"Zur Invariantentheorie der Systeme PFAFF'schen Gleichungen," Ber. Verh. Kon. Ges. Wiss. **50** (1898), 207-229.

On the invariant theory of systems of PFAFF equations

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1. – If a system of PFAFF equations:

$$(1) \qquad \nabla_s \equiv dx_{m+s} - \sum_{i=1}^{m} a_{si}\, dx_i = 0 \qquad (s = 1, \ldots, n - m)$$

in the variables $x_1, x_2, \ldots, x_n$ is given, and if $dx_1, dx_2, \ldots, dx_n$ ($\delta x_1, \delta x_2, \ldots, \delta x_n$, resp.) are two independent systems of variations for the x then from a remark of **Engel** ([1]), the system of equations:

$$(2) \qquad dx_{m+s} = \sum_{i=1}^{m} a_{si}\, dx_i\,, \qquad \delta x_{m+s} = \sum_{i=1}^{m} a_{si}\, \delta x_i\,,$$

$$(3) \qquad \sum_{i=1}^{m} (da_{si}\, \delta x_i - \delta a_{si}\, dx_i) = 0 \qquad (s = 1, \ldots, n - m)$$

will be invariantly coupled with the system (1). Equations (3), which we would like to refer to briefly as the *bilinear covariants* of the system (1), will reduce to the form:

$$(4) \qquad \sum_{k=1}^{m} \sum_{i=1}^{m} a_{iks}\, dx_i\, \delta x_k = 0 \qquad (s = 1, \ldots, n - m)$$

by means of (2). In this, one has:

$$(5) \qquad a_{iks} \equiv - a_{kis} \equiv A_i\, a_{sk} - A_k\, a_{si}\,,$$

when one sets:

$$(6) \qquad A_i f \equiv \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} a_{si} \frac{\partial f}{\partial x_{m+s}} \qquad (i = 1, \ldots, m),$$

to abbreviate. The rank ([2]) K that the matrix:

([1]) These Berichte (1890), pp. 207.
([2]) That is, the order of the highest minor that does not vanish identically.

(7) $$\left\| \sum_{k=1}^{m} a_{iks} \lambda_k \right\| \qquad (i = 1, \ldots, m, s = 1, \ldots, n - m)$$

possesses for arbitrary values of the $x_1, \ldots, x_n$, and the quantity λ is consequently an invariant of the system of equations (1). In what follows, we will refer to the number K as the *character* of the system (1). If $K = 0$ – i.e., all a_{iks} vanish – then the system (1) will be integrable without restriction, and when one introduces suitable new variables, it will take on the form:

$$dy_1 = 0, \qquad dy_2 = 0, \qquad \ldots, \qquad dy_{n-m} = 0.$$

The number $n - m$ of equations (1) is therefore the only further invariant of the system (1). The study of the next-higher case $K = 1$ defines the subject of the present article.

2. – We start with a consideration that is valid for any arbitrary **Pfaff** system (1). Because of (6), the most general infinitesimal transformation:

(8) $$X f \equiv \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i}$$

that satisfies the conditions:

(9) $$\xi_{m+s} \equiv \sum_{i=1}^{m} a_{si}\, \xi_i \qquad (s = 1, \ldots, n - m)$$

will have the form:

(10) $$\sum_{i=1}^{m} \xi_i\, A_i f\,,$$

in which the ξ_i mean arbitrary functions of the x. From **Engel** ([1]), the family of infinitesimal transformation is invariantly coupled with the system (1). We would next like to determine those transformations $X f$ of the family (10) that take the system of equations (1) to itself ([2]); i.e., that satisfy the identities:

(11) $$X\, \nabla_s \equiv 0 \qquad (s = 1, \ldots, n - m)$$

by means of (1). Since the left-hand side of (11) can be written in the form:

$$d\xi_{m+s} - \sum_{i=1}^{m} a_{si}\, d\xi_i - \sum_{i=1}^{m} X\, a_{si} \cdot dx_i \equiv d\left(\xi_{m+s} - \sum_{i=1}^{m} a_{si}\, \xi_i\right) + \sum_{i=1}^{m} (da_{si} \cdot \xi_i - X\, a_{si} \cdot dx_i)\,,$$

the conditions (11) will go to the following ones:

([1]) These Berichte (1889), pp. 158, *et seq.*
([2]) *Ibidem*, pp. 160, *et seq.*

(12) $$\sum_{k=1}^{m} \xi_k \, a_{iks} = 0 \qquad (i = 1, \ldots, m \,;\, s = 1, \ldots, n - m)$$

by means of (1) and (9). Hence, $m - h$ is the rank of the matrix:

(13) $$\begin{Vmatrix} 0 & a_{121} & \cdots & a_{1m1} & 0 & a_{122} & \cdots & a_{1m2} & 0 & \cdots \\ a_{211} & 0 & \cdots & a_{2m1} & a_{212} & 0 & \cdots & a_{2m2} & a_{213} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m11} & a_{m21} & \cdots & 0 & a_{m12} & a_{m21} & \cdots & 0 & a_{m13} & \cdots \end{Vmatrix}$$

iff there exist h linearly-independent transformations:

(14) $$X_1 f, X_2 f, \ldots, X_h f$$

of the family (10) that take the system of equations (1) to itself. **Engel** showed that the equations:

(15) $$X_1 f = 0, \qquad X_2 f = 0, \ldots, \qquad X_h f = 0$$

define a complete system ([1]). That assertion can also be verified in this way ([2]): If:

$$X_i f = \sum_{l=1}^{n} \xi_{il} \frac{\partial f}{\partial x_l}, \qquad X_k f = \sum_{l=1}^{n} \xi_{kl} \frac{\partial f}{\partial x_l}$$

are two infinitesimal transformations of the family (10) that take the system (1) to itself then that will also be true for the transformation $(X_i X_k)$. In order to show that $(X_i X_k)$ again belongs to the family:

(16) $$\lambda_1 X_1 f + \lambda_2 X_2 f + \ldots + \lambda_h X_h f ,$$

it will suffice to prove that $(X_i X_k)$ is included in (10). One now has:

$$(X_i X_k) \equiv \left(\sum_{l=1}^{m} \xi_{il} \, A_l f , \sum_{l=1}^{m} \xi_{kl} \, A_l f \right)$$

$$\equiv \sum_{k=1}^{m} \sum_{l=1}^{m} (\xi_{il} A_l \, \xi_{kj} \cdot A_j f - \xi_{kl} A_l \, \xi_{ij} \cdot A_j f) + \sum_{j,l=1}^{m} \xi_{ij} \, \xi_{kl} (A_j A_l)$$

$$\equiv \sum_{j=1}^{m} A_j f \left[\xi_{il} A_l \, \xi_{kj} - \xi_{kl} A_l \, \xi_{ij} \right] + \sum_{j,l=1}^{m} \xi_{ij} \, \xi_{kl} (A_j A_l) .$$

([1]) *Loc. cit.*, pp. 165, Theorem 3.
([2]) Cf., the analogous proof for a **Pfaff** equation by **Engel**, these Berichte (1896), pp. 418.

However, since one has:

$$(A_j\, A_l) \equiv \sum_{s=1}^{n-m} a_{jls} \frac{\partial f}{\partial x_{m+s}} \tag{17}$$

identically, the second term in the expression for $(X_i\, X_k)$ will vanish identically, due to (12), which was to be shown.

If we assume, to fix ideas, that equations (15) are soluble for $\frac{\partial f}{\partial x_{m-h+1}}, \ldots, \frac{\partial f}{\partial x_m}$ then the $n - h$ solutions $y_1, y_2, \ldots, y_{n-h}$ of the system (15) will be mutually-independent with respect to $x_1, \ldots, x_{m-h}, x_{m+1}, \ldots, x_n$. If we write:

$$y_{n-h+k} = x_{n-h+k} \qquad (k = 1, 2, \ldots, h) \tag{18}$$

then we can introduce the y as new variables in place of the x, which will make the family (16) take on the form:

$$\sum_{k=1}^{h} \sigma_{n-h-k} \frac{\partial f}{\partial y_{n-h+k}}, \tag{19}$$

in which one understands the σ to mean arbitrary functions of $y_1, \ldots, y_n$. If the system (1) now goes to:

$$\sum_{i=1}^{n} \beta_{si}\, dy_i \qquad (s = 1, \ldots, n - m) \tag{20}$$

under our transformation of variables and the most general transformation (8) of the family (10) goes to:

$$\sum_{i=1}^{h} \eta_i (y_1, \ldots, y_n) \frac{\partial f}{\partial y_i}$$

then one must have:

$$\sum_{i=1}^{n} \beta_{si}\, \eta_i \equiv 0 \qquad (s = 1, \ldots, n - m), \tag{21}$$

since the relations (9) express an invariant relationship between the infinitesimal transformation (8) and the system (1). However, since the infinitesimal transformation (19) belongs to the family (10) for any arbitrary form of the σ, equations (21) must be fulfilled for:

$$\eta_i \equiv 0 \qquad (i = 1, \ldots, n - h), \qquad \eta_k \equiv \sigma_k \qquad (k = n - h + 1, \ldots, n)\,;$$

i.e., the coefficients of the differentials of the variables in the system (20) must all vanish. The β_{si} can depend upon only $y_1, \ldots, y_{n-h}$, since equations (20) admit all transformations (19), moreover. Conversely, if the system (20) possesses that property then it will admit all transformations (19); i.e., equations (12) have h and only h systems of solutions. That then implies the:

Theorem I:

*In order for a **Pfaff** system* (1) *to be reducible to a system of equations in n – h variables (and no less) by introducing new variables, it is necessary and sufficient that the matrix* (13) *must possess the rank n – h.*

3. – From now on, we introduce the assumption that the system (1) has character *one*. Now, since not all bilinear covariants vanish identically, to fix ideas, we can assume that the first of the covariants (4) is not zero. One will then have identities of the form:

$$a_{iks} \equiv \mu_s\, a_{ik1} \qquad (s = 2, \ldots, n-m,\ i, k = 1, \ldots, m) \tag{22}$$

(which are always fulfilled in the case $m = 2$). Equations (12) reduce to the following ones:

$$\sum_{i=k}^{m} \xi_k\, a_{ik1} = 0 \qquad (i = 1, 2, \ldots, m)\ . \tag{23}$$

Hence, if 2ν is the rank of the skew-symmetric matrix:

$$\| a_{ik1} \| \qquad (i, k = 1, \ldots, m) \tag{24}$$

then corresponding to the system of $m - 2\nu$ independent solutions of equations (23):

$$\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_m^{(k)}, \qquad (k = 1, 2, \ldots, m - 2\nu) \tag{25}$$

there will be an $(m - 2\nu)$-parameter family of infinitesimal transformations that is invariantly coupled with (1), namely:

$$\rho_1\, X^{(1)} \cdot f + \rho_2\, X^{(2)} \cdot f + \ldots + \rho_{m-2\nu}\, X^{(m-2\nu)} \cdot f\ , \tag{26}$$

which will take the system (1) to itself and will be contained in the family (10). In this, one sets:

$$X^{(k)} f \equiv \sum_{i=1}^{n} \xi_i^{(k)} \frac{\partial f}{\partial x_i}\ ,$$

and the $\xi_{m+s}^{(k)}$ will be included in the quantities (25) by means of the relations (9). We briefly refer to the number 2ν as the *rank* of the system (1).

4. – If one introduces the solutions:

$$y_1\, ,\ y_2\, ,\ \ldots,\ y_{n-m+2\nu} \tag{27}$$

of the complete system:

(28) $$X^{(1)} f = 0, \qquad X^{(2)} f = 0, \qquad \ldots, \qquad X^{(m-2\nu)} f = 0$$

as new variables in equations (1) then those equations will reduce to a system of $n - m$ equations in the variables (27) that we will think of as being solved in the form:

(29) $$dy_{2\nu+s} = \sum_{i=1}^{2\nu} \beta_{si}(y_1, \ldots, y_{n-m+2\nu})\, dy_i \qquad (s = 1, 2, \ldots, n-m).$$

The bilinear covariants of (29) naturally reduce by means of (29) to a single one; i.e., if one sets:

$$B_i f \equiv \frac{\partial f}{\partial y_i} + \sum_{i=1}^{n-m} \beta_{si} \frac{\partial f}{\partial y_{2\nu+s}} \qquad (i = 1, 2, \ldots, 2\nu),$$

$$B_k \beta_{si} - B_i \beta_{sk} \equiv b_{iks}$$

then one will have:

$$b_{iks} \equiv \rho_s\, b_{ik1} \qquad (s = 2, \ldots, n-m;\ i, k = 1, \ldots, 2\nu),$$

and the rank of the system (29), which agrees with that of the matrix:

(30) $$\| b_{ik1} \| \qquad (i, k = 1, 2, \ldots, 2\nu),$$

will be 2ν. If one sets:

(31) $$y_1 = c_1$$

in (29), in which c_1 means an arbitrary constant, then one will get a **Pfaff** system in $n - m + 2\nu - 1$ variables of character one *that has rank* $2\nu - 2$. In fact, the rank of the matrix:

$$\| b_{ik1}(c_1, y_2, \ldots, y_{n-m+2\nu}) \| \qquad (i, k = 1, 2, \ldots, 2\nu),$$

cannot be $> 2\nu - 2$, since it is skew-symmetric and of odd order, but it also cannot be $< 2\nu - 2$ as long as c_1 remains arbitrary, since otherwise the determinant (30) would have rank $< 2\nu$.

The function y_1 in this is an arbitrary solution of the complete system (28) (in the case of $m = 2n$, it will be an arbitrary function of x) that is subject to only the one constraint that the relation $dy_1 = 0$ must not impair the linear independence of equations (1); i.e., y_1 cannot be a common solution of the equations:

(32) $$A_1 f = 0, \qquad A_2 f = 0, \qquad \ldots, \qquad A_m f = 0.$$

The possible integrals of (32) are naturally included among those of (28).

The invariance of the rank 2ν now implies the following:

Theorem II:

If the ***Pfaff*** *system* (1) *has character* **one** *and rank* 2ν, *and one further expresses one of the variables* x *– say,* x_1 *– in terms of the* $x_2, \dots, x_n$ *and the arbitrary constant* c_1 *by means of the relation* (31), *in which* y_1 *is an arbitrary solution of the system* (28) *that is independent of the integrals of the system* (32), *but an arbitrary function of the* x_i *in the case* $m = 2\nu$, *and one substitutes the value thus-obtained in the system* (1) *then it will be converted into a system* (1′) *with* $n - 1$ *variables that has character one and rank* $2\nu - 2$.

5. – We can once more apply the same process to the system (1′). We first form the $m - 2\nu + 1$-parameter complete system in the $n - 1$ variables $x_2, \dots, x_n$ that belongs to (1′) and is analogous to equations (28). Let $y_2^{(1)}$ be an arbitrary solution of it that satisfies the condition that y_1 did before, and naturally depends upon c_1, in addition to $x_2, \dots, x_n$. If we then eliminate the variable x_2 from (1′) by means of the relation $y_2^{(1)} = c_2$ then we will obtain a system (1″) that consists of $n - m$ equations in $n - 2$ variables and has rank $2\nu - 4$. We can then once more reduce it to a system (1‴) of rank $2\nu - 6$ by a relation $y_3^{(1)} = c_3$, etc. After the ν^{th} step, we will arrive at a system $(1^{(\nu)})$ in $n - \nu$ variables that has rank zero; i.e., it is integrable without restriction and can take the form:

$$d\varphi_s\,(c_1, c_2, \dots, x_{\nu+1}, \dots, x_n) = 0 \qquad (s = 1, \dots, n - m).$$

The equations:

$$y_1 = c_1, \; y_2^{(1)} = c_2, \; \dots, \; y_\nu^{(\nu-1)} = c_\nu, \qquad \varphi_s = c_{s+\nu} \qquad (s = 1, 2, \dots, n - m) \tag{33}$$

will then define an integral equivalent to (1) for arbitrary values of the c_i ; i.e., equations (1) will be a consequence of the relations:

$$dy_1 = 0, \qquad dy_2^{(1)} = 0, \qquad \dots, \qquad dy_\nu^{(\nu-1)} = 0, \qquad d\varphi_s = 0.$$

If one then understands y_i ($y_{\nu+s}$, resp.) to mean those functions of $x_1, \dots, x_n$ that arise from $y_i^{(i-1)}$ (φ_s , resp.) when one eliminates the c_i from them in succession by means of equations (33) then the system (1) can obviously be brought into the following reduced form:

$$dy_{\nu+s} = \sum_{i=1}^{\nu} \eta_{si}\, dy_i \qquad (s = 1, 2, \dots, n - m), \tag{34}$$

in which the functions η_{is} of the variables x can be calculated by means of linear equations by comparing the two systems (1) and (34) as soon as the $y_1, y_2, \dots, y_\nu, y_{\nu+1}, \dots, y_{\nu+n-m}$ are known. In order to exhibit that reduced form, one must ascertain an integral of:

an $m - 2\nu$-parameter complete system in n variables,

″ $m - 2\nu + 1$- ″ ″ ″ $n - 1$ ″,
……………………………………………………………....,

″ $m - \nu + 1$- ″ ″ ″ $n - \nu + 1$ ″,

and finally, all integrals of an $(m - \nu)$-parameter complete system in $n - \nu$ variables. The orders of the operators that are necessary for this are equal to:

$n - m + 2n$, $n - m + 2n - 2$, …, $n - m + 2$, $n - m$, $n - m - 1$, …, 3, 2, 1,

resp., when one uses the **Lie-Mayer** method.

6. – The two cases $\nu > 1$ and $\nu = 1$ exhibit a completely different behavior in the further treatment of the reduced form (34). Namely, one has the:

Theorem III:

A system of n – m ***Pfaff*** *equations whose character is* ***one*** *and whose rank is* $2\nu > 2$ *always possesses n – m – 1 independent integrable combinations.*

Since one has:

$$(A_i\, A_k) \equiv a_{ik1} \cdot Bf \qquad (i, k = 1, \dots, m) \tag{35}$$

identically due to (17) and (22), in which one sets:

$$B\,f \equiv \frac{\partial f}{\partial x_{m+1}} + \sum_{s=2}^{n-m} \mu_s \frac{\partial f}{\partial x_{m+s}}, \tag{36}$$

clearly our theorem can also be expressed:

If one forms all equations:

$$(A_i\, A_k) = 0 \tag{37}$$

from a system of linear partial differential equations (32) *and obtains just one new equation:*

$$B\,f = 0 \tag{38}$$

then equations (32), (38) *will represent a complete system, assuming that the rank of the matrix* (24) *is greater than two.*

Since the assumption that was just made will break down in the case of $m = 2$, we can assume that $m \geq 3$. If i, k, l are any three indices in the sequence 1, …, m then one will have the **Jacobi** identity:

(39) $$((A_i\, A_k)\, A_l) + ((A_k\, A_l)\, A_i) + ((A_l\, A_i)\, A_j) \equiv 0,$$

or, from (35):

$$(a_{ik1}\, Bf, A_l) + (a_{kl1}\, Bf, A_i) + (a_{li1}\, Bf, A_k) \equiv 0,$$

so:

(40) $$a_{ik1}\,(B, A_l) + a_{kl1}\,(B, A_i) + a_{li1}\,(B, A_k) \equiv Bf \cdot (A_l\, a_{ik1} + A_i\, a_{kl1} + A_k\, a_{li1})$$

However, one has:

$$((A_i\, A_k)\, A_l) \equiv \left(\sum_{s=1}^{n-m} a_{iks} \frac{\partial f}{\partial x_{m+s}}, A_l\right)$$

identically, so if one replaces the other two terms in (39) with corresponding expressions then it will follow from evaluating the bracket symbols that the expression:

$$\sum_{s=1}^{n-m}\left[A_l\, a_{iks} - \sum_{r=1}^{n-m} a_{ikr} \frac{\partial a_{sl}}{\partial x_{m+r}}\right]\frac{\partial f}{\partial x_{m+s}}$$

$$+\sum_{s=1}^{n-m}\left[A_i\, a_{iks} - \sum_{r=1}^{n-m} a_{klr} \frac{\partial a_{si}}{\partial x_{m+r}}\right]\frac{\partial f}{\partial x_{m+s}}$$

$$+\sum_{s=1}^{n-m}\left[A_k\, a_{lis} - \sum_{r=1}^{n-m} a_{lik} \frac{\partial a_{sk}}{\partial x_{m+r}}\right]\frac{\partial f}{\partial x_{m+s}}$$

will vanish identically. In particular, when one recalls (22) and (36), the identical vanishing of the coefficients of $\frac{\partial f}{\partial x_{m+1}}$ will imply that:

$$A_i\, a_{ik1} + A_i\, a_{kl1} + A_k\, a_{li1} \equiv a_{ik1}\, B\, a_{1l} + a_{kl1}\, B\, a_{1i} + a_{li1}\, B\, a_{1k}\,,$$

so the identity (40) can be written:

(41) $$\Phi_{0i}\, a_{kl1} + \Phi_{0k}\, a_{kl1} + \Phi_{0l}\, a_{li1} \equiv 0 \qquad (i, k, l = 1, 2, \ldots, m)$$

when one sets:

(42) $$\Phi_{0l} \equiv -\Phi_{0l} \equiv (B\, A_l) - Bf \cdot B\, a_{1l}\,,$$

to abbreviate. All of the four-rowed principal sub-determinants in the skew-symmetric $(m + 1)$-rowed determinant:

$$(43) \qquad \begin{vmatrix} 0 & \Phi_{01} & \Phi_{02} & \cdots & \Phi_{0m} \\ \Phi_{10} & 0 & a_{121} & \cdots & a_{1m1} \\ \Phi_{20} & a_{211} & 0 & \cdots & a_{2m1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Phi_{m0} & a_{m11} & a_{m21} & \cdots & 0 \end{vmatrix}$$

that include elements of the first row and column will vanish then. From known theorems on skew-symmetric matrices, either absolutely all four-rowed principal sub-determinants of (43) will vanish – i.e., the rank of the matrix (24) will be two – or all Φ_{0i} will be identically zero; i.e., one will have:

$$(44) \qquad (B\,A_i) = Bf \cdot B\,a_{1i} \qquad (i = 1, 2, \ldots, m),$$

which was to be proved.

7. – Since equations (32), (38) always defines a complete system in the case $\nu > 1$, the equations:

$$(45) \qquad \nabla_s - m_s \nabla_1 = 0 \qquad (s = 2, 3, \ldots, n - m)$$

will represent a system that is integrable without restriction. Let:

$$(46) \qquad z_{2\nu+2} = c_1\,, \qquad z_{2\nu+3} = c_2\,, \qquad \ldots, \qquad z_{2\nu+n-m} = c_{n-m-1}$$

be the integral equations of (45). Their left-hand sides will then be mutually-independent with respect to x_{m+2} , x_{m+3} , …, x_n . If one eliminates the latter from the system (1) by means of (46) then it will be converted into a single **Pfaff** equation:

$$(47) \qquad dx_{m+1} = \sum_{i=1}^{m} \beta_i (x_1, x_2, \ldots, x_{m+1}, c_1, c_2, \ldots, c_{n-m-1})\, dx_i\,.$$

Now, that can be brought into the normal form:

$$(48) \qquad d\varphi_{2\nu+1} = \varphi_{\nu+1}\, d\varphi_1 + \varphi_{\nu+2}\, d\varphi_2 + \ldots + \varphi_{2\nu}\, d\varphi_\nu\,,$$

in which the φ means $2\nu + 1$ independent functions of x_1 , …, x_{m+1} . If one lets z_k denote what φ_k becomes when one replaces the c_i with their values in (46) then the system of equations (1) will take the normal form:

$$(49) \qquad \left\{\begin{array}{ll} a) & dz_{2\nu+1} = z_{\nu+1}\,dz_1 + z_{\nu+2}\,dz_2 + z_{2\nu}\,dz_\nu\,, \\ b) & dz_{2\nu+2} = 0, \quad dz_{2\nu+3} = 0, \quad \cdots \quad dz_{2\nu+n-m} = 0, \end{array}\right.$$

which will make the functions:

$$(50) \qquad z_1\,,\, z_2\,,\, \ldots,\, z_{n-m+2}$$

mutually independent. The fact that the **Pfaff** equation (47) can, in fact, be brought into the normal form (48) already follows from the fact that any other assumption on the number of terms in the right-hand side of (49a) would be incompatible with the rank 2ν of the system (1). One will then have the:

Theorem IV:

*A system of **Pfaff** equations with character one possesses no other invariants besides the number n – m of equations and the rank* 2ν *in the case where* $2\nu > 2$.

If one considers the functions (50) and any other $m - 2\nu$ variables z_i to be new independent variables then the system (28) will take the form:

$$\frac{\partial f}{\partial z_{n-m+2\nu+s}} = 0 \qquad\qquad (s = 1, 2, \ldots, m - 2\nu).$$

The functions (50) are then the solution to the system (28). As in no. **5**, one can choose z_1 to be an integral of equations (28) that is independent of the solutions (46) of the complete system (32), (38). The further reduction follows just as it did in no. **5**. However, if one determines the common integral (46) of the system (32), (38) in advance then, as is known from the theory of the **Pfaff** problem, the determination of z_1 , z_2 , …, z_ν, $z_{2\nu+1}$, will generally require:

$$2\nu + 1,\ 2\nu - 1,\ \ldots,\ 5,\ 3,\ 1$$

operations, respectively, which represents a simplification in comparison to the method in no. **5**.

8. – When written in terms of the independent variables z_1 , …, z_n , the family (10) will take the following form:

$$\sum_{h=1}^{\nu} \sigma_h \left(\frac{\partial f}{\partial z_h} + z_{\nu+h} \frac{\partial f}{\partial z_{2\nu+1}} \right) + \sum_{h=\nu+1}^{2\nu} \tau_h \frac{\partial f}{\partial z_{\nu+k}} + \sum_{k=n-m+2\nu+1}^{n} \omega_h \frac{\partial f}{\partial z_k}\,,$$

in which the σ, τ, ω mean arbitrary functions of all z_i . That family includes the following one:

(51) $$\tau_1 \frac{\partial f}{\partial z_{\nu+1}} + \sum_{k=n-m+2\nu+1}^{n} \omega_k \frac{\partial f}{\partial z_k}.$$

The infinitesimal transformations of that family are the only ones that will change the left-hand side of a **Pfaff** equation that is obtained from (49) by linear combination into an expression of the form $\lambda\, dz_1$ by means of (49). The complete system:

(52) $$\frac{\partial f}{\partial z_{\nu+1}} = 0, \qquad \frac{\partial f}{\partial z_k} = 0 \qquad (k = n - m + 2\nu + 1, \ldots, n)$$

possesses the integrals:

(53) $$z_1, z_2, \ldots, z_\nu, z_{\nu+2}, z_{\nu+3}, \ldots, z_{2\nu+n-m}.$$

If we then wish to determine the most general infinitesimal transformation (8) of the family (10) that fulfills the $n - m$ identities:

(54) $$X\, \nabla_s \equiv \sigma_s\, dz_1 \qquad (s = 1, 2, \ldots, n - m)$$

by means of (1) then due to the invariance of the stated properties of Xf, we must obtain an expression for the most general transformation of the family (51) and the complete system (52) in the original variables $x_1, \ldots, x_n$. Similarly to what was done in no. **2**, the linear equations:

(55) $$\sum_{k=1}^{m} \xi_k\, a_{ik1} = \sigma_1 A_i z_1 \qquad (i = 1, \ldots, m)$$

will follow from the conditions (54), and one will have:

(56) $$\sigma_s \equiv \mu_s\, \sigma_1 \qquad (s = 2, 3, \ldots, n - m),$$

in which the μ_s are defined by (22). Since z_1 satisfies the system (28), by assumption, one will have the identities:

$$\sum_i \xi_i^{(k)} A_i z_1 \equiv 0 \qquad (i = 1, 2, \ldots, m - 2\nu).$$

It follows immediately from this that the rank of the matrix that belongs to the system of equations (55) is 2ν. Those equations then possess $m - 2\nu + 1$ linearly-independent systems of solutions, namely, the $m - 2\nu$ systems (25) that correspond to the assumption $\sigma_1 \equiv 0$, and another one:

(57) $$\xi_1^{(m-2\nu+1)}, \ldots, \xi_m^{(m-2\nu+1)}, \sigma_1 \qquad (\sigma_1 \not\equiv 0).$$

The equations:

(58) $$0 = X^{(k)} f \equiv \sum_{i=1}^{n} \xi_i^{(k)} \frac{\partial f}{\partial x_i} \qquad (k = 1, 2, \ldots, m - 2\nu + 1)$$

then represent the complete system with the integrals (53). If z_2 is an arbitrary solution of it that is independent of z_1 and the functions (46) then the rank of the matrix that belongs to the system of equations:

$$\sum_{k=1}^{m} \xi_k \, a_{ik1} = \tau_1 \, A_i \, z_1 + \tau_2 \, A_i \, z_2 \qquad (i = 1, \ldots, m)$$

will be, in turn, 2ν. Those equations will then possess $m - 2\nu + 2$ linearly-independent solutions, namely, the system (25), for which the τ_1 and τ_2 vanish, and furthermore, (57), in which $\tau_1 = \sigma_1$, $\tau_2 \equiv 0$, and finally, another one for which $\tau_2 \not\equiv 0$. One will then get the complete system:

$$X^{(k)} f = 0 \qquad (k = 1, 2, \ldots, m - 2\nu + 2),$$

which has the functions $z_1, \ldots, z_\nu$, $z_{\nu+3}$, z_{n-m+2}. The proofs of those assertions then follows by analogy with the previous ones from the existence of a family of transformations that change any arbitrary combination of equations (49) into an expression of the form $\lambda_1 \, dz_1 + \lambda_2 \, dz_2$. In general, then will imply the following: If the functions $z_1, \ldots, z_r$ $(r \leq \nu)$ are determined from the normal form (49) then the equations:

$$\sum_{k=1}^{m} \xi_k \, a_{ik1} = \omega_1 \, A_i \, z_1 + \omega_2 \, A_i \, z_2 + \ldots + \omega_r \, A_i \, z_r$$

will possess $m - 2\nu + r$ independent systems of solutions. If $\xi_1^{(k)}, \ldots, \xi_m^{(k)}$ are the associated system of values for ξ_i then the complete system:

$$\sum_{i=1}^{n} \xi_i^{(k)} \frac{\partial f}{\partial x_n} \equiv X^{(k)} f = 0 \qquad (k = 1, 2, \ldots, m - 2\nu + r)$$

[in which the $\xi_{m+s}^{(k)}$ are ascertained from (9)] will have the solutions $z_1, z_2, \ldots, z_\nu, z_{\nu+r+1}, z_{\nu+r+2}, \ldots, z_{n-m+2\nu}$, and the function z_{r+1} (z_{2r+1}, resp., in the case $r = \nu$) will be a solution of it that is independent of $z_1, \ldots, z_n$, and the functions (46). That process, which allows one to exhibit the successive complete systems that serve to determine the $z_1, \ldots, z_r, z_{2\nu+1}$ *explicitly*, is, as one sees, nothing but a modification of the known **Frobenius** method for exhibiting the normal form of *one* **Pfaff** equation.

9. – We now turn to a consideration of the case $\nu = 1$. From no. **5**, we next get a reduced form for the system (1) here:

(59) $$dy_{s+1} - \varphi_s \, dy_1 = 0 \qquad (s = 1, 2, \ldots, n - m),$$

in which the functions $y_1, y_2, \ldots, y_{n-m+1}$ are independent of each other, so one can introduce $m - 1$ other arbitrary variables:

(60) $$y_{n-m+2}\,,\; y_{n-m+3}\,,\; \ldots,\; y_n$$

along with them in place of the x as new variables. Due to (59), the bilinear covariant will then assume the form:

$$\sum_{i=1}^{m-1} \frac{\partial \varphi_s}{\partial y_{n-m+i+1}} (dy_{n-m+i+1}\, \delta y_1 - \delta y_{n-m+i+1}\, dy_1) = 0\,.$$

Since that should reduce to *just one* independent one, one must have:

$$\frac{\partial \varphi_s}{\partial y_{n-m+i+1}} = \rho_s \frac{\partial \varphi_1}{\partial y_{n-m+i+1}} \qquad (i = 1, 2, \ldots, m - 1;\; s = 2, \ldots, n - m)\,.$$

Hence, the functions φ_s will reduce to *just one* independent one with respect to the variables (6), which we would like to denote by y_{n-m+2} ; the system (1) can then be brought into the form:

(61) $$\begin{cases} dy_2 - y_{n-m+2}\, dy_1 = 0, \\ dy_{2+s} - \varphi_s(y_1, y_2, \ldots, y_{n-m+2})\, dy_1 = 0 \quad (s = 1, 2, \ldots, n - m - 1). \end{cases}$$

If equations (1) possess integrable combinations then one can make just as many functions φ_s vanish.

The functions $y_1, \ldots, y_{n-m+2}$ are the solutions of the complete system (28). One will obtain the complete system that possesses only the integrals $y_1, y_2, \ldots, y_{n-m+1}$ in the form (58) as above. One sets $\nu = 1$ in it, and the quantities $\xi_1^{(k)}, \ldots, \xi_m^{(k)}$ will be the system of solutions of the equations (55) when one replaces z_1 with y_1 on the right.

10. – We now address the question: Under what conditions will the system (1) possess the normal form:

(62) $$\begin{cases} a) \quad dz_2 = 0, \quad dz_3 = 0, \quad \cdots \quad dz_{r+1} = 0, \\ b) \quad dz_{r+2} - z_{r+3}\, dz_1 = 0, \quad dz_{r+3} - z_{r+4}\, dz_1 = 0, \quad \cdots \quad dz_{n-m+1} - z_{n-m+2}\, dz_1 = 0, \end{cases}$$

in which the functions:

(63) $$z_1,\, z_2,\, \ldots,\, z_{n-m+2}$$

are independent of each other?

It is self-explanatory that only one **Pfaff** system with character one, rank two, and having r integrable combinations can possess such a normal form, and that the functions (63) are the integrals of the complete system (28).

Should the normal form (62) exist then, above all, it must be possible for the system of equations (1) to be reducible to the following form:

$$(64)\qquad \begin{cases} a) & dz_{n-m+1} - z_{n-m+1}\, dz_1 = 0, \\ b) & \nabla'_s \equiv dz_s - \psi_s\, dz_1 = 0, \quad (s = 2, 3, \ldots, n-m), \end{cases}$$

in which the quantities (63) are independent functions of x, and the ψ_s depend upon only the variables:

$$(65)\qquad z_1, z_2, \ldots, z_{n-m}, z_{n-m+1},$$

but not upon z_{n-m+2}. If we introduce the functions (63) and any other $m - 2$ variables $z_{n-m+3}, \ldots, z_n$ as new independent variables then the family of infinitesimal transformations that were denoted by (10) will take the form:

$$\sigma\left(\frac{\partial f}{\partial z_1} + z_{n-m+2}\frac{\partial f}{\partial z_{n-m+1}} + \sum_{s=2}^{n-m} \psi_s \frac{\partial f}{\partial z_s}\right) + \tau \frac{\partial f}{\partial z_{n-m+2}} + \sum_{i=n-m+3}^{n} \tau_i \frac{\partial f}{\partial z_i},$$

in which the σ, τ mean arbitrary functions of $z_1, \ldots, z_n$. The following one is included in that family:

$$(66)\qquad \tau \frac{\partial f}{\partial z_{n-m+2}} + \sum_{i=n-m+3}^{n} \tau_i \frac{\partial f}{\partial z_i}.$$

An arbitrary infinitesimal transformation of that family leaves any linear combination of the equations (64.b) unchanged *by means of just those equations*, while it will change an arbitrary combination of equations (64.a), (64.b) by an expression of the form $\sigma\, dz_1$ by means of the system (64.a, b). In order to get necessary conditions for the existence of a reduced form (64), we must observe when there exists an $m - 1$-parameter family among the transformations (10) whose transformations change each equation $\nabla_s = 0$ by an expression $\sigma_s\, dz_1$ by means of (1), and at the same time leave a system of $n - m - 1$ **Pfaff** equations that is contained in (1) invariant.

Should an infinitesimal transformation (8) that satisfies the conditions (9) fulfill the identities:

$$(67)\qquad X\, \nabla_s \equiv \sigma_s\, dz_1 \qquad (s = 1, 2, \ldots, n-m)$$

because of (1) then one will get the relations (55) and (56), as in no. **8**. Now, σ_1 cannot vanish, in general, since otherwise $X f$ would leave the system (1) invariant, so it would belong to the family (26). Now, if $X f$ satisfies the relations (67), in which $\sigma_1 \not\equiv 0$, then the system of equations:

(68) $$\nabla'_s = \nabla_s - \mu_s \nabla_1 = 0 \qquad (s = 2, 3, \ldots, n - m)$$

will remain invariant under the transformation $X f$ *by means of* (1); i.e., one will have the identities:

$$X(\nabla_s - \mu_s \nabla_1) \equiv 0 \qquad \text{[by means of (1)]},$$

and there will obviously be no other $n - m - 1$-parameter **Pfaff** system in (1) that has that property. If a reduced form (64) exists at all then the system of equations (64.b) must be equivalent to the system (68). The latter system is invariantly coupled with the given equations (1), since it has the form:

$$dx_{m+s} - \mu_s \, dx_{m+1} - \sum_{i=1}^{m} (a_{si} - \mu_s \, a_{1i}) \, dx_i = 0 \qquad (s = 2, \ldots, n - m),$$

and the adjoint system of linear partial differential equations will then consist of the equations:

(69) $$A_1 f = 0, \qquad A_2 f = 0, \qquad \ldots, \qquad A_m f = 0, \qquad B f = 0,$$

but since those equations are equivalent to the set of relations:

(70) $$A_i f = 0, \qquad (A_i A_k) = 0 \qquad (i, k = 1, \ldots, m),$$

our statement will follow immediately from the theorem of **Engel** ([1]) that the system (70) is invariantly coupled with the system of equations $A_i f = 0$.

We would like to call equations (68) the *derived system of* (1).

11. – The elimination of the ξ_i from (55) then shows that z_1 must be an integral of the system (28). If that is the case then equations (55) will possess $m - 1$ independent systems of solutions. We must now observe whether and how z_1 can be determined such that the $m - 1$ infinitesimal transformations $X f$ of the **Pfaff** system (68) that one obtains from (55) are left invariant not only by means of (1), but *even by means of* (68) *alone.* To that end, we must next write out the conditions for the system (68) to admit $m - 1$ infinitesimal transformations of the family (10) at all.

If we write:

$$b_{s,\,m+1} \equiv \mu_s\,, \qquad b_{si} \equiv a_{si} - \mu_s \, a_{1i} \qquad (s = 2, 3, n - m)$$

$$B f \equiv B_{m+1} f \equiv \frac{\partial f}{\partial x_{m+1}} + \sum_{s=1}^{n-m} \mu_s \frac{\partial f}{\partial x_{m+s}}\,,$$

([1]) These Berichte (1889), pp. 165.

$$B_i f \equiv B_i f - a_{1i}\, B_{m+1} f \equiv \frac{\partial f}{\partial x_i} + \sum_{s=1}^{n-m} (a_{si} - \mu_s\, a_{1i}) \frac{\partial f}{\partial x_{m+s}},$$

$$b_{iks} \equiv B_i\, b_{sk} - B_k\, b_{si} \qquad (i, k = 1, 2, \ldots, m + 1)$$

then we will have:

$$(71) \qquad b_{m+1,\,i,\,s} = B_{m+1}\, a_{si} - \mu_s\, B_{m+1}\, a_{1i} - A_i\, \mu_s\,,$$

$$(72) \qquad b_{iks} \equiv - b_{kis} \equiv a_{1k}\, b_{m+1,\,i,\,s} - a_{1i}\, b_{m+1,\,k,\,s} \qquad (i, k = 1, 2, \ldots, m)$$

identically, due to (22).

From no. **2**, we now write the conditions for the system (68) to admit the infinitesimal transformation (8) as follows:

$$\sum_{k=1}^{m+1} \xi_k\, b_{iks} = 0 \qquad (s = 2, \ldots, n - m\,;\; i = 1, \ldots, m + 1)$$

or if one recalls (72) and (9):

$$a_{1i} \cdot \sum_{k=1}^{m} \xi_k\, b_{m+1,k,s} = 0 \qquad (i = 1, \ldots, m\,;\; s = 1, \ldots, n - m),$$

$$(73) \qquad \sum_{k=1}^{m} \xi_k\, b_{m+1,k,s} = 0 \qquad (s = 2, \ldots, n - m).$$

If all $b_{m+1,\,k,\,s}$ vanish identically then the derived system (68) will be integrable without restriction, and we will get the normal form (49) for the system (1) by means of the method of no. **7**, in which one sets $\nu = 1$; we can then ignore that case. Now, since equations (73) should possess $m - 1$ independent systems of solutions, by assumption, they must all be equivalent to one of them – say, the first one; i.e., one must have:

$$(74) \qquad b_{m+1,\,k,\,s} \equiv \mu'_s\, b_{m+1,\,k,\,2} \qquad (s = 3, 4, \ldots, n - k\,;\; k = 1, \ldots, m).$$

As a result of this, due to (72), all b_{iks} (the quantities b_{iks}, resp.) will be proportional; i.e., the derived system (68) has character one. Conversely, if that is the case then the conditions (74) will be fulfilled.

We have then obtained as a necessary condition for the existence of a reduced form (64) *that the derived system* (68) *of the equations* (1) *must, in turn, have character one,* or in other words, *that the equations:*

$$(A_1\, B) = 0, \qquad (A_2\, B) = 0, \qquad \ldots, \qquad (A_m\, B) = 0$$

must, in turn, yield one and only one equation by means of (69).

In fact, due to (71), that will be expressed by the conditions (74).

12. – We would like to further assume that the derived system fulfills that condition. Its rank is obviously two then, and it will admit all transformations of the $(m - 1)$-parameter family:

(75) $$\rho_1 X^{(1)} f + \ldots + \rho_{m-1} X^{(m-1)} f,$$

in which we have set:

$$X^{(m-1)} f = \sum_{i=1}^{n} \xi_i^{(k)} \frac{\partial f}{\partial x_i},$$

and the $\xi_1^{(k)}, \ldots, \xi_m^{(k)}$ mean the system of solutions to the linear equation:

(76) $$\sum_{k=1}^{m} \xi_k \, b_{m+1,k,2} = 0.$$

Among those solutions, one also finds the $m - 2$ systems (25). In order to show that, it will suffice to prove that the relation (76) is a consequence of equations (23). Now, since one has:

$$\Phi_{0l} \equiv \sum_{s=2}^{n-m} b_{m+1,l,s} \frac{\partial f}{\partial x_{m+s}}$$

identically, when one recalls (71) and (42), and since the rank of the matrix (43), as well as the matrix (24) is equal to two, moreover, one will have, from known theorems, the following table for it:

$$\left\| \begin{matrix} \Phi_{01} & \Phi_{02} & \cdots & \Phi_{0m} \\ 0 & A_{121} & \cdots & A_{1m1} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m11} & A_{m21} & \cdots & 0 \end{matrix} \right\|,$$

which will immediately go to the matrix of the system of equations (23), (76) when one replaces f with x_{m+2} in it; (76) will then be a consequence of (23), and we can then identify the first $m - 2$ terms in (75) with (26).

Now, if $X f$ is an arbitrary infinitesimal transformation of the family (75) for which $\rho_{m-1} \not\equiv 0$ then one can determine a function z_1 such that the relations (55) are fulfilled. In fact, due to the fact that $\rho_{m-1} \not\equiv 0$, one will also have $\rho_1 \not\equiv 0$, so $\xi_1, \ldots, \xi_m$ will be a system of solutions of (55) that is linearly independent of the $m - 2$ systems of values (25). The elimination of σ_1 will then yield $m - 1$ conditions for z_1 that can be written in the form:

$$\sum_{i=1}^{m} \xi_i A_i z_1 = 0, \qquad \sum_{i=1}^{m} \xi_i^{(k)} A_i z_1 = 0 \qquad (k = 1, 2, \ldots, m-2),$$

as one will see when one multiplies equations (55), first by $\xi_1, \ldots, \xi_m$, and then by $\xi_1^{(k)}$, …, $\xi_m^{(k)}$, and adds them each time. The function z_1 is then an integral of the system:

$$X^{(1)} f = 0, \qquad X^{(2)} f = 0, \qquad \ldots, \qquad X^{(m-1)} f = 0, \tag{77}$$

which is complete, from no. **2**.

13. – If one introduces the integrals $z_1, z_2, \ldots, z_{m-n+1}$ of the system (77) as new variables, along with other variables $z_{m-n+2}, \ldots, z_n$, then that will convert the derived system (68) into equations of the form:

$$\sum_{i=1}^{n-m+1} \beta_{si}(z_1, \ldots, z_{n-m+1})\, dz_i = 0 \qquad (s = 1, 2, \ldots, n-m-1).$$

If one brings that into the reduced form (64.b), from no. **5**, then the still-remaining equation $\nabla_1 = 0$ will take on the form:

$$dz_{n-m+1} - \psi\, dz_1 = 0$$

due to (64.b).

After introducing the z_i, the differentials $dz_{n-m+2}, \ldots, dz_n$ cannot actually appear in equations (1) any more, since the family (75) that is contained in (10) will now assume the form:

$$\sigma_1 \frac{\partial f}{\partial z_{n-m+2}} + \cdots + \sigma_{m-1} \frac{\partial f}{\partial z_n}.$$

Obviously, the function ψ does not depend upon just $z_1, \ldots, z_{n-m+1}$, since otherwise the system (1) would be integrable without restriction; we can then write z_{n-m+2}, instead of y, and have the:

Theorem V:

In order for a ***Pfaff*** *system* (1) *of character one and rank two to possess a reduced form* (64), *it is necessary and sufficient that its derived system* (68) *must, in turn, have character one (or zero).*

14. – We now denote the given system (1) by S, its derived system by S', and assume that S' has character one, so it will again possess a derived system S'', about which, we make the same assumption, etc., until we arrive at a system $S^{(n-m-r-1)}$ whose derived system has character zero; i.e., it is integrable without restriction. Therefore, from no. **7**, $S^{(n-m-r-1)}$, which then consists of $r + 1$ equations, can then take on the normal form:

$$(78) \qquad dz_2 = 0, \quad dz_3 = 0, \; \ldots, \quad dz_{r+1} = 0, \quad dz_{r+2} - z_{r+3}\, dz_1 = 0.$$

From no. **13**, the system $S^{(n-m-r-2)}$ will then arise from (78) by appending a relation of the form:

$$dz_{r+3} - z_{r+4}\, dz_1 = 0,$$

and likewise $S^{(n-m-r-3)}$ will arise by appending an equation:

$$dz_{r+4} - z_{r+5}\, dz_1 = 0,$$

etc. The assumptions that were made about the successive systems $S, S', S'', S''', \ldots$ will then be sufficient for the existence of a normal form (62), and as one sees immediately, also necessary. We then have the:

Theorem VI:

A ***Pfaff*** *system* (1) *with character one and rank two will have a normal form* (62) *iff the relations:*

$$(B\,A_1) = 0, \qquad (B\,A_2) = 0, \qquad \ldots, \qquad (B\,A_m) = 0$$

yield one and only one new equation $B\,f = 0$ *by means of* (32), (38), *so when the equations:*

$$(B'B) = 0, \qquad (B'A_i) = 0 \qquad (i = 1, 2, \ldots, m)$$

likewise reduce to only one new equation $B''f = 0$ *by means of the foregoing, etc., until one arrives at an equation* $B^{(n-m-r-1)}\, f = 0$ *that defines a complete system along with the foregoing one.*

The fact that these conditions express an invariant property of the system (1) follows immediately from the theorem of **Engel** that was cited in no. **10**.

As one sees, the number $n - m$ of equations and the number r of integrable combinations are the only invariants of a **Pfaff** system of that kind.

From no. **7**, exhibiting the normal form for the system S will require:

$$r, r - 1, \ldots, 3, 2, 1; 3, 1$$

operations, resp., which are required to exhibit the normal form (78) for the system $S^{(n-m-r-1)}$, but no others, in addition.

15. – A normal form (62) always exists in the case of a system (1) of character one and rank two that consists of *two* equations, so in particular, for any arbitrary twoparameter **Pfaff** system in four variables, which **Engel** has shown already ([1]). In order for a system of three equations in five variables that is not integrable without restriction:

$$dx_{2+s} = a_{s1}\, dx_1 + a_{s2}\, dx_2 \qquad (s = 1, 2, 3)$$

to possess one of three normal forms:

$$\begin{aligned}
&dz_2 = 0, && dz_3 = 0, && dz_4 - z_5\, dz_1 = 0,\\
(79)\qquad &dz_2 = 0, && dz_3 - z_4\, dz_1 = 0, && dz_4 - z_5\, dz_1 = 0,\\
&dz_2 - z_3\, dz_1 = 0, && dz_3 - z_4\, dz_1 = 0, && dz_4 - z_5\, dz_1 = 0,
\end{aligned}$$

it is necessary and sufficient for the identities (74) to exist, so when one sets $n = 5$, $m = 2$ in them. Upon eliminating μ_3', one will then get a condition:

$$0 \equiv \begin{vmatrix} B a_{21} - \mu_2 B a_{11} - A_1 \mu_2 & B a_{31} - \mu_3 B a_{11} - A_1 \mu_3 \\ B a_{22} - \mu_2 B a_{12} - A_2 \mu_2 & B a_{32} - \mu_3 B a_{12} - A_2 \mu_3 \end{vmatrix},$$

in which one sets:

$$A_i f \equiv \frac{\partial f}{\partial x_i} + \sum_{s=1}^{3} a_{si} \frac{\partial f}{\partial x_{2+s}},$$

$$A_1 a_{s2} - A_2 a_{s1} \equiv \mu_s (A_1 a_{12} - A_2 a_{11}) \qquad (s = 2, 3),$$

$$B f \equiv \frac{\partial f}{\partial x_3} + \mu_2 \frac{\partial f}{\partial x_4} + \mu_3 \frac{\partial f}{\partial x_5}.$$

Among the **Pfaff** systems of character one for which $m = 3$, one will find many special cases that are important for the theory of partial differential equations with two independent variables. For instance, if one consider the three-parameter system in six variables x, y, z, p, q, r :

$$\begin{aligned}
&dz = p\, dx + q\, dy,\\
(80)\qquad &dp = r\, dx + s\, dy,\\
&dq = s\, dx + t\, dy,
\end{aligned}$$

in which s and t are defined by two equations of the form:

$$(81)\qquad s = \varphi(x, y, z, p, q, r), \qquad t = \psi(x, y, z, p, q, r),$$

([1]) These Berichte (1889), pp. 175.

then the conditions for (80) to have character one will then express the idea that the two partial differential equations (81) define a system in involution. The system (80) will then possess one and only one normal form (79) when the two functions φ and ψ are completely linear in r ; i.e., when the five-fold infinitude of common second-order characteristic strips of the two equations (81) always contain a simple infinitude of first-order strips of them ([1]).

Munich, in May 1898.

([1]) I have explained the meaning of the bilinear covariant for the theory of partial differential equations in two independent variables in the papers "Ueber simultane partielle Differentialgleichungen II O. mit 3 variabeln," Sitz. math.-phil. Classe der kön. bayer. Akad. Wiss **25** (1895), pp. 101 and "Ueber gewisse Systeme PFAFF'scher Gleichungen," *ibidem*, pp. 432.