, p_n, q_n) = 0 \qquad \left(i = 1, \ldots, \nu;\, p_i = \frac{\partial z_i}{\partial x};\, q_i = \frac{\partial z_i}{\partial y}\right)$$

that are independent with respect to the quantities p, q define a system in involution ([303]), and not all n-rowed determinants will vanish identically ([303.a]) by means of (180) in the *characteristic matrix:*

$$\left\| \frac{\partial F_i}{\partial q_k} - \lambda \frac{\partial F_i}{\partial p_k} \right\| \qquad (i = 1, \ldots, \nu;\, k = 1, \ldots, n),$$

then the latter will have precisely $2n - \nu$ linear factors $\lambda - \lambda_i$ in common ([303.b]). Correspondingly, there are $2n - \nu$ systems of characteristic strips ([304]) that are defined by just as many systems Σ_i of

([302]) See my paper in Münchener Ber. **25**, pp. 423; cf., the special cases of the theory that were treated by *M. Falk* (footnote 276), *M. Hamburger*, J. f. Math. **93** (1822), pp. 201 and *V. Sersawy*, Wiener Denkschr. 49^2, pp. 33-60.

([303]) See my article, J. f. Math. **118** (1897), pp. 123. For the special case $p = 0$, cf., *M. Hamburger*, J. f. Math. **93** (1882), pp. 188.

([303.a]) The case in which the number n and rank of the characteristic matrix are arbitrary can be reduced to the one above; see my article in Münchener Ber. **29**, (1899), pp. 231.

([303.b]) For the case of multiple roots and their relationship to elementary divisors, see my previously-cited work; cf., also *Hamburger* (footnotes 303 and 305).

([304]) For $n = 2$, $\nu = 2$, cf., also *Bäcklund*, Math. Ann. **19**, pp. 389, *et seq.*

Pfaff equations in x, y, z_i, p_k, q_k . One will obtain those systems when one expresses the idea that the $2n$ equations:

$$dp_k = r_k\, dx + s_k\, dy\,, \qquad dp_k = r_k\, dx + s_k\, dy \qquad \left(r_k = \frac{\partial^2 z_k}{\partial x^2}, \text{etc.}\right),$$

together with the 2ν derived equations of (180), leave the quantities r_k, s_k, t_k undetermined. In a similar way, one can define characteristic strips of order two and higher, as well as $2n - \nu$ systems of *characteristic curves* in the space $R_{n+2} = (x, y, z_1, \ldots, z_n)$ in the case where equations (180) are linear in the p_k, q_k ([305]), by systems of total differential equations. The possible integrable combinations of the latter lead to an integration procedure that is entirely analogous to the *Darboux-Lévy* method. In the case of $\nu = 2n - 1$, one and only one first-order characteristic strip will go through any common surface element x, y, z_i, p_k, q_k of equations (180), and the integration of (180) will be completed, just as in the *Cauchy* method (no. **34**), with the help of a system of ordinary differential equations ([306]).

According to *A. V. Bäcklund* ([307]), one will obtain, in general, two systems J_1, J_2 of two involutory third-order partial differential equations that are linear in the highest derivatives for the unknowns z_1 and z_2 by derivations and eliminations that are applied to two equations of the form:

$$F_i\,(x, y, z_1, \ldots, z_n, p_1, q_1, p_2, q_2) = 0 \qquad (i = 1, 2),$$

resp. An invertible single-valued relationship will then exist between the second-order surface elements of the two spaces (z_1, x, y) and (z_2, x, y) such that each integral surface of J_1 will correspond to such a thing for J_2, and conversely. However, each of the systems J_1 and J_2 can also reduce to a second-order partial differential equation. Every integral of J_1 will then correspond to at most ∞^1 integrals of J_2, and conversely ([308]). Analogous statements are true for every relation between the surfaces elements x, y, z, p, q and x', y', z', p', q' of two spaces that are mediated by four relations ([309]).

([305]) The case of linear systems with n equations and n unknowns was treated by *M. Hamburger*, J. f. Math. **81** (1876), pp. 243. Systems of characteristic curves also exist for certain more general differential systems that are linear in the quantities $p_i, q_i, p_i\, q_k - p_k\, q_i$, and define an analogous of the *Ampère* and *Natani* equations (nos. **43** and **48**). A system of that type can possess one (and only one) general integral of the form:

$$\varphi_1\,(u_1, v_1) = 0\,, \quad \varphi_2\,(u_2, v_2) = 0\,, \quad \ldots, \quad \varphi_n\,(u_n, v_n) = 0\,,$$

in which the φ_i mean arbitrary functions, and the u_i, v_i mean functions of $x, y, z_1, \ldots, z_n$; cf., *Hamburger*, *loc. cit.* The system of two linear first-order differential equations with two unknowns was investigated by *L. Königsberger*, Math. Ann. **41**, pp. 260.

([306]) See also *Lie*, Christ. Forh. (1880), pp. 1; Leipziger Ber. (1895), pp. 85.

([307]) Footnote 52, For a generalization, see *Goursat B* 2, pp. 292.

([308]) For that relationship between two second-order equations, cf., *Goursat B* 2, art. 194, *et seq*.

([309]) *Bäcklund*, *loc. cit*.; cf., *G. Darboux*, *Surfaces* 3, pp. 438; *Goursat B*, art. 202, *et seq*.

53. The Laplace method and its generalizations. – If one introduces the integrals of the differential equations $dy = \Lambda_i\, dx$ (i = 1, 2, cf., no. **46**) into the linear second-order partial differentia equation:

$$R\,r + S\,s + T\,t + P\,p + Q\,q + Z\,z = 0 \qquad (R, \dots, Z \text{ are functions of } x, y) \tag{181}$$

as independent variables then it will take on the form:

$$\frac{\partial^2 z}{\partial x\,\partial y} + a\frac{\partial z}{\partial x} + b\,\frac{\partial z}{\partial y} + c\,z = 0\,. \tag{A}$$

Two equations of the form (A) are called equivalent ([310]) when one of them can be converted into the other one by substituting an expression $z' = \rho(x, y) \cdot z$. In order for that to be true, it is necessary and sufficient that the *invariants* ([310]):

$$h \equiv \frac{\partial a}{\partial x} + a\,b - c\,, \qquad k \equiv \frac{\partial b}{\partial y} + a\,b - c\,,$$

are the same for the two equations, resp. The equation that is adjoint to (A) ([311]):

$$\frac{\partial^2 z}{\partial x\,\partial y} - a\frac{\partial z}{\partial x} - b\,\frac{\partial z}{\partial x} + \left(c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) z = 0 \tag{A$'$}$$

has the invariants h, k. (A) is equivalent to (A′) for $h \equiv k$, and can take the form $s = h\,z$.

The application of the *Darboux-Lévy* theory to (A) leads to the *cascade method* that *P. S. Laplace* ([312]) gave before. By means of the formulas:

$$z_1 = \frac{\partial z}{\partial y} + a\,z, \tag{182}$$

$$z_{-1} = \frac{\partial z}{\partial x} + b\,z\,, \tag{183}$$

after eliminating z from (A) and (182) [(A) and (183), resp.], one will get an equation (A_1), (A_{-1}) for z_1 (z_{-1}, resp.) that has the same form ([313]) as (A) and whose invariants h_1, k_1 (h_{-1}, k_{-1}, resp.) depend upon h, k in a simple way. Repeating the process will generally produce a two-sided infinite series of equations:

$$\dots, (A_{-2}), (A_{-1}), (A), (A_1), (A_2), \dots \tag{184}$$

([310]) *Darboux*, *Surfaces* 2, pp. 23, *et seq.*
([311]) *Darboux*, *loc. cit.*, pp. 71, *et seq.*
([312]) Paris Hist. (1773), pp. 341; see, e.g., *Darboux*, *loc. cit.*, pp. 23. *Goursat B* 2, Chap. 5.
([313]) *Goursat B* 2, art. 187-193, considered the most-general equation (153), which will once more yield a second-order equation for z_1 under the substitution $z_1 = q$. The *Laplace* method seems to be a special case of that theory.

with the invariants …, h_{-2}, k_{-2} ; h_{-1} , k_{-1} ; h, k ; h_1 , k_1 ; …, resp. The application of the same method to any equation (184) or an equation that is equivalent to it will give an equivalent series. A series …, (A'_{-1}), (A′), (A'_1), …, in which (A'_{-i}) is adjoint to (A_i), will likewise follow from (A′). The general integral of any equation (184) will imply those of all remaining ones by quadratures and differentiations. The series (184) will truncate on the right with (A_ν) if and only if $h_\nu \equiv 0$, so the series that belongs to (A′) will truncate on the left with $(\mathrm{A}'_{-\nu})$, and (A) will possess a general intermediate integral ([314]):

$$a_0(x, y)\cdot z + \sum_{i=1}^{\nu+1} a_i(x, y)\frac{\partial^i z}{\partial y^i} = \varphi(y) \qquad (\varphi \text{ arbitrary}),$$

as well as a general integral that is obtained by quadratures:

$$z = \xi_0 \cdot X + \xi_1 \cdot X' + \cdots + \xi_\nu \cdot X^{(\nu)} + B,$$

where X means an arbitrary function of x, and the ξ_i are well-defined functions of x, y, while an arbitrary function Y of y will generally enter into B by partial quadratures. Conversely, if the general integral of (A) has that form then the series (184) will truncate with (A_ν) at the latest. In order for that to be true, it is necessary and sufficient ([315]) that a system of $\nu + 1$ particular solutions must exist whose coefficients depend upon only x, without all being constant.

Analogous statements will be true when the symbols that refer to x are switched with the ones that refer to y in the foregoing.

h_ν, $k_{-\mu}$ (so also $\mu = \nu$ in the case of $h \equiv k$) will vanish if and only if the series (184) truncates on the right with (A_ν) and on the left with ($\mathrm{A}_{-\mu}$), and (A) will have a general integral:

$$z = a_0\, X + a_1\, X' + \cdots + a_\nu\, X^{(\nu)} + b_0\, Y + b_1\, Y' + \cdots + b_\mu\, Y^{(\mu)},$$

so it will belong to *Ampére*'s first class (no. **4**), just like (A′).

Following *Moutard* ([316]) and *Darboux* ([317]), for any pair of numbers μ, ν, one can represent the most general equation (A) of that type explicitly, as well as its general integral and the associated *Laplace* series. In particular, for $h \equiv k$, one can derive it by a recursion process ([318]) from the simplest type $\dfrac{\partial^2 z}{\partial x\, \partial y} = 0$.

If equation (153) can be integrated by the *Darboux-Lévy* method then one will get the general integral of the auxiliary equation (no. **5**, footnote 33) by means of the *Laplace* method ([319]).

([314]) *Goursat B* 2, art. 168.

([315]) *Goursat B*, art. 109-112; C. R. Acad. Sci. Paris **122** (1896), pp. 169.

([316]) In an article that has since been lost, see. C. R. Acad. Sci. Paris **70**, pp. 834; *J. Bertrand*, *ibidem*, (1870), pp. 1068.

([317]) *Surfaces* 2, pp. 46 and 122; cf., *O. Nicoletti*, Rend. Lincei (5) **6** (1897), pp. 307.

([318]) *Ibidem*, pp. 157; *Moutard*, J. éc. polyt. **28**, cah. 28 (1878), pp. 1.

([319]) *Goursat B* 2, note 1.

The *Laplace* method can be adapted to equations (A) whose right-hand sides consist of an arbitrary function of x, y, and according to *A. M. Legendre* ([320]), they can be adapted directly to equations of the form (181) ([321]).

If ζ is a particular integral of (A), and z is the general one, then the function $\zeta\frac{\partial z}{\partial y} - z\frac{\partial \zeta}{\partial y}$ will again satisfy a second-order equation of the form (A) whose integration will come down to that of (A) ([322]). Repeating it and the *Laplace* transformation will lead to the most-general expression:

$$Az + \sum_{i=1}^{m} B_i \frac{\partial^i z}{\partial x^i} + \sum_{i=1}^{n} C_i \frac{\partial^i z}{\partial y^i} \qquad (A, B_i, C_i \text{ are functions of } x, y)$$

that represents the general integral of a linear second-order equation of the same form as a result of (A) ([323]). That, and other, methods ([324]) for deriving a new equation of the same form from an equation (A), whose integration will revert to that of (A), will imply relations between two second-order partial differential equations of the type that *Bäcklund* studied (no. **52**, footnote 308 and 52) as special cases.

A second-order partial differential equation that possesses a general integral of the form:

$$z = f(x, y, X, X', \ldots, X^{(k)}, Y, Y', \ldots, Y^{(l)})$$

can (except for trivial cases) be brought into the form (A), or to the equation $s = a\,e^{bx}$ (a, b constant) that *J. Liouville* integrated ([325]), or finally into the form:

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}[M(x, y)e^z] - \frac{\partial}{\partial y}[N(x, y)e^z] \,,$$

by introducing new variables ([326]), where the coefficients M, N are subject to certain conditions, and whose integration comes down to that of a *Laplace* equation (A) ([327]).

([320]) Paris Hist. (1787), pp. 319; cf., *Imschenetsky*, Arch. Math. **54**, art. 67-74.

([321]) According to *Darboux* (*Surfaces* 4, pp. 267), it can also be adapted to certain systems in involution of linear second-order equations with several independent variables.

([322]) *L. Lévy*, J. éc. polyt. **56** (1886), pp. 63; for a geometric interpretation, see *Darboux*, *Surfaces* 2, pp. 219. The *Laplace* transformation is included in that as a limiting case, *Darboux*, *loc. cit.*, pp. 177.

([323]) *Darboux*, *Surfaces* 2, pp. 164, *et seq.*

([324]) *Loc. cit.*, pp. 179, *et seq.*; *Goursat B*, art., 194, *et seq.*; *O. Nicoletti*, Ann. Pisa (1897); Tor. Atti **32** (1897).

([325]) J. de math. (1) **18** (1853), pp. 71. It admits an infinite group of point-transformations and is the only equation of the form $s = f(z)$ that is *Darboux-Lévy*-integrable; *Lie*, Norw. Arch. **6** (1881), pp. 112.

([326]) *Moutard* (footnote 316); *H. W. L. Tanner*, Mess. Math. **5** (1876); *É. Cosserat* in *Darboux*, *Surfaces* 4, pp. 405.

([327]) Cf., *Goursat B* 2, pp. 250.

54. Applying the concept of group to differential equations. – All transformations of the infinite group of point-transformations:

$$z' = \rho(x, y) \cdot z, \qquad x' = \xi(x, y), \qquad y' = \eta(x, y), \tag{185}$$

and only them, will take *any* linear partial differential equation (181) into another such thing again ([328]). The question of whether two equations of the form (181) can be transformed into each other by a transformation (185) was resolved by *Cotton* ([329]) and *Burgatti* ([330]) by considering two invariants h, k that were analogous to the *Darboux* invariants. In all cases, the question of whether two differential problems can or cannot be converted into each other by a transformation of a given group is decided by comparing a finite number of differential invariants ([331]) that they possess under the group. That is the basis for, e.g., a classification ([332]) of all differential system in three variables x, y, z under the group of all point transformation or all contact transformations of the R_3.

Not every system of partial differential equations with the unknown z and the independent variables $x_1, x_2, \ldots, x_m$ admits an infinitesimal contact transformation of the space $R_{m+1} = (z, x_1, x_2, \ldots, x_m)$ ([333]). The existence of such a transformation is generally connected with the existence of certain relations ([334]) between the associated differential invariants. Any known infinitesimal point or contact transformation of the R_{m+1} that takes a differential system to itself will yield a first-order partial differential equation that has invariant integral structures in common with the system under the infinitesimal transformation ([335]). The determination of the latter comes down to the integration of a differential system with fewer than m independent variables. Similar statements are true for a differential system with several known infinitesimal transformations that commute with each other, or even define a group ([336]). *Lie* ([335]) had developed analogous theories for first-order differential systems with several unknowns. For differential problems with known infinitesimal transformations into themselves, the existence of certain types of particular integrals can be established *a priori* ([335]) in many cases.

If an n^{th}-order partial differential equation with three variables x, y, z admits an infinite group of contact transformations of $R_3 = (x, y, z)$ ([337]) then one can use the differential invariants of the group to exhibit other differential systems that define a *Darboux* system with the given equation.

([328]) Analogous statements are true for linear partial differential equations of each order and number of independent variables; *P. Stäckel*, J. f. Math. **114** (1895), pp. 116; cf., *Lie*, Leipziger Ber. (1894), pp. 322.

([329]) C. R. Acad. Sci. Paris **123** (1896), pp. 936.

([330]) Rend. Lincei (5) 5^2 (1896), pp. 433.

([331]) *Lie*, Math. Ann. **24** (1884), pp. 537; *A. Tresse*, Acta math. **18** (1894), pp. 1.

([332]) *Lie*, *loc. cit.*, pp. 572.

([333]) *A. V. Bäcklund*, Math. Ann. **15**, pp. 63.

([334]) *Lie*, Math. Ann. **24**, pp. 578.

([335]) *Lie*, Leipziger Ber. (1895), pp. 90-112.

([336]) Second-order *Laplace* equations with infinitesimal transformation were classified and integrated by *Lie*, Norw. Arch. **6** (1881), pp. 328.

([337]) *J. Beudon*, C. R. Acad. Sci. Paris **118** (1894), pp. 1188; *Goursat B*, art. 173.

In particular, in the case of $n = 2$ ([338]), a second-order equation can be integrated by the *Darboux-Lévy* method ([339]).

Far-reaching integration theories ([340]) can be obtained for differential systems with infinite groups when one introduces a complete system of differential invariants of the latter as new variables. Thus, the integration of a *Darboux* system of class k (no. **50**) that admits an infinite group of contact transformations that depend upon l arbitrary functions can, under certain assumptions, reduce to the integration of ordinary differential equations and a *Darboux* system of class $k - l$. Another example comes from the differential problems with a system of *fundamental solutions* ([341]) $\zeta_1, \ldots, \zeta_n$ by means of which the general integral $z_1, \ldots, z_n$ will be obtained from relations of the form $z_i = \varphi_i(\zeta_1, \ldots, \zeta_n)$ that represent a transformation group.

2) *Differential systems with m independent variables*.

55. Characteristics of an n^{th}-order partial differential equation. – An $m - 1$-fold extended manifold of elements will be defined on an integral surface $z = f(x_1, x_2, \ldots, x_m)$ for an n^{th}-order partial differential equation:

$$F(x_1, \ldots, x_m, z, z_{1,0,\ldots,0}, \ldots, z_{\beta_1 \cdots \beta_m}, \ldots) = 0 \qquad (\textstyle\sum \beta_i \leq n \text{ ; no. } \mathbf{1}) \tag{186}$$

by adding a relation $x_1 = \varphi(x_2, \ldots, x_m)$. The integral surface will then be established uniquely when the latter is given. However, φ satisfies the first-order partial differential equation that is obtained setting the *characteristic form:*

$$\sum \frac{\partial F}{\partial z_{\alpha_1 \alpha_2 \cdots \alpha_m}} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_m^{\alpha_m} \qquad (\alpha_1 + \ldots + \alpha_m = n) \tag{187}$$

equal to zero, once one has represented z in it and its derivatives as functions of the x by means of $z = f$ and has replaced $\xi_1, \ldots, \xi_m$ with $-1, \frac{\partial \varphi}{\partial x_2}, \ldots, \frac{\partial \varphi}{\partial x_m}$, resp., along with x_1 with φ, if and only if the corresponding element-M_{m-1} is common to infinitely-many integral surfaces of (186), since all of its associated surface elements of order $n + 1$ fulfill the first m derived equations of (186). It will be called ([342]) an $m-1$-*dimensional characteristic* of the partial differential equation (186) and denoted by C_{m-1} .

([338]) *P. Medolaghi* classified the second-order partial differential equations that admit infinite groups of point-transformations in Ann. di mat. (3) **1** (1898), pp. 229-263.

([339]) The converse of this theorem is not true, *Goursat B*, 2, pp. 196.

([340]) *Lie*, Leipziger Ber. (1895), pp. 122.

([341]) *Ibidem*, pp. 282; *J. Drach*, C. R. Acad. Sci. Paris **116** (1893), pp. 1041.

([342]) *A. V. Bäcklund*, Math. Ann. **13** (1878), pp. 411; cf., also *J. Beudon*, C. R. Acad. Sci. Paris **124** (1897), pp. 671; *É. Goursat*, *ibid.* **126** (1898), pp. 1332, but the priority of *Bäcklund*'s presentation should be expressly emphasized.

For special equations (186), it can happen ([343]) that every C_{m-1} is composed of ∞^1 $m-2$-*dimensional characteristic manifolds* C_{m-2} whose totality is then defined by a system of first-order partial differential equations with $m-2$ independent variables. An even-more-specialized class of equations (186) will be obtained when each C_{m-2} can be generated by ∞^1 characteristics C_{m-3}, etc. Meanwhile, one knows of no example of an equation (186) whose most general integral surface can be composed of $\infty^{m-\nu}$ characteristics C_ν ($m-1>\nu>1$), but not $\infty^{m-\nu+1}$ characteristics $C_{\nu-1}$ ([344]).

If the characteristic form (187) decomposes into n linear factors ([345]), so the partial differential equations S that belongs to an arbitrary integral surface (cf., *supra*) will decompose into n linear partial differential equations, then the "characteristic curves" of each of the latter will determine just as many systems of n^{th}-order *strips* on the integral surfaces that are called *one-dimensional characteristics* or *characteristic strips* of (186) and shall be denoted by C_1. The characteristics C_1 are defined by n systems of linear total differential equations in the variables x_i, z, $z_{\alpha_1\cdots\alpha_m}$ $\left(\sum \alpha_i \le n\right)$, independently of the integral surface considered. The number of equations in such a system is m smaller than the number of differentials dx_i, dz, $dz_{\alpha_1\cdots\alpha_m}$ $\left(\sum \alpha_i \le n\right)$. Any non-singular integral surface of (186) consists of an $m-1$-fold infinitude of C_1 from each of the n characteristic systems. The most-general C_ν ($\nu = 2, 3, \ldots, m-1$) is generated by $\infty^{\nu-1}$ characteristics. For equations (186) of that type, characteristic strips of each arbitrary order $n+k$ can be defined by systems of total differential equations ([346]). As in the case of $m = 2$, the possible integrable combinations of those systems lead to partial differential equations that are in involution with (186). However, a corresponding adaptation of the *Darboux-Lévy* method has not been carried out up to now ([347]).

56. Systems in involution with *one* unknown. – A system of μ ($\le m$) partial differential equations (186) is involutory ([348]) if and only if $\mu\, m$ of the $(n+1)^{\text{th}}$-order equations that follow from them by derivation reduce to only $\mu\, m - \frac{1}{2}\mu\,(\mu-1)$ independent equations. The m characteristic forms then have a common factor of degree $n-1$. Accordingly, there is a system of common characteristics C_{m-1} on any common integral surface that is defined by a first-order partial differential equation of degree $n-1$ in the derivatives; in particular, there are ∞^{m-1} common one-dimensional characteristics in the case of $n = 2$. A system in involution of m second-order equations

([343]) *Bäcklund*, *loc. cit.*

([344]) The example of a second-order equation that was given by *Bäcklund* (Math. Ann. **15**, pp. 83, *et seq.*) has the stated property only in regard to a particular family of integrals, cf., Math. Ann. **17**, pp. 326.

([345]) See *Bäcklund*, *loc. cit.*; *V. Sersawy*, Wiener Denkschr. 49^2, pp. 60, *et seq.*, esp. pp. 81.

([346]) *A. R. Forsyth*, Trans. London Math. Soc. A (1898), pp. 1 gave the defining equations of the characteristic strips of orders 2 and 3 for the case of $n = 2$, $m = 3$.

([347]) Cf., the Ansätze of *Sersawy*, *loc. cit.* (with a method and result that are not free from objections) and the suggestions of *Forsyth*, *loc. cit.*

([348]) *Bäcklund*, Math. Ann. **13**, pp. 104-107, 423, 427; Math. Ann. **15**, pp. 78.

in m independent variables possesses ([349]) a family of second-order characteristics that depends upon a finite number of constants, and from which all integral surfaces of the system can be generated. The integration of the latter then comes down to exhibiting the most-general integral-M_{m-1} and integrating a system of ordinary differential equations. If k second-order equations define a $k + 1$-parameter system in involution, along with each of the second-order partial differential equations: $u_1 = c_1, \ldots, u_{m-k+1} = c_{m-k+1}$, then they will define such a thing with all equations of the form $\varphi(u_1, \ldots, u_{m-k+1}) = 0$.

J. Beudon ([350]) considered involutory equations of the form (186) whose number is smaller by $\nu\,(< m)$ than the number of n^{th} derivatives of z. Such a system possesses a finite-parameter family of characteristics C_{m-1} such that *one* of those $C_{m-\nu}$ is established by each common surface element of order n of the given equations. The ∞^ν characteristics $C_{m-\nu}$, resp., that go through the surface elements of an integral-M_ν of the system generate the most-general integral-M_m. The system in involution of first-order partial differential equations (no. **38**) and the *Darboux* system of class one (no. 50) are special cases of that theory.

If two equations (183), without being involutory, possess enough common integral surfaces that each of their common surface elements of order $n + 1$ belong to *at least one* integral surface ([351]), i.e., the given equations and their $2m$ derived equations collectively define an unrestricted-integrable system in the *Lie* sense (footnote 41), then the two first-order partial differential equation that define the characteristics C_{m-1} of the given equation on a common integral surface, resp., will be involutory. Every common integral surface will then be generated by a simple infinitude of *common* characteristics C_{m-1} of the given second-order equations in the case of $m = 2$ and ∞^∞ of them in the case of $m > 3$.

57. Generalization of the Monge-Ampère theory ([352]) **(nos. 43-45).** – In order for a second-order partial differential equation:

$$F(x_1, \ldots, x_m, z, p_1, \ldots, p_m, r_{11}, r_{12}, \ldots, r_{mm}) = 0 \qquad \left(r_{ik} = \frac{\partial^2 z}{\partial x_i\, \partial x_k}\right) \tag{188}$$

to possess a general integral of the form:

$$\varphi(u_1, u_2, \ldots, u_m) = 0\,, \tag{189}$$

([349]) *Bäcklund*, Math. Ann. **13**, pp. 107. *J. Beudon* determined and integrated the system of that type that is *linear* in the second derivatives in Ann. éc. norm. sup. (1898), pp. 229. The results in question can be adapted to m-parameter systems in involution of linear equations of *arbitrary* order with no further discussion.

([350]) Ann. éc. norm. sup. (1896), Suppl. = Thesis, Paris, 1896. In J. de math. (5) **5** (1899), pp. 351, *Beudon* considered those differential systems with *one* unknown whose general integral included *one* function of ρ ($< m$) variables. Cf., the special cases that were treated by *Bäcklund*, Math. Ann. **19**, pp. 410, *et seq*.

([351]) *Bäcklund*, Math. Ann. **15**, pp. 69-74.

([352]) *Bäcklund*, Math. Ann. **11**, pp. 236; **13**, pp. 99, *et seq.*; *H. W. L. Tanner*, Proc. London Math. Soc. **7** (1875-76), pp. 43, 75; *G. Vivanti*, Rend. Lombardo (2) **29** (1896), pp. 777; *ibidem*, **32** (1899); *A. R. Forsyth*, Trans. Camb. Phil. Soc. 16[2] (1898), pp. 191; *É. Goursat*, Bull. soc. math. **27** (1899).

in which φ means an arbitrary function, and the u_i are functions of $x_1, \ldots, x_m, z, p_1, \ldots, p_m$, its left-hand side must be a linear function of the determinants $| r_{ik} |$ and its minors of order $m - 1, m - 2, \ldots, 1$. Moreover, their coefficients must satisfy certain relations that say that equation (188) must possess two different systems of *first-order characteristic strips* that are defined by two systems Σ_1, Σ_2, resp., of $m + 1$ *Pfaff* equations in x_k, z, p_i. Finally, Σ_1 must admit m integrable combinations $du_1, \ldots, du_m$. One can always arrange that Σ_1 has the form:

$$dz = \sum_{h=1}^{m} p_h \, dx_h \, ; \qquad dp_i + \sum_{h=1}^{m} \alpha_{ih} \, dx_h = 0$$

$$(i = 1, \ldots, m \, ; \; \alpha_{ik} \text{ are functions of the } x, z, p)$$

by a contact transformation. Σ_2 will arise from that when one switches α_{ik} and α_{ki} everywhere, and (188) has the form ([353]):

$$| r_{ik} + \alpha_{ik} | = 0 \qquad (i, k = 1, 2, \ldots, m).$$

If Σ_2 also admits m integrable combinations dv_i then equation (188) will possess a second general intermediate integral:

$$\psi(v_1, v_2, \ldots, v_m) = 0 \,. \tag{190}$$

Any first-order partial differential equation of that form is then in involution with any equation (189). If the system is unrestricted-integrable, i.e., it admits $m + 1$ integrable combinations of $du_1, \ldots, du_{m+1}$, then it will be identical to Σ_2. Conversely, if that is the case and there are m integrable combinations du_i then there will always exist an $(m + 1)^{\text{th}}$ one ([354]). The equations $u_1 = c_1, \ldots, u_{m+1} = c_{m+1}$ then represent ∞^{m+1} manifolds M_m of first-order surface elements, and the most-general integral-M_m of (188) is the enveloping structure of any ∞^1 of them ([355]).

A third-order partial differential equation with m independent variables and one first integral (189) ([356]), in which the u_i mean functions of x_i, z, p_i, r_{ik}, has a particular form that raises the degree of the third derivatives by m. If an equation of that form possesses a second intermediate integral (190) then every equation (190) will define a second-order system in involution with any equation (189), and there will be three different systems of second-order characteristic strips, such that every integral-M_m of the given third-order equation will include ∞^{m-1} strips from each of the three systems. Two of those systems are defined by linear total differential equations that admit the integrable combinations du_i and dv_i, resp. Analogous statements are true for equations of arbitrary order with intermediate integrals. In particular, if the characteristic form of an n^{th}-order equation

([353]) *Goursat, loc. cit.*
([354]) For the case of $m = 3$, cf., *G. Vivanti*, Math. Ann. **48**, pp. 474.
([355]) *Vivanti, loc. cit.*; *J. Kurschak* (footnote 256).
([356]) *Bäcklund*, Math. Ann. **13**, pp. 104.

that is linear in its highest derivatives decomposes into n linear factors ([357]) then there will be just as many system of $(n-1)^{\text{th}}$-order characteristic strips that are defined by *Pfaff* equations, and there can be, correspondingly,1, 2, 3, …, and even n itself, different general integrals that appear. Any k integrals from different systems will define an $(n-1)^{\text{th}}$-order system in involution.

Bäcklund ([358]) had also considered equations of the form (188) that admit a first integral with m arbitrary constants (for $n = 2$, $n = 2$, cf., no. **45**).

58. First-order linear differential systems with *n* unknowns. – *M. Hamburger* ([359]) gave a theory of integration for the differential system:

$$\sum_{k=1}^{n}\sum_{s=1}^{m} P_{iks}\, p_{ks} = Q_i \tag{191}$$

$$\left(i = 1, \dots, n;\ p_k = \frac{\partial z_k}{\partial x_s};\ P_{iks}, Q_i \text{ are functions of } x_1, \dots, x_m, z_1, \dots, z_n\right)$$

with the unknowns $z_1, \dots, z_n$, and the independent variables $x_1, \dots, x_m$, under the assumption that the functions P_{iks} are subject to certain algebraic conditions in the case of $m > 2$ that say, *inter alia*, that the determinants:

$$\left|\sum_{s=1}^{m} P_{iks}\,\lambda_s\right| \qquad (i, k = 1, \dots, n) \tag{192}$$

decompose into n factors that are linear in $\lambda_1, \dots, \lambda_m$. Those factors then correspond to just as many systems Σ_i of $m + 1$ *Pfaff* equations in the x, z. A one-dimensional point-manifold in space R_{m+n} $(x_1, \dots, x_m, z_1, \dots, z_n)$ that satisfies such a system Σ_i can be referred to as a "characteristic curve" of the differential system (191). Any integral structure:

$$z_1 = \zeta_1 (x_1, \dots, x_m), \qquad \dots, \qquad z_n = \zeta_n (x_1, \dots, x_m)$$

includes ∞^{m-1} characteristic curves from each of the n systems. If every system Σ_i possesses m integrable combinations $du_{i1}, du_{i2}, \dots, du_{im}$ then one will get the general integral of (191) by solving the equations:

$$\varphi_i (u_{i1}, u_{i2}, \dots, u_{im}) = 0 \qquad (i = 1, \dots, n\ ;\ \varphi_1, \dots, \varphi_n \text{ are arbitrary functions}) \tag{193}$$

for $z_1, \dots, z_n$. The method is also applicable to the case of multiply-counted linear factors of (192), assuming that each k-fold factor appears $k - l$ times in all $n - l$-rowed subdeterminants of (192). In

([357]) That case was considered already by *Monge*, Paris Hist. (1784), pp. 161; cf., *A. M. Legendre*, *ibidem* (1787), pp. 323; *M. Falk*, Tidskr. f. Mat. Upsala **4** (1871); *E. Combescure*, C. R. Acad. Sci. Paris **74** (1872), pp. 798.

([358]) Math. Ann. **11**, pp. 240.

([359]) J. f. Math. **100** (1887), pp. 401, *et seq.* For the case of $m = 1$, cf., footnote 305.

particular, if $k = n$ under that assumption then the system (191) will have the form that *Jacobi* ([63]) considered:

$$P_1 \frac{\partial z_i}{\partial x_1} + \cdots + P_m \frac{\partial z_i}{\partial x_m} = Q_i \qquad (i = 1, \ldots, m),$$

and from no. **11**, its integration comes down to that of a single system of ordinary differential equations ([360]).

In order for n first-order equations in n unknowns z_i to possess a general integral (193) ([359]), their left-hand sides must have a particular form that raises the degree of the p_{ik} by at most m, and its coefficients must satisfy certain identities. The concept of "characteristic curve" and the integration method above can be adapted to that class of differential systems with no further analysis.

59. Nonlinear first-order differential systems with *n* unknowns. Normal systems. – The system of N ($< m \cdot n$) involutory first-order equations with n unknowns $z_1, \ldots, z_n$ and m independent variables $x_1, \ldots, x_m$ that are independent with respect to the p_{ik} :

$$\text{(J)} \qquad F_i (x_1, \ldots, x_m, z_1, \ldots, z_m, p_{11}, \ldots, p_{n,m}) = 0 \qquad \left(i = 1, 2, \ldots, N;\ p_{ik} = \frac{\partial z_i}{\partial x_k} \right)$$

has been more closely examined ([361]) for the case in which it can be solved in the form ([362]):

$$\text{(194)} \qquad p_{ik} = f_{ik} (x_1, \ldots, x_m, z_1, \ldots, z_m, p_{rs}, \ldots)$$

$$(k = 1, \ldots, \mu;\ i = 1, \ldots, n \text{ and } k = \mu + 1,\ i = 1, 2, \ldots, \nu),$$

in which one sets $N = \mu n + \nu$ ($\mu < m,\ \nu < n$). The conditions of passivity (no. **2**) of (194) have, *inter alia*, the consequence that the equations that are obtained by setting all n-rowed determinants in the *characteristic matrix* ([363]):

$$\left\| \sum_{s=1}^{m} P_{iks} \lambda_s \right\| \qquad \left(i = 1, 2, \ldots, N;\ p_{ik} = \frac{\partial z_i}{\partial x_k} \right)$$

([360]) Analogous statements are true for one of the classes of nonlinear first-order systems in n unknowns that *Hamburger* considered, J. f. Math. **110**, pp. 167-173.

([361]) See my articles: C. R. Acad. Sci. Paris **123** (1892), pp. 292; Leipziger Ber. (1897), pp. 329; Math. Ann. **49**, pp. 543.

([362]) The existence of solutions to a passive system of the form (194) has been investigated in the case of $\nu = 0$ by *J. König* (Math. Ann. **23**, pp. 520), and in general, by *C. Bourlet* [Ann. éc. norm. sup. (1891), Suppl., pp. 43].

([363]) That concept can be adapted to any arbitrary differential system; cf., *De Pistoye*, C. R. Acad. Sci. Paris **78** (1874), pp. 1102.

equal to zero represents an $m - \mu - 1$-dimensional point-manifold M of degree $n - \nu$ in the space R_{m-1} with the homogeneous point-coordinates $\lambda_1, \ldots, \lambda_m$. Each of the μ *subsystems:*

$$(\mathrm{J}_s) \qquad p_{is} = f_{is} \,; \qquad p_{j,m+1} = f_{j,m+1} \qquad (i = 1, \ldots, n \,;\, j = 1, \ldots, \nu)$$

defines a system in involution by itself when $x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_m$ are parameters. The system (J) is called a *normal system* when the P_{iks} satisfy certain algebraic condition that have the consequence, *inter alia*, that M will decompose into $n - \nu$ linear point-manifolds. Any system (J_s) then likewise defines a normal system, and (J) generally possesses $n - \nu$ different systems of *characteristics*, which are μ-fold-extended integral manifolds C_μ that are defined by linear first-order partial differential equations with the independent variables $x_1, \ldots, x_\mu$ and the dependent variables $x_{\mu+1}, \ldots, x_m, p_{ik}$ in the case $\mu = 1$, so by $n - \nu$ systems Σ_i *Pfaff* equations in x_k, z, p_{ik} . Any integral structure of the normal system (J) is generated by $\infty^{m-\mu}$ characteristics C_μ of each of the $n - \nu$ systems. The individual C_μ include $\infty^{\mu-1}$ characteristics C_1 from each subsystem (J_s), and the systems (J_1), (J_2), …, (J_μ) will then have a relationship to each other that is similar to how the equations of an m-parameter system in involution of first-order partial differential equations with *one* unknown relate to each other.

In the case of $\nu = n - 1$, *only one* C_μ goes through each surface element of the normal system (J), and the integration of (J) will be completed by means of ordinary differential equations once the most-general integral-$M_{m-\mu}$ has first been ascertained by integrating a differential system in $m - \mu$ independent variables. The integration of (194) will be reduced to the case of $\mu = 1$ with the help of the *Lie-Mayer transformation* (no. **17**) ([364]). If the $u - \nu$ *Pfaff* systems Σ_i possess a sufficient number of integrable combinations in the latter case then the integration of (J) can be reduced to a problem in fewer than m independent variables, and possibly to systems of ordinary differential equations.

Most of the theories of integrating partial differential problems, in particular, the *Darboux-Lévy* theory and its generalizations (no. **51**), as a well as the *Beudon* system (no. **56**, footnote 350), are included in the theory of normal systems as special cases.

60. Systems of Pfaff equations. – The adjoint family of infinitesimal transformations (no. **14**) that is invariantly linked with the system of *Pfaff* equations:

$$(\mathrm{S}) \qquad dx_{m+s} = a_{1s}\, dx_1 + \ldots + a_{ms}\, dx_m$$

$$(s = 1, \ldots, n - m \,;\, a_{is} \text{ are functions of } x_1, \ldots, x_m)$$

has the form:

$$(195) \qquad \rho_1 A_1 f + \rho_2 A_2 f + \ldots + \rho_m A_m f$$

([364]) *König*, *loc. cit.*, pp. 525; *Bourlet*, *loc. cit.*, pp. 43.

$$\left(A_i f \equiv \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} a_{si} \frac{\partial f}{\partial x_{m+s}} \right).$$

If one writes:

$$a_{iks} \equiv a_{kis} \equiv A_i\, a_{ks} - A_k\, a_{is} ,$$

and if $m - h$ is the rank of the m-rowed matrix:

$$\begin{Vmatrix} 0 & a_{121} & \cdots & a_{1m1} & 0 & a_{122} & \cdots & a_{1m2} & 0 & \cdots \\ a_{211} & 0 & \cdots & a_{2m1} & a_{212} & 0 & \cdots & a_{2m2} & a_{213} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m11} & a_{m21} & \cdots & 0 & a_{m12} & a_{m22} & \cdots & 0 & a_{m13} & \cdots \end{Vmatrix}$$

then, corresponding to the h systems of solutions $\xi_1^{(i)}, \dots, \xi_m^{(i)}$ to the equations:

$$\sum \xi_k\, a_{iks} = 0 ; \qquad \xi_{m+s} = \sum a_{ks}\, \xi_k \qquad (s = 1, \dots, n - m;\ i = 1, \dots, m),$$

there will be h linearly-independent infinitesimal transformations that belong to the family (196):

$$X_i f = \sum \xi_k^{(i)} \frac{\partial f}{\partial x_k} \qquad (i = 1, 2, \dots, h)$$

and leave the system unchanged (no. **14**). The family:

$$\rho_1 X_1 f + \dots + \rho_n X_h f$$

is invariantly coupled with the system (S) ([365]). If one introduces the integrals $y_1, \dots, y_{n-1}$ of any of the partial differential equations $X_i f = 0$ instead of just as many x as new variables in (S) then it will be converted into a system that no longer contains the variables $y_1, \dots, y_{n-1}$ ([366]). The equations $X_1 f = 0, \dots, X_h f = 0$ define a complete system. If one introduces its integrals $y_1, \dots, y_{n-h}$ into (S) then a system will arise that will no longer include the $y_1, \dots, y_{n-h}$. Any further reduction of the number of variables is impossible ([367]).

The system of equations:

([365]) *F. Engel*, Leipziger Ber. (1889), pp. 157.

([366]) *M. Hamburger*, J. f. Math. **110** (182), pp. 158; *W. de Tannenberg*, C. R. Acad. Sci. Paris **120** (1895), pp. 674. The theorem includes the *Pfaff-Grassmann* reduction method for *one Pfaff* equation as a special case (see no. **27**). For the conditions that express the fact that the number that was denoted by h above is > 0, one can also cf., *Grassmann*, *Werke* 1^2, no. 511, and *Engel*'s remarks in it, *ibidem*, pp. 479, *et seq.*.

The (trivial) fact that it makes no sense to wish to adapt the methods that lead to one's goal in the theory of the *Pfaff* problem in a purely formal way to arbitrary *systems* of *Pfaff* equations was result of the investigations of *O. Biermann*, Zeit. Math. Phys. **30** (1885), pp. 234 and *A. R. Forsyth* (*Theory of Diff. Equations* 1, Chap. 13).

([367]) See my article in Leipziger Ber. (1898), pp. 207.

$$dx_{m+s} = \sum a_{ks}\, dx_k\,, \qquad \delta x_{m+s} = \sum a_{ks}\, \delta x_k\,, \qquad \sum\sum a_{iks}\, dx_i\, \delta x_k = 0 \qquad (s = 1, 2, \ldots, n-m)$$

is invariantly coupled with (S) ([368]). Therefore, the ranks K, 2ν that the matrices:

$$(196) \qquad \begin{aligned} &\left\| \sum_{k=1}^{m} a_{iks} \lambda_s \right\| \qquad (i = 1, \ldots, m,\ s = 1, \ldots, n-m), \\ &\left\| \sum_{s=1}^{n-m} a_{iks}\, \mu_s \right\| \qquad (i, k = 1, \ldots, m), \end{aligned}$$

resp., take on for arbitrary x, λ, μ, are *invariants* of the system (S); K is called the *character*, and 2ν is the *rank* of (S) ([367]). If one lets $D_1, D_2, \ldots, D_p$ denote the K-rowed subdeterminants of (196) that do not vanish identically and replaces the λ_k with dx_k in the D_i then the equations:

$$(197) \qquad dx_{m+1} = \sum a_{ks}\, dx_k\,; \quad D_i = 0 \qquad (s = 1, \ldots, n-m\,;\ i = 1, \ldots, p)$$

will define a system of total differential equations that is invariantly coupled with (S) ([369]).

The differential system:

$$(198) \qquad A_i f = 0\,, \qquad A_i (A_k f) - A_k (A_i f) = 0 \qquad (i, k = 1, 2, \ldots, m)$$

is likewise invariantly coupled with (S) ([370]). The same thing will then be true for the *Pfaff* system (S_1) that is adjoint to (198), which is included in the system (S) and might be called the *derived system* to the latter.

In the case of $K = 0$, one also has $2\nu = 0$, while the a_{iks} vanish, so (S) is identical to (S_1) and integrable without restriction.

In the case of $K = 1$, (S_1) consists of $n - m - 1$ equations, so the differential system (198) consists of $m + 1$ independent equations, and indeed the system is *complete*, so (S_2) will be integrable without restriction ([367]) when the rank $2\nu > 2$. (S) can then take the normal form:

$$dz_{2\nu+1} = z_{\nu+1}\, dz_1 + z_{\nu+2}\, dz_2 + \ldots + z_{2\nu}\, dz_\nu\,; \qquad dz_{2\nu+2} = 0\,, \ldots, dz_{2\nu+n-m} = 0\,,$$

in which the functions $z_1, \ldots, z_{2\nu+n-m}$ are independent, so it will possess no other invariants besides the numbers 2ν and $n - m$.

By contrast, if $K = 1$, $2\nu = 2$ then (S) will possess a reduced form:

([368]) *F. Engel*, Leipziger Ber. (1890), pp. 192.
([369]) *Ibidem*, pp. 197.
([370]) In regard to this and other invariant constructions and their geometric interpretation, cf., the two cited works by *Engel*.

$$dy_2 = y_{n-m+2}\, dy_1\, ; \qquad dy_{2+s} = \varphi\,(y_1, y_2, \ldots, y_{n-m+2})\, dy_1 \qquad (s = 1, 2, \ldots, n-m-1).$$

(S_2) is the derived system of (S_1), (S_3) is that of (S_2), etc., and all of those systems possess a character of 1 [i.e., they consist of $n - m - 2$, $m - m - 3$, …, resp., equations until one arrives at a system ($S_{n-m-r-1}$) whose derived system is integrable without restriction] if and only if (S) possesses a normal form ([367]):

$$dz_2 = 0\,, \quad dz_3 = 0\,, \quad \ldots, \quad dz_{r+1} = 0\,,$$
$$dz_{r+2} = z_{r+3}\, dz_1\,, \quad dz_{r+3} = z_{r+4}\, dz_1\,, \ldots, dz_{n-m+1} = z_{n-m+2}\, dz_1\,,$$

in which the z_i are independent functions of the x, and there are no further invariants of (S) besides the numbers r and $n - m$. In particular, each two-term *Pfaff* system in four variables that possesses no integrable combination can be brought into the normal form ([365]):

$$dz_2 = z_3\, dz_1\, ; \qquad dz_3 = z_4\, dz_1\, .$$

The question of the complete system of invariants of (S) in the general case of $K = 1$, $2\nu = 2$, as well as for the cases $K > 1$, still awaits resolution ([371]).

A *classification of differential systems* [the systems of *Pfaff* equations (no. **8**) that are equivalent to them, resp.] is based upon the number K. The case of $K= 0$ subsumes, e.g., all *Mayer* systems (no. **3**), while the case of $K = 1$, $m = 3$ subsumes all *Darboux* systems of class one (no. **50**) ([372]).

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([371]) *Duport* [Liouville's Jour. (5) **3** (1897), pp. 17] gave reduced forms for the case $n = 6$, $n - m = 2$.

([372]) For the meaning of the bilinear covariants in the theory of differential problems with two independent variables, cf., my articles: Münchener Ber. **25** (1895), pp. 101; *ibidem*, pp. 423.