# On de Rham's theorems 

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The most satisfying proof of de Rham's famous theorems at present is the one that results from H. Cartan's theory of homology, which includes it, as well as the Poincare duality theorem, as special cases. However, that theory has been the subject of only some partial publications in the form of mimeographed course notes $\left({ }^{1}\right)$. Moreover, at its origins, one finds, on the one hand, a paper by Leray, and on the other hand, there is a proof of de Rham's theorems that I communicated to Cartan in 1947. Aside from any other use it might have, it can also serve to introduce Cartan's methods, and that above all is the reason why I shall present it here, with some of the improvements that I have G. de Rham and N. Hamilton to thank for. I shall also add a proof (that also dates back to 1947) of the fact that any space possesses a covering of a certain type (which is called "topologically simple") with the same homotopy type as the nerve of that covering.

## § 1. - Construction of a simple covering.

Let $\left(X_{i}\right)_{i \in I}$ be a family of subsets of a space $E$ with an set of arbitrary indices $I$. As one knows, one says that this family is locally-finite if any point of $E$ has a neighborhood that meets only a finite number of $X_{i}$. If $E$ is locally compact then it would amount to the same thing to say that any compact subset of $E$ meets only a finite number of the $X_{i}$. We agree, once and for all, that if $\left(X_{i}\right)_{i \in I}$ is a locally-finite family and $J \subset I$ then we will set $X_{j}=\bigcap_{i \in J} X_{i}$. The set $N$ of non-vacuous subsets $J$ of $I$ such that $X_{J}$ is non-vacuous is called the nerve of the family $\left(X_{i}\right)$. If $J \in N$ then $J$ will be finite.

The subject of our subject will be an $n$-dimensional differentiable manifold $V$ that is "paracompact," i.e., every connected component of it is denumerable at infinity. That amounts to the same thing as saying that $V$ must admit a locally-finite covering by "charts," i.e., by open subsets that are each endowed with a differentiable isomorphism onto an open subset of $\mathbb{R}^{n}$. The word "differentiable" will always be taken to mean "indefinitely differentiable" (or "of class $C^{\infty}$ "). That is not a true restriction when one recalls Whitney's theorem that says that for $n \geq 1$ any

[^0]manifold of class $C^{n}$ will admit a homeomorphism of class $C^{n}$ onto a manifold of class $C^{\infty}$. Moreover, the method that will be presented will also apply to manifolds of class $C^{n}$ for $n \geq 2$.

Our main tool will be a locally-finite covering $\mathfrak{U}=\left(U_{i}\right)_{i \in I}$ of $V$ by relatively-compact open subsets $U_{i}$ that must have the following property, moreover: Every non-vacuous set $U_{J}=\bigcap_{i \in J} U_{i}$ possesses a "differentiable retraction," i.e., a differentiable map $\varphi_{J}$ of $U_{J} \times \mathbb{R}$ into $U_{J}$ such that $\varphi_{J}(x, t)=x$ whenever $x \in U_{J}$ and $t \geq 1$, and $\varphi_{J}$ is constant on $\left.U_{J} \times\right]-\infty, 0[$. When such a covering is endowed with retractions $\varphi_{J}$, it will be called differentiably simple.

In order to construct such a covering, one can do as de Rham did $\left(^{2}\right.$ ) and appeal to a $d s^{2}$, it is perhaps more elementary to proceed as follows: Start with a locally-finite covering of $V$ by relatively-compact open charts $V_{i} . V_{i}$ will then be attached to a differentiable isomorphism of $V_{i}$ onto an open subset of $\mathbb{R}^{n}$ by means of "local coordinates" $t_{1}^{(i)}, \ldots, t_{n}^{(i)}$. For each $i$, one can then define open subsets $W_{i}, W_{i}^{\prime}$, and a differentiable function $f_{i}$ on $V$ in such a manner that the $W_{i}$ will again form a covering of $V$, so one will have $\bar{W}_{i} \subset W_{i}^{\prime}$ and $\bar{W}_{i}^{\prime} \subset V_{i}$, and $f_{i}$ has the value 1 on $\bar{W}_{i}$ and 0 outside of $W_{i}^{\prime}$. Set $f_{i 0}=f_{i}$ and let $f_{i j}$ denote the function that is equal to $f_{i} t_{j}^{(i)}$ inside of $V_{i}$ and 0 outside of $V_{i}$. The set of functions $f_{i j}$ for $0 \leq j \leq n$ and for all values of $i$ determines a map of $V$ into the space $\mathbb{R}^{(A)}$, where $A$ is the set of pairs $(i, j)$. One knows that one thus denotes the vector space of maps from $A$ into $\mathbb{R}$ that take the value 0 everywhere except for a finite number of elements of $A$. Moreover, the map $\left(f_{i j}\right)$ of $V$ into $\mathbb{R}^{(A)}$ will determine a differentiable isomorphism of any relatively-compact open subset $Z$ of $V$ onto a submanifold of a finite-dimensional vector subspace of $\mathbb{R}^{(A)}$. One can then simplify the language by identifying $V$ with its image in $\mathbb{R}^{(A)}$. We define a ("pre-Hilbertian") metric space structure on $\mathbb{R}^{(A)}$ by means of the distance $d(x, y)=$ $\left[\sum_{i, j}\left(x_{i j}-y_{i j}\right)^{2}\right]^{1 / 2}$. It will make any finite-dimensional subspace of $\mathbb{R}^{(A)}$ into a Euclidian space. From the preceding, the distance from $\bar{W}_{i}$ to $V-W_{i}^{\prime}$ is $\geq 1$, since the coordinate $x_{i 0}$ has the value 1 on the first set and 0 on the second one.

For any $x \in V$, let $T_{x}$ denote the linear manifold that is tangent to $V$ at $x$, and let $P_{x}$ denote the orthogonal projection of $\mathbb{R}^{(A)}$ onto $T_{x}$, which is considered to be a linear map of $\mathbb{R}^{(A)}$ onto $T_{x}$, and let $U(x, r)$ denote the intersection of $V$ with the open ball with center $x$ and radius $r$. If $x \in \bar{W}_{i}$ and $r<1$ then one will have $U(x, r) \subset W_{i}^{\prime}$. Therefore, $U(x, r)$ will be relatively compact provided that $r<1$.
$\left(^{2}\right)$ Cf., G. de Rham, "Complexes à automorphismes et homéomorphie différentiable," Ann. Gren. 2 (1950), pp. 51. The latter discussion, like my 1947 proof, remains limited to the compact case. However, it was de Rham himself who pointed out to me the possibility of extending either method to non-compact manifolds.

Let $x \in \bar{W}_{i}$. Let $E$ be a finite-dimensional vector space that contains $\bar{W}_{i}^{\prime}$. Upon taking orthogonal coordinates in $E$ with their origin at $x$, so the first $n$ coordinate vectors will be chosen in $T_{x}$, one will see that $x$ possesses an open neighborhood $U$ that is contained in $W_{i}^{\prime}$ and has the following properties:
(a) For any $y \in U, P_{y}$ will induce a differentiable isomorphism on $U$ (i.e., a bijective map of rank $n$ everywhere) that takes $U$ onto its image $U_{y}=P_{y}(U)$ in $T_{y}$.
(b) For any $y, z_{1}, z_{2}$ in $\bar{U}$, one will have $d\left(z_{1}, z_{2}\right)<2 d\left(P_{y}\left(z_{1}\right), P_{y}\left(z_{2}\right)\right)$.
(c) For any $z_{0} \in U, d\left(z_{0}, z\right)^{2}$ will be a convex function of $P_{y}(z)$ into $U_{y}$.

Indeed, the last condition signifies that the matrix of second derivatives of $d\left(z_{0}, z\right)^{2}$ with respect to the coordinates of $P_{y}(z)$ in $T_{y}$ is the matrix of a positive-definitive quadratic form. Now, as long as $U$ is sufficiently small, that matrix will be as close as one likes from its value for $y=z_{0}=z=x$, which is a value that is nothing but the identity matrix. Let $K$ be a compact subset of $V$ then. Cover $K$ by a finite number of sets $U_{\alpha}$ that have the properties $(a),(b),(c)$, and take $0<r(K)<1$ such that $U(x, r(K))$ is contained in one of the $U_{\alpha}$ for any $x \in K$. Moreover, for $x \in K$ and $r=r(K)$, the projection $P_{x}[U(x, r)]$ of $U(x, r)$ onto $T_{x}$ will contain all of the points of $T_{x}$ that are at a distance $<r / 2$ from $x$. Indeed, if $z^{\prime}$ is a frontier point of that projection then $z^{\prime}$ will be a limit point of the points $z_{v}^{\prime}=P_{x}\left(z_{v}\right)$ with $z_{v} \in U(x, r)$. Since $U(x, r)$ is relatively compact on $V$, one can replace the $z_{v}$ with a partial sequence that has a limit $z$ on $V$. Since $P_{x}$ is a differentiable isomorphism of $U(x$, $r$ ) onto its image, any interior point of $U(x, r)$ will project onto an interior point of $P_{z}[U(x, r)]$. Therefore, $z$ is a frontier point of $U(x, r)$, and one will have $d(x, z)=r$, so $d\left(z, z^{\prime}\right)>r / 2$ by virtue of $(b)$. Now show that if $x \in K, 0<r \leq r(K) / 4$, and $y \in U(x, r)$ then $P_{x}$ will induce a differentiable isomorphism on $U(y, r)$ of $U(y, r)$ into a convex subset of $T_{z}$. Since one has $U(y, r) \subset U(x, 2 r)$, the only point that must be proved is the convexity of $P_{x}[U(y, r)]$. Now, that will be the set of points $z^{\prime}=P_{x}(z)$ when $z \in U(x, r(K))$ and $d(y, z)^{2}<r^{2}$. Consider two such points $z_{1}^{\prime}=P_{x}\left(z_{1}\right)$, $z_{2}^{\prime}=P_{x}\left(z_{2}\right)$. For $h=1$ and $h=2$, one has $d\left(z_{h}, x\right)<2 r$, so $d\left(z_{h}^{\prime}, x\right)<2 r$, thus one will also have $d\left(z_{h}^{\prime}, x\right)<2 r \leq r(K) / 2$ for any $z^{\prime}$ on the line segment that connects $z_{1}^{\prime}$ and $z_{2}^{\prime}$ in $T_{x}$. That segment is then contained in $P_{x}\left[U(x, r(K)]\right.$. Since $d(y, z)^{2}$ is a convex function of $z^{\prime}=P_{x}(z)$ in the latter set, it will be a convex function of $z^{\prime}$ on the segment that connects $z_{1}^{\prime}$ and $z_{2}^{\prime}$. Since the value of that function is $<r^{2}$ at the extremities of the segment, that will also be true all along the segment. It will then be indeed contained in $P_{x}[U(y, r)]$.

Having done that, for each $i$, choose a finite number of points $x_{i \lambda}$ in $\bar{W}_{i}$ such that the sets $U_{i \lambda}=$ $U\left(x_{i \lambda}, r\left(\bar{W}_{i}^{\prime}\right) / 4\right)$ form a covering of $\bar{W}_{i}$. I say that the $U_{i \lambda}$ form a differentiably simple covering of $V$. Since one has $x_{i \lambda} \in \bar{W}_{i}$ and $r\left(\bar{W}_{i}^{\prime}\right) / 4<1$, one will have $U_{i \lambda} \subset W_{i}^{\prime}$, so the $U_{i \lambda}$ will be relatively compact and form a locally-finite covering of $V$. Let $x$ be a point common to the sets $U_{i \lambda}, U_{j \mu}, U_{k v}$,
$\ldots$, which are necessarily finite in number. Let $r$ be the greatest of the numbers $r\left(\bar{W}_{i}^{\prime}\right), r\left(\bar{W}_{j}^{\prime}\right)$, $r\left(\bar{W}_{k}^{\prime}\right), \ldots$ Suppose, for example, that one has $r=r\left(\bar{W}_{i}^{\prime}\right)$. Each of the sets will then have the form $U_{i \lambda}, U_{j \mu}, \ldots$ will then have the form $U\left(y, r^{\prime}\right)$, with $y \in U\left(x, r^{\prime}\right)$ and $r^{\prime} \leq r\left(\bar{W}_{i}^{\prime}\right) / 4$. They are all contained in $U\left(x, r\left(\bar{W}_{i}^{\prime}\right)\right)$, and since $x \in \bar{W}_{i}^{\prime}, P_{x}$ will induce a differentiable isomorphism on $U\left(x, r\left(\bar{W}_{i}^{\prime}\right)\right)$ for which the image of each of the $U_{i \lambda}, U_{j \mu}, \ldots$ will be a convex open subset of $T_{x}$, from what was proved above. $P_{x}$ will also induce a differentiable isomorphism of their intersection onto a convex open subset $U^{\prime}$ of $T_{x}$. It admits the retraction $\left(z^{\prime}, t\right) \rightarrow x+\lambda(t)\left(z^{\prime}-x\right)$, in which $\lambda$ $(t)$ is a differentiable function on $\mathbb{R}$ that is equal to 0 for $t \leq 0$ and to 1 for $t \geq 1$. By virtue of the isomorphism that is induced by $P_{x}$, that retraction will be transported to the intersection of the $U_{i \lambda}$, $U_{j \mu}, \ldots$, which completes the proof.

In reality, we have made use of only the fact that when $V$ is embedded in $\mathbb{R}^{(A)}$, any compact subset of $V$ will have bounded curvature, or rather that any point of $V$ will have a neighborhood that can be represented parametrically by means of functions of class $C^{1}$ whose first-order derivatives will have bounded derived numbers. Already for a manifold $V$ of class $C^{1}$, it does not seem to be easy to construct a simple covering without first defining a structure of class $C^{2}$ on $V$ by means of the theorem of Whitney that was cited before, and the problem of the existence of a simple covering will remain open insofar as the manifolds of class $C^{0}$ are concerned. Of course, for such a manifold, we can impose nothing more upon the retractions $\varphi_{J}$ than that they should be continuous. On the other hand, any locally-finite simplicial complex will trivially admit one such covering that is composed of the open stars of their vertices. For the sake of what will follow, we shall briefly recall some definitions that relate to those complexes. We intend the term abstract simplicial complex to mean a set $N$ of non-vacuous finite subsets of an arbitrary set $I$ such that if $J$ $\in N$ then any non-vacuous subset of $J$ will also belong to $N . N$ is called locally-finite (or star-finite) if any $i \in I$ belongs to at most a finite number of elements of $N$. We agree to identify the abstract complex $N$ with its "geometric realization," i.e., the set of points $x=\left(x_{i}\right)_{i \in I}$ of the space $\mathbb{R}^{(I)}$ such that $\sum_{i \in I} x_{i}=1, x_{i} \geq 0$ for any $i$, and that the set of $i \in I$ such that $x_{i} \neq 0$ belongs to $N$. With no loss of generality, we can suppose that $I$ is the union of the sets in $N$ (otherwise we could replace $I$ with that union). For each $i$, let $e_{i}$ be the point of $\mathbb{R}^{(I)}$ whose coordinate with the index $i$ is equal to 1 and the others are all zero. The elements $i$ of $I$, or also the points $e_{i}$ that correspond to them, will be called the vertices of $M$. We will make any $J \in N$ correspond, on the one hand, to the simplex $\Sigma_{J}$, which is the set of point $x=\left(x_{i}\right)$ in $N$ such that $x_{i}=0$ when $i$ does not belong to $J$, and on the other hand, to the open star $S t t_{J}$, which is the set of points $x=\left(x_{i}\right)$ in $N$ such that $x_{i}>0$ for $i \in J$. If $J=\{i\}$ then $\Sigma_{J}$ will reduce to the vertex $e_{i}$ of $N$, and $S t_{J}$, which we will write as $S t_{i}$, will be called the open star of $e_{i}$. For $J \in N$, we will have $S t_{J}=\bigcap_{i \in J} S t_{i}$. If $J$ has $m$ elements, so $\Sigma_{J}$ is $(m-1)$-dimensional,
then the center of gravity (or barycenter) of $\Sigma_{J}$ will be the point $e_{J}=\left(x_{i}\right)$, with $x_{i}=1 / m$ for $i \in J$ and $x_{j}=0$ when $j$ does not belong to $J$. If the function $\lambda(t)$ is defined as above then $(x, t) \rightarrow e_{J}+\lambda$ $(t)\left(x-e_{J}\right)$ will be a retraction of $S t_{J}$. The $S t_{i}$ will then indeed form a simple covering of $N$.

## § 2. - Differential forms.

One always intends the term "differential form" to mean a form whose coefficients will be functions of class $C^{\infty}$ of the coordinates when it is expressed locally. A form $\omega$ is called closed when $d \omega=0$. It is called homologous to 0 on the manifold where it is defined if there exists a form $\eta$ on that manifold such that $\omega=d \eta$.

Let $U$ be an open subset of a differentiable manifold $V$ that is endowed with a retraction $\varphi$. Let $\omega$ be a form of degree $m$ on $U$. Consider the form $\omega[\varphi(x, t)]$ on $U \times \mathbb{R}$, which is the reciprocal image of $\omega$ by $\varphi$. If $x_{1}, \ldots, x_{n}$ are local coordinates on a neighborhood $U$ of a point then one can write:

$$
\omega[\varphi(x, t)]=\sum_{(i)} f_{(i)}(x, t) d x_{i_{1}} \wedge \cdots \wedge d x_{i_{m}}+\sum_{(j)} g_{(j)}(x, t) d t \wedge d x_{j_{1}} \wedge \cdots \wedge d x_{j_{m-1}}
$$

in which $\wedge$ denotes the exterior product. In that same neighborhood, consider the form $I \omega$ of degree $m-1$ that is defined by:

$$
I \omega=\sum_{(j)}\left(\int_{0}^{1} g_{(j)}(x, t) d t\right) d x_{j_{1}} \wedge \cdots \wedge d x_{j_{m-1}} .
$$

One immediately verifies that this operator is compatible with the changes in local coordinates and can then be considered to be defined globally in $U$. If $m=0$ then one has $I \omega=0$. One can use the local expression for $I$ to quickly verify that that one has $\omega=I d \omega+d I \omega$ if $m>0$. If $m=0$ then one will have $\omega=I d \omega+\omega(a)$ if $a$ is the constant value of $\varphi(x, t)$ for $t \leq 0$. It will then follow that if $m>0$ then $d \omega=0$ will imply that $\omega=d I \omega$.

Now suppose that a differentiably simple covering $\mathfrak{U}=\left(U_{i}\right)_{i \in I}$ of $V$ is given once and for all. Let $N$ be the nerve of $\mathfrak{U}$. If $H=\left(i_{0}, i_{1}, \ldots, i_{p}\right)$ is a arbitrary sequence of elements of $I$ (whether distinct or not) then one lets $|H|$ denotes the set of distinct $i_{v}$. One intends the term differential co-element of bidegree ( $m, p$ ) to mean a system $\Omega=\left(\omega_{H}\right)=\left(\omega_{i_{0} i_{1} \cdots i_{p}}\right)$ of $p+1$ elements of $I$ such that $|H| \in N$, in which $\omega_{H}$ is a form that is defined in $U_{|H|}=\bigcap_{0 \leq v \leq p} U_{i_{v}}$ for any $H$. The co-element $\Omega$ will be called finite if it includes only a finite number of forms $\omega_{H} \neq 0$. It will be called alternating if $\omega_{H}=\omega_{i_{0} \cdots i_{p}}$ is an alternating function of the indices $i_{0}, \ldots, i_{p}$, which implies that this form will be zero if not all of the $i_{\nu}$ are distinct.

If $\Omega=\left(\omega_{H}\right)$ is a co-element of bidegree $(m, p)$ then $d \Omega=\left(d \omega_{H}\right)$ will be a co-element bidegree ( $m+1, p$ ). Since one is given a retraction $\varphi_{J}$ of $U_{J}$ for every $J \in N$, by hypothesis, one can define, as above, an operator $I_{J}$ for every $J \in N$ such that $\omega=d I_{J} \omega$ for any closed form $\omega$ of degree $m>$ 0 that is defined in $U_{J}$. If $\Omega=\left(\omega_{H}\right)$ is a co-element of bidegree $(m, p)$ then $I \Omega=\left(I_{|H|} \omega_{H}\right)$ will be a co-element of bidegree $(m-1, p)$. If $m>0$ then one will have $\Omega=I d \Omega+d I \Omega$, and as a result $d \Omega=0$ will imply that $\Omega=d I \Omega$. If $m=0$ then $\Omega=\left(f_{H}\right)$ will be a system of functions. Since the $U_{H}$ are retractable, and as a result connected, the $f_{H}$ will be constants if $d \Omega=0$. Therefore, $\Omega$ is nothing in this case but a system $\left(\xi_{H}\right)$ of real numbers that are attached to the sequences $H$ of $p+$ 1 elements of $I$ such that $|H| \in N$. As one knows, it is what one calls a cochain in $N$ (with real coefficients), which will be finite or alternating if $\Omega$ is finite or alternating, resp. It is clear that the operators $d, I$ transform the finite co-elements into finite co-elements and alternating co-elements into alternating co-elements.

Moreover, let $\Omega=\left(\omega_{H}\right)=\left(\omega_{i_{0} i_{1} \cdots i_{p}}\right)$ be a co-element of bidegree ( $m, p$ ). One says the coboundary of $\Omega$, and denotes it by $\delta \Omega$, to mean the co-element $\delta \Omega=\left(\eta_{i_{0} \cdots i_{p+1}}\right)$ of bidegree $(m, p+1)$ that is defined by:

$$
\eta_{i_{0} \cdots i_{p+1}}=\sum_{v=0}^{p+1}(-1)^{v} \omega_{i_{0} \cdots i_{v-1} i_{v+1} \cdots i_{p+1}},
$$

in which one intends that each of the terms on the right-hand side is replaced with the form that it induces on $U_{\left|i_{0} \cdots i_{p+1}\right|}$, which has meaning since the latter set is the intersection of the sets in which those terms are defined. Similarly, if $\omega$ is a form of degree $m$ that is defined on $V$, and if $\omega$ induces the form $\omega_{i}$ on $U_{i}$ then one can set $\delta \omega=\left(\omega_{i}\right) . \delta \omega$ is then an alternating co-element of bidegree ( $m$, 0 ) that is finite if $\omega$ has compact support, and only in that case. It is clear that $\delta$ commutes with $d$ and transforms any finite co-element into a finite co-element and any alternating co-element into an alternating co-element, and one immediately verifies that $\delta^{2}=0$.

In order to define the latter operator that we have need for, we give ourselves, once and for all, a differentiable partition of unity that is subordinate to the covering $\mathfrak{U}$. As we know, we intend that to mean a family $\left(f_{i}\right)_{i \in I}$ of differentiable functions that are $\geq 0$ on $V$ and are such that $\sum_{i \in I} f_{i}=1$ and the support of $f_{i}$ (i.e., the adherence to the set of points where $f_{i}>0$ ) is contained in $U_{i}$ for every $i$ $\in I$. Having said that, let $J \in N, i \in I$ and $J^{\prime}=J \cup\{i\}$. If $\omega$ is a form that is defined in $U_{J^{\prime}}$ then one agrees to let $f_{i} \omega$ denote the form that is defined in $U_{J}$ that is equal to $f_{i} \omega$ in $U_{J^{\prime}}$ and to 0 in $U_{J} \cap \mathrm{C}\left(U_{J^{\prime}}\right)$. Indeed, it is immediate that it is in fact a form (with differentiable coefficients) in $U_{J .}$. If $U_{J^{\prime}}$ does not belong to $N$, i.e., if $U_{J^{\prime}}=\varnothing$, then that definition will imply that $f_{i} \omega=0$. With that convention, if $\Omega=\left(\omega_{H}\right)$ is a co-element of bidegree ( $m, p$ ) with $p>0$ then we set $\left({ }^{3}\right) K \Omega=$ $\left(\zeta_{i_{0} \cdots i_{p-1}}\right)$, with:

[^1]$$
\zeta_{i_{0} \cdots i_{p-1}}=\sum_{k \in I} f_{k} \omega_{k i_{0} \cdots i_{p-1}}
$$
in which the terms on the right-hand side must be interpreted in the way that we just spoke of. Similarly, if $\Omega=\left(\omega_{i}\right)$ is a co-element of bidegree $(m, 0)$ then one lets $K \Omega$ denote the form $\omega=$ $\sum_{k \in I} f_{k} \omega_{k}$, in which must interpret $f_{k} \omega_{k}$ as the form that is defined on $V$ and is equal to $f_{k} \omega_{k}$ on $U_{k}$ and to 0 outside of it. Therefore, $\omega$ is a form that is defined on $V$. If $\Omega$ has bidegree $(m, p)$ and is finite then $K \Omega$ will be finite if $p>0$, and it will have compact support if $p=0$. If $\Omega$ is alternating and $p>0$ then $K \Omega$ will be alternating. One immediately verifies that one has $\Omega=K \delta \Omega+\delta K \Omega$, so $\delta \Omega=0$ will imply that $\Omega=\delta K \Omega$ for $p \geq 0$. If $\omega$ is a form that one will have $\omega=K \delta \omega$ and $\delta \omega$ $=0$ will then imply that $\omega=0$.

Under those conditions, consider all of the sequences $\left(\omega, \Omega_{0}, \Omega_{1}, \ldots, \Omega_{m-1}, \Xi\right)$, in which $\omega$ is a form of degree $m>0$ on $V, \Omega_{h}$ is a co-element of bidegree $(m-h-1, h)$ for $0 \leq h \leq m-1$, and $\Xi$ is a co-element of bidegree $(0, m)$ that satisfies the relations:

$$
\begin{equation*}
\delta \omega=d \Omega_{0}, \quad \delta \Omega_{h}=d \Omega_{h+1} \quad(0 \leq h \leq m-2) \quad \delta \Omega_{m-1}=\Xi \tag{I}
\end{equation*}
$$

If that is true then one will have $d \delta \Omega_{h}=0$ for $0 \leq h \leq m-1, d \Xi=0$, and $\delta \Xi=0$, and $\delta d \omega=0$, so $d \omega=K \delta d \omega=0$. Therefore $\omega$ belongs to the vector space $\mathfrak{F}_{m}($ on $\mathbb{R})$ of closed forms on $V$. $\Omega_{h}$ belongs to the vector space $\mathfrak{F}_{m, h}$ of co-elements of bidegree $(m-h-1, h)$ that satisfy $d \delta \Omega=0$. As for $\Xi$, since one has $d \Xi=0$, so as one has seen, one can consider it to be a cochain on $N$. Since $\delta \Xi=0$, it will be a cocycle. Therefore, $\Xi$ belongs to the vector space of cocycles of dimension $m$ on $N$ (with real coefficients). Suppose that $\Omega_{h}$ is given in $\mathfrak{F}_{m, h}$ and $h<m-1$. The relation $\delta \Omega_{h}=$ $d \Omega_{h+1}$ will then be satisfied for $\Omega_{h+1}=I \delta \Omega_{h}$. Suppose that $\Omega_{h}$ is in the sum $\mathfrak{H}_{m, h}$ of the subspaces of $\mathfrak{F}_{m, h}$, respectively, that are defined by the conditions $d \Omega=0$ and $\delta \Omega=0$. One will then have $\Omega_{h}=X+Y, d X=0, \delta Y=0$. Since $X$ has bidegree $(m-h-1, h)$, and one has $m-h-1>0, d X=$ 0 will imply that $X=d I X$. One will then have $\delta \Omega_{h}=\delta d(I X)$, so one will have $d Z=0$ when one sets $Z=\Omega_{h+1}-\delta I X$. Since one has $\Omega_{h+1}=\delta(I X)+Z, d Z=0, \Omega_{h+1}$ will be in $\mathfrak{H}_{m, h+1}$. In exactly the same way, one sees that if $\Omega_{h+1}$ is given in $\mathfrak{H}_{m, h+1}$ then the relation $\delta \Omega_{h}=d \Omega_{h+1}$ will be satisfied by $\Omega_{h}=K d \Omega_{h+1}$, and then that $\Omega_{h+1} \in \mathfrak{H}_{m, h+1}$ will imply that $\Omega_{h} \in \mathfrak{H}_{m, h}$. It will then follow that the relation $\delta \Omega_{h}=d \Omega_{h+1}$ determines an isomorphism between the vector spaces $\mathfrak{F}_{m, h} / \mathfrak{H}_{m, h}$ and $\mathfrak{F}_{m, h+1} / \mathfrak{H}_{m, h+1}$.

Similarly, if $\Omega_{0}$ is given in $\mathfrak{F}_{m, 0}$ then one will satisfy $\delta \omega=d \Omega_{0}$ when one takes $\omega=K d \Omega_{0}$. If $\Omega_{0}$ is in $\mathfrak{H}_{m, 0}$ then one will have $\Omega_{0}=X+Y, d X=0, \delta Y=0, Y=\delta(K Y)$, and $\delta \omega=d Y=\delta d(K Y)$, so upon setting $\eta=K Y, \delta(\omega-d \eta)=0$, so $\omega=d \eta$. Conversely, if $\omega$ is given in $\mathfrak{F}_{m}$ then one will satisfy $\delta \omega=d \Omega_{0}$ by taking $\Omega_{0}=I \delta \omega$. If $\omega=d \eta$ then one will have $\delta d \eta=d \Omega_{0}$, so upon setting
$X=\Omega_{0}-\delta \eta$, one will have $\Omega_{0}=X+\delta \eta, d X=0$, and therefore $\Omega_{0} \in \mathfrak{H}_{m, 0}$. Upon denoting the vector space of forms of degree $m$ that are homologous to 0 on $V$ by $\mathfrak{H}_{m}$, one will then see that the relation $\delta \omega=d \Omega_{0}$ will determine an isomorphism between the "de Rham group" $\mathfrak{F}_{m} / \mathfrak{H}_{m}$ and $\mathfrak{F}_{m, 0} / \mathfrak{H}_{m, 0}$. Finally, if $\Omega_{m-1}=X+Y, d X=0, \delta Y=0$ then one will have $\Xi=\delta X$, and $X$ will be a cochain in $N$, so $\Xi$ will be a coboundary in $N$. Conversely, if $\Xi$ is given and $\delta \Xi=0$ then one can satisfy $\delta \Omega_{m-1}=\Xi$ by taking $\Omega_{m-1}=K \Xi$. If $\Xi=\delta X$, in which $X$ is a cochain, i.e., a co-element that satisfies $d X=0$, then one will have $\Omega_{m-1}=X+Y, d X=0, \delta Y=0$. Hence, the relation $\delta \Omega_{m-1}=\Xi$ will determine an isomorphism between $\mathfrak{F}_{m, m-1} / \mathfrak{H}_{m, m-1}$, and the $m$-dimensional cohomology group $H^{m}(N)$ in $N$ with real coefficients. By definition, (I) will then establish an isomorphism between the de Rham group $\mathfrak{F}_{m} / \mathfrak{H}_{m}$ of $V$ and the group $H^{m}(N)$, and that isomorphism is determined canonically by the single given of the simple covering $\mathfrak{U}$.

One will see, moreover, that if one is given the closed form $\omega$ then one take $\Omega_{h}=(I \delta)^{h+1} \omega, \Xi$ $=\delta(I \delta)^{m} \omega$. Conversely, if one is given the cocycle $\Xi=\left(\xi_{i_{0} \cdots i_{m}}\right)$ then one can take $\Omega_{h}=$ $K(d K)^{m-h-1} \Xi, \omega=K(d K)^{m} \Xi$, i.e.:

$$
\omega=(-1)^{m(m-1) / 2} \sum_{i_{0}, i_{1}, \ldots, i_{m}} \xi_{i_{0} \cdots i_{m}} f_{i_{m}} d f_{i_{0}} \wedge \cdots \wedge d f_{i_{m}} .
$$

For $m=0$, one substitutes the single relation $\delta \omega=\Xi$ for the relations (I), and it will then be trivial for one to deduce the same results.

There will be nothing that has to be changed in the preceding discussion if one desires to consider the co-elements and alternating cochains exclusively. There will nothing that has to be changed if one desires to consider "currents" (which are the forms whose coefficients will be distributions instead of differentiable functions when one expresses them in terms of local coordinates) instead of forms. Finally, there will no longer be anything that has to be changed when one desires to consider the finite co-elements and cochains and forms with compact support exclusively. Of course, in that case, one will not arrive at the same groups as before, but one will obtain an isomorphism between the de Rham groups with compact support and the cohomology groups of $N$ relative to finite cochains.

Finally, suppose that one is given two closed forms $\omega, \omega^{\prime}$ of degrees $m, r$, respectively, and that one has formed two sequences $\left(\omega, \Omega_{0}, \ldots, \Omega_{m-1}, \Xi\right)$ and ( $\omega^{\prime}, \Omega_{0}^{\prime}, \ldots, \Omega_{r-1}^{\prime}, \Xi^{\prime}$ ) that satisfy (I). One can form a sequence ( $\omega^{\prime \prime}, \Omega_{0}^{\prime \prime}, \ldots, \Omega_{r-1}^{\prime \prime}, \Xi^{\prime \prime}$ ) that satisfies (I) and begins with the exterior product $\omega^{\prime \prime}=\omega \wedge \omega^{\prime}$ with no new integrations. Indeed, set $\Omega_{h}=\left(\omega_{i_{0} \cdots i_{h}}^{h}\right), \Xi=\left(\xi_{i_{0} \cdots i_{h}}\right)$, and similarly for $\Omega_{k}^{\prime}, \Xi^{\prime}$. One can then take:

$$
\begin{array}{ll}
\Omega_{h}^{\prime \prime}=\left(\omega_{i_{0} \cdots i_{h}}^{h} \wedge \omega^{\prime}\right) & (0 \leq h \leq m-1), \\
\Omega_{m+k}^{\prime \prime}=\left(\xi_{i_{0} \cdots i_{m}} \omega_{i_{m} \cdots i_{m+k}}^{\prime k}\right) & (0 \leq k \leq r-1), \\
\Xi^{\prime \prime}=\left(\xi_{i_{0} \cdots i_{m}} \xi_{i_{m} \cdots i_{m+r}}\right), &
\end{array}
$$

i.e., $\Xi^{\prime \prime}=\Xi \cup \Xi^{\prime}$. If one appeals to alternating co-elements exclusively then the formulas above will have to be modified. The simplest way of doing that is to order all of the $i \in I$ once and for all and agree that $\Omega_{m+k}^{\prime \prime}$ and $\Xi^{\prime \prime}$ are alternating and components that are given by the formulas above of $i_{0}<i_{1}<\ldots<i_{m+r}$. That will give Whitney's "cup product."

## § 3. - Singular cycles.

The hypotheses and notations are still the same as they were in § 2, and we now move on to the study of differentiable singular cycles.

Consider $m+1$ points $a_{0}, \ldots, a_{m}$ in an affine space. Let $K$ be the smallest convex set that contains the $a_{\mu}$, and let $L$ be the linear manifold that carries $K . L$ is the set of points $\sum_{\mu=0}^{m} x_{\mu} a_{\mu}$ for which $\sum_{\mu} x_{\mu}=1$, and $K$ is the set of points of that form for which $\sum_{\mu} x_{\mu}=1$ and $x_{\mu} \geq 0$ for any $\mu$. If $L$ has dimension $m$ then $K$ will be an $m$-dimensional Euclidian simplex whose vertices are $a_{0}$, $\ldots, a_{m}$. In particular, if $e_{\mu}$ is the vector in $\mathbb{R}^{m+1}$ whose $\mu^{\text {th }}$ component is 1 , while the others are zero, then one denotes the simplex with its vertices at $e_{0}, \ldots, e_{m}$ by $\Sigma^{m}$, i.e., the set of $x \in\left(x_{\mu}\right)$ in $\mathbb{R}^{m+1}$ such that $\sum_{\mu} x_{\mu}=1$ and $x_{\mu} \geq 0$ for any $\mu$.

Following S. Eilenberg $\left({ }^{4}\right)$, one intends the term $m$-dimensional differentiable singular simplex in $V$ to mean the restriction of a differentiable map $f$ of a neighborhood of $\Sigma^{m}$ into $V$ to $\Sigma^{m} . f\left(\Sigma^{m}\right)$ will be called the support of that simplex. Moreover, if $K$ and $L$ are defined as above by starting with the points $a_{0}, \ldots, a_{m}$, and $f$ is a differentiable map of a neighborhood of $K$ (in the ambient space or only in $L$ ) into $V$ then the restriction of the map $\left(x_{0}, \ldots, x_{m}\right) \mapsto f\left(\sum_{\mu} x_{\mu} a_{\mu}\right)$ to $\Sigma^{m}$ will be a differentiable singular simplex that will be denoted by $\left[f ; a_{0}, \ldots, a_{m}\right]$. It will be degenerate if $L$ has a dimension $<m$.

The word "differentiable" will generally be implicit in what follows. One interprets the term $m$-dimensional chain (or more explicitly, a differentiable singular chain) in $V$ with coefficients in an Abelian group $G$ to mean any expression of the form $t=\sum_{\rho} c_{\rho} s_{\rho}$, in which the $c_{\rho}$ belong to $G$ and the $s_{\rho}$ are $m$-dimensional singular simplexes in $V$ whose supports form a locally-finite family. Such an expression will be called reduced if all of the $s_{\rho}$ are distinct and all of the $c_{\rho}$ are non-zero. Any chain possesses one and only one reduced expression. The support $|t|$ of a chain $t$ will be the union of the supports of the simplexes that appear in the reduced expression for $t$. One says that $t$ is contained in a subset $U$ of $V$ if $|t| \subset U$. A chain is called finite if its reduced expression is a finite

[^2]sum or, what amounts to the same thing, its support is compact. If $t=\sum_{\rho} c_{\rho} s_{\rho}$ is a finite chain then one sets $\operatorname{deg}(t)=\sum_{\rho} c_{\rho}$.

If $s=\left[f ; a_{0}, \ldots, a_{m}\right]$ and one sets $s_{\mu}=\left[f ; a_{0}, \ldots, a_{\mu-1}, a_{\mu+1}, \ldots, a_{m}\right]$ then the finite chain $b s=$ $\sum_{\mu=0}^{m}(-1)^{\mu} s_{\mu}$ is called the boundary of $s$. That operator extends to chains by linearity. A chain with boundary zero is called a cycle. One will have $b^{2}=0$, which permits one to define the homology groups of $V$ by means of $b$ and the group of chains (or rather, the group of finite chains) with coefficients in $G$. If $t$ is 0 -dimensional then one will have $b t=0$, but one sets $b_{0} t=\operatorname{deg}(t)$ when $t$ is finite. One will have $b_{0} b t=0$ if $t$ is one-dimensional. More generally, one will have $\operatorname{deg}(b t)=$ $\operatorname{deg}(t)$ if $t$ has even dimension $m>0$, and $\operatorname{deg}(b t)=0$ in any other case.

Let $s$ be a singular simplex that is defined by a differentiable map $f$ of a neighborhood $W$ of $\Sigma^{m}$ into $V$. If $\omega$ is a form of degree $m$ in $V$ then its reciprocal image $\omega[f(x)]$ under $f$ will be a form of degree $m$ in $W$ whose integral over $\Sigma^{m}$ is, by definition, the integral $\int_{s} \omega$ of $\omega$ over $s$. That definition extends by linearity to finite chains with real coefficients if $\omega$ has compact support. One has Stokes's formula $\int_{s} d \omega=\int_{b s} \omega$, which is valid whenever $t$ is a finite chain or $\omega$ has compact support. Finite chains associated with forms by duality and chains with forms with compact support by means of $\int_{s} \omega$, which is a bilinear form in $t$ and $\omega$, which permits one to transpose the operations and results of § $\mathbf{2}$ to chains by duality. However, we shall give it an independent discussion in a manner that does not have to suppose that $G=\mathbb{R}$.

First let $U$ be an open subset of $V$ endowed with a differentiable retraction $\varphi$. Let $p$ be the constant value of $\varphi(x, t)$ for $t \leq 0$. One lets $\bar{s}_{m}$ denote the degenerate $m$-dimensional simplex [ $f$; $a, a, \ldots, a]$, in which $f(a)=p$, or what amounts to the same thing, the simplex that is defined by restriction of the constant map of $\mathbb{R}^{m+1}$ to $p$ to $\Sigma^{m}$. One has $b \bar{s}_{m}=\bar{s}_{m-1}$ is $m>0$ is even, and $b \bar{s}_{m}$ $=0$ if $m$ is odd or 0 . Consider a singular simplex $s=\left[f ; a_{0}, \ldots, a_{m}\right]$ in $U$, in which the $a_{\mu}$ are points in an affine space $E$. Let $a_{\mu}^{0}, a_{\mu}^{1}$ denote the points $\left(a_{\mu}, 0\right)$ and $\left(a_{\mu}, 1\right)$ in $E \times \mathbb{R}$. By definition, $f$ is a differentiable map of a neighborhood of the smallest convex set $K$ that contains the $a_{\mu}$ into $U$. If one sets $f^{\prime}(x, t)=\varphi[f(x), t]$ then $f^{\prime}$ will be a differentiable map of a neighborhood of $K \times \mathbb{R}$ into $U$. Under those condition, set:

$$
P s=\sum_{\mu=0}^{m}(-1)^{\mu}\left[f^{\prime} ; a_{0}^{0}, \ldots, a_{\mu}^{0}, a_{\mu}^{1}, \ldots, a_{m}^{1}\right]+\bar{s}_{m+1}
$$

and extend that operator by linearity to the finite chains in $U$. An easy calculation will give $b P s$ $+P b s=s$ for $m>0$ and $b P s+P b s=s-\bar{s}_{0}$ for $m=0$, so for any $m$-dimensional finite chain, $t$
$=b P t+P b t$ if $m>0$ and $t=b P t+\left(b_{0} t\right) \bar{s}_{0}$ if $m=0$. Therefore, $b t=0$ will imply $t=b P t$ if $m$ $>0$ and $b_{0} t=0$ will imply $t=b P t$ if $m=0$.

One intends the term $\mathfrak{U}$-simplex to mean a singular simplex that is contained in at least one of the sets $U_{i}$ of the covering $\mathfrak{U}$. One intends a $\mathfrak{U}$-chain to mean a chain whose simplexes are all $\mathfrak{U}$ simplexes. The application of our method demands that we should restrict ourselves to $\mathfrak{U}$-chains. From a theorem of S. Eilenberg $\left({ }^{5}\right)$, that will not change the homology groups. Let us recall the main points of his proof. Let $s=\left[f ; a_{0}, \ldots, a_{m}\right]$ be a singular simplex. Set $I_{\mu}=\{0,1, \ldots, \mu\}$ for $0 \leq$ $\mu \leq m$, and if $I=\left\{\mu_{1}, \ldots, \mu_{k}\right\}$ is an arbitrary subset of $I_{m}$, suppose that $a_{I}=\sum_{h=1}^{k}(1 / k) a_{\mu h}$. One will then call the finite chain:

$$
\sigma s=\sum_{\pi} \varepsilon_{\pi}\left[f ; a_{\pi\left(I_{0}\right)}, \ldots, a_{\pi\left(I_{m}\right)}\right]
$$

in which the sum extends over all permutations $\pi$ of $I_{m}$ and $\varepsilon_{\pi}= \pm 1$ according to whether $\pi$ is even or odd the barycentric subdivision of $s$. One extends the operator to chains by linearity. One verifies that one has $b \sigma=\sigma b$. On the other hand, Eilenberg (loc. cit., Note 5, pp. 429) defined another operator $\rho$ that is analogous, but its explicit expression is more complicated, such that $b \rho+\rho b=$ $\sigma-1 . \rho s$ is an $(m+1)$-dimensional finite chain that is a sum of terms of the form $\pm\left[f ; b_{0}, \ldots\right.$, $\left.b_{m+1}\right]$, in which each of the $b_{\mu}$ is one of the $a_{I}$. Having said that, if $s$ is a singular simplex then one can find an integer $v$ that is large enough that $\sigma^{v} s$ will be a $\mathfrak{U}$-chain. Let $n(s)$ be the smallest integer that has that property. Let $\tau$ be the operator on the singular simplexes that is defined by:

$$
\tau s=\rho\left(1+\sigma+\ldots+\sigma^{\nu(s)-1}\right) s,
$$

and extend it to chains by linearity. If one sets $b s=\sum_{\mu}(-1)^{\mu} s_{\mu}$, as above, then one will verify immediately that one has:

$$
(b \tau+\tau b) s=\left(\sigma^{\nu(s)-1}-1\right) s-\sum_{\mu}(-1)^{\mu} \sum_{j=\nu\left(s_{\mu}\right)}^{\nu(s)-1} \rho \sigma^{j} s_{\mu},
$$

which shows that $(1+b \tau+\tau b) s$ is a finite $\mathfrak{U}$-chain whose support is contained in that of $s$. Therefore, if $t$ is a chain then $\bar{t}=(1+b \tau+\tau b) t$ will be a $\mathfrak{U}$-chain that will be finite if $t$ is finite. If $t$ is a cycle then one will have $\bar{t}=t+b \tau t$, so $\bar{t}$ will be a cycle and $t$ will be homologous to $\bar{t}$. Moreover, the formula above shows that if $v(s)=0$, i.e., if $s$ is a $\mathfrak{U}$-simplex, then $(b \tau+\tau b) s=0$, so if $t$ is a $\mathfrak{U}$-chain then $\bar{t}=t$. Suppose that a $\mathfrak{U}$-chain $t^{\prime}$ is the boundary of a chain $t$. One will have $t^{\prime}=b t$, and $b \bar{t}=b t+b \tau b t=\vec{t}=t^{\prime}$, so $t^{\prime}$ is also the boundary of a $\mathfrak{U}$-chain. It will then
$\left(^{5}\right)$ S. Eilenberg, "Singular homology theory," Ann. Math. 45 (1944), pp. 407.
follow that the restriction to $\mathfrak{U}$-chains will not change the homology groups. From now on, we shall consider only them, and to abbreviate we shall say "chain" instead of " $\mathfrak{U}$-chain."

One intends the term a singular element of bidegree $(m, p)$ to mean a system $T=\left(t_{H}\right)=\left(t_{i_{0} \cdots i_{p}}\right)$ of $m$-dimensional finite chains $t_{H}$ that are attached to sequences $H=\left(i_{0}, \ldots, i_{p}\right)$, respectively, of $p$ +1 elements of $I$ such that $|H| \in N$ and $t_{H}$ is contained in $U_{|H|}$ for every $H$. The element $T$ will be called finite when the $t_{H}$ are all zero except for a finite number of them and alternating when $t_{i_{0} \cdots i_{p}}$ is an alternating function of its indices.

If $T=\left(t_{H}\right)$ is an element of bidegree ( $m, p$ ) then $b T=\left(b T_{H}\right)$ will be an element of bidegree ( $m$ $-1, p$ ) that is finite if $T$ is finite and alternating if $T$ is alternating; one has $b^{2}=0$. Moreover, if $m$ $=0$ then $b_{0} T=\left(b_{0} t_{H}\right)$ will make $H$ correspond to an element $b_{0} t_{H}$ of the group of coefficients $G$. That is what one calls a chain in $N$ with coefficients in $G$. If $m=1$ then one will have $b_{0} b T=0$.

Since the covering $\mathfrak{U}$ is simple, one can define operators $P_{J}$ on the $U_{J}$ that have the properties that were described above by means of retractions $\varphi_{J}$ that are attached to any $J \in N$, and in particular, if $t$ is a finite chain of dimension $m>0$ in $U_{J}$ then $b t=0$ will imply that $t=b P t$. If $T$ $=\left(t_{H}\right)$ is an element of bidegree ( $m, p$ ) then one sets $P T=\left(P_{\mid H} \mid t_{H}\right)$. It is an element of bidegree ( $m$ $+1, p$ ). If $m>0$ then $b T=0$ will imply that $T=b P T$. If $m=0$ then $b_{0} T=0$ will imply that $T=b$ $P T$. In general, one will have $T=b P T+P b T$ if $m>0 . P T$ is finite if $T$ is finite and alternating if $T$ is alternating.

On the other hand, if $T=\left(t_{i_{0} \cdots i_{p}}\right)$ is an element of bidegree ( $m, p$ ) and $p>0$ then we can define an element $\partial T=\left(u_{i_{0} \cdots i_{p-1}}\right)$ of bidegree $(m, p-1)$ by means of the formulas:

$$
u_{i_{0} \cdots i_{p-1}}=\sum_{\mu, k}(-1)^{\mu} t_{i_{0} \cdots i_{\mu-1} k i_{\mu} \cdots i_{p-1}},
$$

in which the summation must be extended over all values of $\mu, k$ for which $\left|i_{0} \ldots i_{\mu-1} k i_{\mu} \ldots i_{p-1}\right|$ $\in N$. There are a finite number of those values, and all of the terms in the right-hand side are finite chains in $U_{\left.\right|_{0} \cdots i_{p-1} \mid}$, so those formulas will indeed define an element $\partial T$ that will be finite if $T$ is finite. Similarly, if $T=\left(t_{i}\right)$ is an element of bidegree $(m, 0)$ then one sets $\partial T=\sum_{k} t_{k} . \partial T$ is a chain that will be finite if $T$ is finite. One has $\partial^{2}=0$, and $\partial$ commutes with $b$. On the other hand, one can also interpret the $T$ in the formula that defines $\partial T$ as a chain in $N$ since the $t_{H}$ will then be elements of $G$. That formula, in which the summation extends over the same value of $\mu, k$ as before, will then define $\partial T$ as a chain in $N$. Under those conditions, $\partial$ will commute with $b_{0}$.

One would like to define an operator $L$ such that $\partial T=0$ will imply that $T=\partial L T$. In order to do that, one agrees to choose, once and for all, one of the $U_{i}$ that is contained in any $\mathfrak{U}$-simplex $s$; let $U_{f(s)}$ be that set. Let $T=\left(t_{H}\right)$ be an element of bidegree ( $m, p$ ). Let $t_{H}=\sum_{\rho} c_{H}^{\rho} s_{\rho}$ be the reduced expression for $t_{H}$. If $H=\left(i_{0}, \ldots, i_{p}\right)$ then one sets $i H=\left(i i_{0}, \ldots, i_{p}\right)$. One will then define an element
$L T=\left(v_{H^{\prime}}\right)$ of bidegree $(m, p+1)$ by setting $v_{i H}=\sum_{f\left(s_{\rho}\right)=i} c_{H}^{\rho} s_{\rho}$ whenever $|i H| \in N$. That amounts to saying that the sum extends over all values of $\rho$ such that $f\left(s_{\rho}\right)=i$. Since $t_{H}$ is a finite sum, $v_{i H}$ will also be one, and each simplex $s_{\rho}$ that appears in $v_{i H}$ will be contained in $U_{|H|}$ since it appears in $t_{H}$, and in $U_{i}$ since $i=f\left(s_{\rho}\right)$, so it will also be contained in $U_{|i H|} . L T$ is indeed an element that will be finite if $T$ is finite. Similarly, if $t=\sum_{\rho} c_{\rho} s_{\rho}$ is the reduced expression for an $m$-dimensional $\mathfrak{U}$-chain then one can define an element $L t=\left(v_{i}\right)$ of bidegree ( $m, 0$ ) by way of $v_{i}=\sum_{f\left(s_{\rho}\right)=i} c_{\rho} s_{\rho}$ that will be finite if $t$ is finite. One has $T=\partial L T+L \partial T$ if $T$ is an element and $t=\partial L t$ if $t$ is a chain. Therefore, if $T$ is an element such that $\partial T=0$ then one will have $T=\partial L T$.

It is not true that $L T$ will be alternating whenever $T$ is alternating. If one would like to appeal to alternating elements exclusively then one must replace $\partial, L$ with operators $\partial^{\prime}, L^{\prime}$, resp., that are defined by the formulas:

$$
\partial^{\prime} T=\left(\sum_{k} t_{k_{0} \cdots \cdots i_{p-1}}\right)
$$

in which the summation extends over all $k$ such that $\left|k i_{0} \ldots i_{p-1}\right| \in N$, and:

$$
L^{\prime} T=\left(\sum_{\mu=0}^{p+1} \sum_{f\left(s_{\rho}\right)=i_{\mu}}(-1)^{\mu} c_{i_{0} \cdots i_{\mu-1} i_{\mu+1} \cdots i_{p+1}} s_{\rho}\right),
$$

with the same notations as above. One easily sees that they possess properties that are similar to those of $\partial$ and $L$ when one applies them to alternating elements and that they will transform them into alternating elements.

Now consider all of the sequences $\left(t, T_{0}, \ldots, T_{m}, Z\right)$, in which $t$ is a chain of dimension $m>0$ on $V, T_{h}$ is an element of bidegree $(m-h, h)$ for $0 \leq h \leq m$, and $Z$ is an $m$-dimensional chain of $N$ that satisfies the relations:

$$
\begin{equation*}
t=\partial T_{0}, \quad b T_{h}=\partial T_{h+1}, \quad(0 \leq h \leq m-1), \quad b_{0} T_{m}=Z \tag{II}
\end{equation*}
$$

If that is true then one will have $b \partial T_{h}=0(0 \leq h \leq m-1), b \partial T_{m}=0, b t=0$ and $\partial Z=0$. Therefore, $t$ belongs to the group $\mathfrak{C}_{m}$ of differentiable singular cycles on $V$ with coefficients in $G$, and $Z$ belongs to the group of cycles on $N$ with coefficients in $G . T_{h}$ belongs to the group $\mathfrak{C}_{m, h}$ of elements of bidegree $(m-h, h)$ that satisfy $b \partial T=$ for $h<m$ and $b_{0} \partial T=0$ for $h=m$. Let $\mathfrak{B}_{m}$ be the group of boundaries in $V$, i.e., the group of elements $\mathfrak{C}_{m}$ of the form $b t^{\prime}$. Let $\mathfrak{B}_{m, h}$, for $0 \leq h \leq m$, be the group of elements of $\mathfrak{C}_{m, h}$ of the form $b X+\partial Y$, in which $X, Y$ are elements of bidegrees $(m-h+$ $1, h)$ and ( $m-h, h+1$ ), respectively. One satisfies the relation $b T_{h}=\partial T_{h+1}$ by taking $T_{h}=P \partial T_{h+1}$, if $T_{h+1}$ is given in $\mathfrak{C}_{m, h+1}$, and $T_{h+1}=L b T_{h}$ if $T_{h}$ is given in $\mathfrak{C}_{m, h}$. One will satisfy $t=\partial T_{0}$ by taking
$T_{0}=L t$ if $t$ is given in $\mathfrak{C}_{m}$. Finally, it is clear that one can define a $T_{m}$ that satisfies $b_{0} T_{m}=Z$ if $Z$ is given. If $T_{h} \in \mathfrak{B}_{m, h}$, so if $T_{h}=b X+\partial Y$, then upon setting $U=T_{h+1}-b Y$, one will have $\partial U=0$, so $U=\partial L U$ and $T_{0}=b\left(L t^{\prime}\right)+\partial(L U) \in \mathfrak{B}_{m, 0}$. If $b_{0} T_{m}=Z$ and $T_{m}=b X+\partial Y$ then one will have $Z=\partial\left(b_{0} Y\right)$, so $Z$ is homologous to 0 and if $Z=\partial Z^{\prime}$ and $b_{0} T^{\prime}=Z^{\prime}$ then upon setting $X=T_{m}-$ $\partial T^{\prime}$, one will have $b_{0} X=0$, so $X=b P X$ and $T_{m}=b(P X)+\partial T^{\prime} \in \mathfrak{B}_{m, h}$. By definition, one sees that the relations (II) establish an isomorphism between the differentiable singular homology group $\mathfrak{C}_{m} / \mathfrak{B}_{m}$ for $V$ and the homology group of chains in $N$ for dimension $m$ and coefficient group $G$, and that isomorphism is canonically determined when one is given the simple covering $\mathfrak{U}$.

For $m=0$, one starts from the relations $t=\partial T_{0}, b_{0} T_{0}=Z$, in which $T_{0}$ is an element of bidegree $(0,0)$, and one will arrive at the same result by an argument that is analogous, but much simpler.

If one would like to consider finite elements and chains then nothing in the preceding needs to be changed. One will then obtain an isomorphism between the homology groups of $V$ and $N$ that are obtained by means of finite chains. There is nothing to change when one wishes to appeal to chains of class $C^{k}$, i.e., when the simplexes are defined by maps that are $k$-times continuously differentiable, when $k$ is an arbitrary integer. In that case, it would be sufficient that the retractions $\varphi_{J}$ should be themselves of class $C^{k}$. For $k=0$, one sees that one will obtain the same results by means of continuous singular chains, since the $\varphi_{J}$ are subject to only the condition that they should be continuous. In particular, that result applies to the simple covering of a locally-finite simplicial complex by open stars at the vertices (see § 1) and will then contain a proof of the topological invariance of the combinatorial homology groups of one such complex, which does not appear to be any different from the classical proof, moreover. There is nothing to change in the preceding if one would like to appeal to alternating elements exclusively, and to alternating chains in $N$, except that one must replace $\partial, L$ with $\partial^{\prime}, L^{\prime}$, resp.

If one takes $G=\mathbb{R}$ then the operators that one has defined by the singular elements are duallyrelated to the ones that one has defined on the differential co-elements. Indeed, let $\Omega=\left(\omega_{H}\right)$ and $T$ $=\left(t_{H}\right)$ be a differential co-element and a singular element, resp., both of which have bidegree ( $m$, $p$ ), and one of which is finite. One then sets:

$$
(T, \Omega)=\sum_{H} \int_{t_{H}} \omega_{H},
$$

and if both of them are alternating:

$$
(T, \Omega)^{\prime}=\frac{1}{(p+1)!}(T, \Omega)=\sum_{H}^{\prime} \int_{t_{H}} \omega_{H},
$$

in which $\Sigma^{\prime}$ indicates that one takes, once and for all, each combination $i_{0}, \ldots, i_{p}$ of $p+1$ elements in $I$, when arranged in an arbitrary order. It is $(T, \Omega)^{\prime}$ that one must appeal to in the alternating theory. Stokes's formula will give $(b T, \Omega)=(T, d \Omega)$, and one easily verifies that one has $(\partial T, \Omega)$
$=(T, \delta \Omega)$, and similarly $\left(\partial^{\prime} T, \Omega\right)^{\prime}=(T, \delta \Omega)^{\prime}$ if $T, \Omega$ are alternating. Similarly, if $\omega$ is a form of degree $m$ on $V$ and $T$ is an element of bidegree ( $m, 0$ ), and $\omega$ has compact support or $T$ is finite then one will have $(T, \delta \omega)=\int_{\partial T} \omega$. Finally, if $T$ is an element of bidegree $(0, p)$ and $\Xi$ is a co-element of bidegree $(0, p)$ that satisfies $d \Xi=0$, or in other words a cochain in $N$, and if $T$ or $\Xi$ is finite then one will have $(T, \Xi)=\left(b_{0} T, \Xi\right)$, whose right-hand side includes the scalar product of chains and cochains in $N$ that is defined by $(Z, \Xi)=\sum_{H} z_{H} \xi_{H}$ for $Z=\left(z_{H}\right), \Xi=\left(\xi_{H}\right)$. Those formulas must be modified in an obvious way in the alternating theory.

Therefore, consider two sequences $\left(\omega, \Omega_{0}, \ldots, \Omega_{m-1}, \Xi\right)$ and $\left(t, T_{0}, \ldots, T_{m}, Z\right)$ that satisfy the relations (I) in § 2 and the relations (II) above, respectively. Suppose that $\omega$ has compact support and the $\Omega_{h}$ and $\Xi$ are finite, of that $t$, the $T_{h}$, and $Z$ are finite. By means of the formulas above, one will immediately get:

$$
\int_{t} \omega=\ldots=\left(T_{m-1}, d \Omega_{m-1}\right)=\left(T_{m}, \delta \Omega_{m-1}\right)=(Z, \Xi) .
$$

It will then follow that the de Rham groups and the singular homology groups with real coefficients of $V$ have the same duality relations between them as the cohomology and homology groups of $N$. In particular, there always exists a closed form $\omega$ on $V$ such that $\int_{t} \omega$ is an arbitrarily-given linear function on the finite singular homology group of $V$, or on other words, it is equal to a linear function $L(t)$ that is given on the vector space of finite cycles of $V$ that is zero on the boundaries of the finite chains. On the other hand, if a closed form $\omega$ with compact support on $V$ is such that $\int_{t} \omega=0$ for any cycle $t$, whether finite or not, of $V$ then it will have the form $\omega=d \eta$, where $\eta$ has compact support. Indeed, from the preceding, in order to obtain those results, it would suffice to verify the analogous results for $N$, which is immediate.

The vector spaces that one deals with here will generally be infinite-dimensional if $V$ is not compact. One cannot hope to establish duality relations between them that will be completely satisfactory unless one introduces suitable topologies. That is a territory upon which we shall not trespass. Rather, if $V$ is compact then the covering $\mathfrak{U}$ will be finite. The preceding will then show that all of the homology groups of $V$ will then have finite type and will be zero above a certain dimension. In particular, on $\mathbb{R}$, all of those groups will be finite-dimensional vector space. One will then conclude from the preceding that the bilinear function $\int_{t} \omega$ will exhibit the duality of the de Rham group of degree $m$ and the differential homology group in dimension $m$ with real coefficients.

One can complete those results by considering the following remarks, which we will confine to the compact case so that every chain will be finite. It is immediate that any chain $t$ with real coefficients can be put into the form $t=\sum_{i} \xi_{i} t_{i}$, in which the $t_{i}$ are chains with integer coefficients
and the $\xi_{i}$ are real numbers that are linearly independent over the field $\mathbb{Q}$ of rationals. $b t=0$ will then imply that $b t_{i}=0$ for any $i$, so any real cycle will be a linear combination of integer cycles, and if an integer cycle $t^{\prime}$ is the boundary $b t$ of a real chain $t$ then upon putting $t$ into the form above, one will see that one of the $\xi_{i}$ (for example, $\xi_{1}$ ) must be rational and that one will then have $t^{\prime}=b\left(\xi_{1} t_{1}\right)$, so an integer multiple of $t^{\prime}$ will be the boundary of an integer cycle. Since the integer homology group of dimension $m$ has finite type, it will be the direct sum of a finite group and a free Abelian group that is generated by a finite number of integer homology classes. Let $t_{1}, \ldots, t_{r}$ be integer cycles that belong to those respective classes. From what was said before, the real homology classes of $t_{1}, \ldots, t_{r}$ will then define a basis for the real homology group in dimension $m$, when it is considered to be a vector space over $\mathbb{R}$, and one can identify the linear forms on the latter group with the homomorphisms of the integer homology group into $\mathbb{R}$, so such a form or homomorphism will be determined completely by its values on the classes of the cycles $t_{i}$.

One intends "the period of a form $\omega$ " to means its integral $\int_{t} \omega$ over an integer cycle $t$. For a well-defined choice of cycles $t_{1}, \ldots, t_{r}$, one often says "fundamental periods" of $\omega$ to mean the integrals of $\omega$ over the $t_{i}$. One will then see that it amounts to the same thing to give either the linear form $\int_{t} \omega$ on the real homology group of $V$, or the homomorphism $\int_{t} \omega$ of the integer homology group of $V$ into $\mathbb{R}$, or the fundamental periods of $\omega$. One will then recover "de Rham's theorems" in their classical form:

There exist closed forms on a compact differentiable manifold $V$ whose fundamental periods are given arbitrarily. Any closed form whose fundamental periods are zero will be homologous to 0 on $V$.

As for "de Rham's third theorem," one part of it is contained in the result at the end of § 2, which implied that the "cup product" of cocycles of $N$ will correspond to the exterior product of forms on $V$. In order to go from that statement to the classical statement of the same theorem, one must appeal to the Poincaré duality that was established by the intersection number between real cycles of dimensions $m$ and $n-m$, or rather (what basically amounts to the same thing) one must go to the product of the manifold with itself and then to the diagonal of that product. I shall not belabor those questions, which are already classical. However, it would not be superfluous to point out an important consequence of our results that one customarily deduces from de Rham's third theorem. We confine ourselves to the compact case. Consider a form $\omega$ on $V$ whose periods are all integers. Let $\Xi$ be a cocycle on $N$ that corresponds to $\omega$, which is a cocycle that is well-defined up to an arbitrary coboundary. $(Z, \Xi)$ will be an integer for any integer cycle $Z$. However, the group of integer cycles on $N$ is the subgroup of the group of integer chains that are determined by the condition $\partial Z=0$, so every integer chain such that some multiple of it is a cycle will be itself a cycle. From the theory of elementary divisors, the group of integer chains will then be the direct
sum of the group of integer cycles and another group in such a way that one can extend any homomorphism that is given on the group of cycles to the group of chains. Since any homomorphism of the group of integer chains into the additive group of integers can be written in the form $Z \rightarrow\left(Z, \Xi_{0}\right)$, in which $\Xi_{0}$ is an integer cochain, one will see that there exists an integer cochain $\Xi_{0}$ such that $\left(Z, \Xi_{0}\right)=(Z, \Xi)$ for any integer cycle $Z$, so it will also exist for any real cycle $Z$. It will then follow that $\Xi_{0}-\Xi$ is the coboundary of a real cochain, and therefore that $\Xi_{0}$ is, like $\Xi$, a cocycle that corresponds to $\omega$. As a result, in order for a form $\omega$ to correspond to a cocycle $\Xi$ with integer coefficients, it is necessary and sufficient that all of its periods should be integers. One then concludes from that and the final result in § $\mathbf{2}$ that if $\omega$ and $\omega^{\prime}$ have integer periods then the same thing will be true for their exterior product $\omega \wedge \omega^{\prime}$. Of course, one can also obtain the same result by passing to the product of $V$ with itself and appealing to the Künneth theorem.

## § 4. - Poincaré duality.

Everything that we have done up to now was, in reality, based upon just one property of the covering $\mathfrak{U}$, namely, that the $U_{J}$ are homologically trivial, i.e., they have the homology of a space that reduces to a point. True, we appealed to retractions $\varphi_{J}$, but only in order to obtain a presentation of things that would be both more elementary and more elegant, thanks to the possibility of defining the operators $I$ and $P$ explicitly. Therefore, the presentation above includes a proof (at least for singular homology) of Leray's theorem that said that if a covering $\mathfrak{U}$ of a space $X$ is such that the $U_{J}$ are homologically trivial then the homology of $X$ will be the same as that of the nerve $N$ of $\mathfrak{U}$.

On the other hand, since any simplicial complex admits a simple covering, it is obvious that the existence of such a covering will not imply the Poincaré duality theorem. In order to obtain that theorem on a manifold by means of the covering $\mathfrak{U}$, one must bring into play a property of the $U_{J}$ that has not been used up to now, namely, that their homology modulo their frontier is trivial in all dimensions except for the dimension $n$ of $V$. That is not an "elementary" property except insofar as differential forms are concerned. We also confine ourselves to them and consequently to Poincaré duality with real coefficients. For forms, the property of the $U_{J}$ in question is nothing but the following result, which is well-known and easy to prove by elementary means:

Let $\omega$ be a differential form with compact support that is contained in a convex open subset $U$ of $\mathbb{R}^{n}$. In order for $\omega$ to be the differential d $\eta$ of a form $\eta$ with compact support that is contained in $U$, it will then be necessary and sufficient that one should have $d \omega=0$ if $\omega$ has degree $<n$, and that one would have $\int_{U} \omega=0$ if $\omega$ has degree n.

Since the subsets $U_{J}$ that are formed by means of our simple covering $\mathfrak{U}$ of $V$ are differentiably isomorphic to convex open subsets in $\mathbb{R}^{n}$, the result above would be applicable to them.

Suppose that $V$ is orientable. In the contrary case, one must appeal to forms "of the second type" in the sense of de Rham, i.e., with "local coefficients" that are "twisted real" ones. That will not introduce any difficulty, but it will imply some complications of language that one would do better to avoid here since one would be dealing with only well-known results, moreover. One therefore supposes that all of the $U_{J}$ are oriented in a coherent manner by means of an orientation of $V$ that is chosen once and for all. It is over those $U_{J}$, thus oriented, that one will integrate the differential forms of degree $n$ with supports that are contained in those sets.

One intends a differential element of bidegree $(m, p)$ to mean a system $\Theta=\left(\theta_{J}\right)$ of forms of degree $m$ that are attached to the sequences $H$ of $p+1$ elements of $I$ such that $|H| \in N$, respectively, in which $\theta_{H}$ is a form of compact support that is contained in $U_{|H|}$. The element $\Theta$ will be called finite if the $\theta_{H}$ are zero except for a finite number of them. One sets $d \Theta=\left(d \theta_{H}\right)$; it will be an element of bidegree $(m+1, p)$. If $\Theta=\left(\theta_{H}\right)$ is an element of bidegree $(n, p)$ then one lets $\int \Theta$ denote the chains $Z=\left(z_{H}\right)$ on $N$ that is defined by $z_{H}=\int_{V} \theta_{H}=\int_{U_{|F|}} \theta_{H}$. One has $d^{2}=0$, and $\int d \Theta=0$ if $\Theta$ has bidegree $(n-1, p)$. In order for the element $\Theta$ of bidegree $(m, p)$ to have the form $d \Theta^{\prime}$, in which $\Theta^{\prime}$ has bidegree ( $m-1, p$ ), it is necessary and sufficient that $d \Theta=0$ if $m<n$, and that $\int \Theta$ $=0$ if $m=n$.

If $\Theta=\left(\theta_{H}\right)=\left(\theta_{i_{0} \ldots i_{p}}\right)$ is an element of bidegree ( $m, p$ ) then for $p>0$, one will define an element $\partial \Theta=\left(\eta_{i_{0} \ldots i_{p-1}}\right)$ of bidegree ( $m, p-1$ ) by means of the formula:

$$
\eta_{i_{0} \cdots i_{p-1}}=\sum_{\mu, k}(-1)^{\mu} \theta_{i_{0} \cdots i_{\mu-1} k i_{\mu} \cdots i_{p}},
$$

in which the summation extends over the values of $m, k$ for which one has $\left|i_{0} \ldots i_{\mu-1} k i_{\mu} \ldots i_{p-1}\right|$ $\in N$. There are a finite number of those values, and each term on the right-hand side will be a form with compact support that is contained in $U_{\left|i_{0} \cdots i_{p-1}\right|}$, so that formula will indeed define an element $\partial \Theta$, if $\Theta$ is finite. Similarly, if $\Theta=\left(\theta_{i}\right)$ has bidegree $(m, 0)$ then one sets $\partial \Theta=\sum_{k} \theta_{k} . \partial \Theta$ will then be a form on $V$ with compact support if $\Theta$ is finite. One has that $\partial^{2}=0$ and that $\partial$ commutes with $d$ and $\int$.

If $\left(f_{i}\right)$ once more denotes a differentiable partition of unity subordinate of $\mathfrak{U}$ then one lets $L$ denote the operator that makes any element $\Theta=\left(\theta_{H}\right)$ of bidegree ( $m, p$ ) correspond to the element $L \Theta=\left(\zeta_{H^{\prime}}\right)$ of bidegree $(m, p+1)$ that is defined by $\zeta_{i H}=f_{i} \theta_{H}$. Similarly, if $\theta$ is a form of degree $m$ on $V$ then one lets $L \theta$ denote the element of bidegree $(m, 0)$ that is defined by $L \theta=\left(f_{i} \theta\right)$; one will then have $\theta=\partial L \theta$. If $\Theta$ is an element then one will have $\Theta=\partial L \Theta+L \partial \Theta$. Therefore $\partial \Theta=0$ will imply that $\Theta=\partial L \Theta$.

Having said that, the theory in § $\mathbf{3}$ will apply with no changes if one replaces the differential elements of bidegree $(n-m, p)$ with the singular elements of bidegree ( $m, p$ ), the forms of degree $n-m$ with chains of dimension $m$, and the operators $d, \int, \partial$ with operators $b, b_{0}, \partial$. One then starts from the relations:

$$
\begin{equation*}
\theta=\partial \Theta_{0}, \quad d \Theta_{h}=\partial \Theta_{h+1} \quad(0 \leq h \leq m-1), \quad \int \Theta_{m}=Z, \tag{III}
\end{equation*}
$$

in which $\theta$ is a form of degree $n-m, \Theta_{h}$ is a differential element of bidegree $(n-m+h, h)$ for $0 \leq$ $h \leq m$, and $Z$ is a chain in $N$ in dimension $m$ with real coefficients, and one concludes, as in § 3, that (III) establishes an isomorphism between the de Rham group of $V$ of degree $n-m$ and the homology group of $N$ in dimension $m$ with real coefficients. Nothing needs to be changed if one restricts oneself to forms with compact support on $V$ and to finite elements and chains. Naturally, one can then appeal to alternating elements by modifying $\partial$ and $L$ in the manner that was described in § 3 .

Finally, the duality that was established in § $\mathbf{3}$ between differential co-elements and singular elements can be transported to differential co-elements and elements here. If $\Omega=\left(\omega_{H}\right)$ is a differential co-element of bidegree ( $m, p$ ) and $\Theta=\left(\theta_{H}\right)$ is a differential element of bidegree ( $n-$ $m, p)$, and one of them is finite then one can set $(\Theta, \Omega)=\sum_{H} \int_{U_{|F|}} \theta_{H} \wedge \omega_{H}$. One will then have $(\partial \Theta$, $\Omega)=(\Theta, \delta \Omega)$, but this time the Stokes formula will give $(d \Theta, \Omega)=(-1)^{n-m}(\Theta, d \Omega)$ if $\Theta$ has bidegree $(n-m-1, p)$ and $\Omega$ has bidegree $(m, p)$ in such a way that for two sequences that satisfy (III) and (I), respectively, one will have:

$$
\int_{V} \theta \wedge \omega=(-1)^{m n+m(m-1) / 2}(Z, \Xi) .
$$

The conclusion is that the duality relations between homology and cohomology of $N$ will be transported to the de Rham groups in complementary dimensions on $V$. In particular, if $V$ is compact then one will see that the bilinear form $\int_{V} \theta \wedge \omega$ exhibits the duality between the de Rham groups of degrees $n-m$ and $m$, respectively.

By way of example, consider the groups that relate to dimensions 0 and $n$. To simplify the language, suppose that $V$ is connected, since the general case can be trivially deduced from that by forming direct sums or products according to whether one is or is not dealing with groups with compact support, resp. The groups in dimension 0 are determined immediately. The finite homology group of $V$ in dimension 0 is free and is generated by the class of a cycle that reduces to a point. If $V$ is not compact then the infinite homology group in dimension 0 will be annulled. The de Rham group of degree 0 with arbitrary support is generated by the form 1 , and the same group with compact support will be annulled when $V$ is not compact. From the results of the present $\S$, one then concludes that de Rham group of degree $n$ with compact support is generated by the class of a form $\omega_{0}$ of degree $n$ such that $\int_{V} \omega_{0}=1$, and that the de Rham group of degree $n$ with
arbitrary support will be annulled if $V$ is not compact, and in order for a form $\omega$ of degree $n$ with compact support to be written $\omega=d \eta$, with $\eta$ having compact support, it is necessary and sufficient that one must have $\int_{V} \omega=0$. By means of the results of $\S \mathbf{3}$, one can then conclude that there exists a differentiable singular cycle $t_{0}$ of dimension $n$ such that $\int_{t_{0}} \omega_{0}=1$. Therefore, if $\omega$ has compact support then one will have $\omega=c \omega_{0}+d \eta$, with $c=\int_{V} \omega$ and $\eta$ has compact support, so $\int_{t_{0}} \omega=c$, and as a result, one will have $\int_{t_{0}} \omega=\int_{V} \omega$ for any $\omega$ with compact support, which will imply that the support of $t_{0}$ is $V$. One can also conclude that any finite cycle $t$ such that $\int_{t} \omega_{0}=0$ is the boundary of a finite chain. Therefore, the homology group in dimension $n$ with compact support and real coefficients will be annulled if $V$ is not compact. If one supposes that $V$ is compact then one can conclude, moreover, that the homology group of $V$ in dimension $n$ with real coefficients is generated by $t_{0}$. However, we have not proved that we can take $t_{0}$ to be an integer cycle, nor have we proved that this cycle generates the homology group of $V$ in dimension $n$ with integer coefficients. In order to do that, we must make use of either a triangulation of $V$ or a theory of the degree of the map for differentiable maps or use more powerful topological methods than the ones that Cartan's theory provides, although, of course, it contains the results in question.

## § 5. - The homotopy theorem.

As was remarked before, the fact that the nerve $N$ of $\mathfrak{U}$ has the same homology as $V$ depends upon only the homological properties of the sets $U_{J}$. If one recalls that they are homotopically trivial then one will obtain a result that is much more precise. It is that $N$ has the same homotopy type as $V$. It will then follow that $N$ can be substituted for $V$ in any problem that depends upon only the homotopy type, and for example in most question that are concerned with fiber bundles with base $V$. Under those circumstances, the nerve of a simple covering of $V$ can often serve the same purpose as a triangulation of $V$. It seems that one will have in that an elementary tool that is very practical for the study of manifolds. It also showed its applicability in a study that G. de Rham recently made of the invariants that are called "torsion invariants" $\left({ }^{6}\right)$. It might happen then that the nerves of simple coverings have properties that are even more precise than the ones that will now be pointed out.

The result that follows has a purely-topological nature. In order to state it, recall that one says that a space $B$ has the extension property if any continuous map of a closed subset $X$ of a normal space $A$ into $B$ can be extended to a continuous map of $A$ into $B$.

[^3]Therefore, let $\mathfrak{U}=\left(U_{i}\right)_{i \in I}$ be a locally-finite covering of a space $E$ by open subsets $U_{i}$; let $N$ be its nerve. One says that $\mathfrak{U}$ is topologically simple if the set $U_{J}=\bigcap_{i \in J} U_{i}$ possesses the extension property for every $J \in N$.

Our theorem is then stated as follows $\left({ }^{7}\right)$ :

If $E$ is a space such that $E \times E \times[0,1]$ is normal, and if $\mathfrak{U}$ is a topologically simple covering of $E$ then the nerve $N$ of $\mathfrak{U}$ will have the same homotopy type as $E$.

The proof appeals to the following lemma:

## Lemma:

Let $E$ be a space such that $E \times E \times[0,1]$ is normal. Let $\left(X_{i}\right)_{i \in I}$ be a locally-finite family of closed subsets of $E$; let $N$ be its nerve. For $J \in N$, let $X_{J}=\bigcap_{i \in J} X_{i}$. Let $\left(U_{J}\right)_{J \in N}$ be a family of subsets of $E$ such that $U_{J}$ has the extension property and contains $X_{J}$ for all $J \in N$ and one has $U_{J} \subset U_{J^{\prime}}$ whenever $J \supset J^{\prime}, J \in N, J^{\prime} \in N$. There will then exist a continuous map $F(x, y, t)$ of $\bigcup_{i \in I}\left(X_{i} \times X_{i}\right.$ $\times[0,1])$ into $E$ such that for all $J \in N, x \in X_{J}$ and $y \in X_{J}$ will imply that $F(x, y, t) \in U_{J}$ and $F(x$, $x, t)=x$ for any $t, F(x, y, 0)=x$ and $F(x, y, 1)=y$.

For any subset $N^{\prime}$ of $N$, set $Y\left(N^{\prime}\right)=\bigcup_{J \in N^{\prime}}\left(X_{J} \times X_{J} \times[0,1]\right)$. Consider all continuous maps $F^{\prime}$ of sets $Y\left(N^{\prime}\right)$ into $E$ that satisfy all of the conditions of the lemma wherever they are defined. One orders them by saying that $F^{\prime} \succ F^{\prime \prime}$ if $Y\left(N^{\prime}\right) \supset Y\left(N^{\prime \prime}\right)$ and $F^{\prime}$ coincides with $N^{\prime \prime}$ on $Y\left(N^{\prime \prime}\right)$. When one recalls that $\left(X_{i}\right)$ is locally finite and that, as a result, any $x \in E$ will have a neighborhood that meets a finite number of the $X_{J}$, one will then see immediately that one can apply Zorn's theorem to the $F^{\prime}$, thus-ordered. Suppose that there exist $J \in N$ such that $X_{J} \times X_{J} \times[0,1]$ is not contained in $Y\left(N^{\prime}\right)$. Among the finite number of $J^{\prime} \in N$ that contain $J$, take one of them that has the same property and has the greatest-possible number of elements. Upon replacing $J$ with that one, one will see that one can suppose, moreover, that $X_{J^{\prime}} \times X_{J^{\prime}} \times[0,1] \subset Y\left(N^{\prime}\right)$ for all $J^{\prime} \neq J$ such that $J^{\prime} \supset J$. Since $X_{J} \times X_{J} \times[0,1]$ is a closed subset of $E \times E \times[0,1]$, it will be a normal space. The points $(x, y, t)$ in that space that satisfy $x=y$ at $t=0$ and $t=1$ form closed subsets. Their intersection with $Y\left(N^{\prime}\right)$ is also closed by reason of the locally-finite character of the family $\left(X_{i}\right)$. It will then follow that there is a continuous map $G(x, y, t)$ of $X_{J} \times X_{J} \times[0,1]$ into $U_{J}$ that coincides

[^4]with $F^{\prime}$ in the intersection of the set with $Y\left(N^{\prime}\right)$ and satisfies $G(x, x, t) x, G(x, y, 0)=x, G(x, y$, $1)=y$. Show that the function that coincides with $F^{\prime}$ on $Y\left(N^{\prime}\right)$ and with $G$ on $X_{J} \times X_{J} \times[0,1]$ has all of the properties that were stated in the lemma, contrary to the hypothesis that $F^{\prime}$ cannot be extended. The only point to verify is that if $J^{\prime} \in N$ and if $x$ and $y$ are in $X_{J} \cap X_{J^{\prime}}$ then $G(x, y, t)$ will be in $U_{J^{\prime}}$. It is obvious that if $J^{\prime} \subset J$ then one will have $U_{J^{\prime}} \supset U_{J}$. In the contrary case, one sets $J^{\prime \prime}=J \cup J^{\prime}$. One will have $J^{\prime \prime} \neq J^{\prime}$ and $J^{\prime \prime} \in N$, so by virtue of the hypothesis that was made on $J$, one will have $X_{J^{\prime \prime}} \times X_{J^{\prime \prime}} \times[0,1] \subset Y\left(N^{\prime}\right)$, so $F^{\prime}(x, y, t) \in U_{J^{\prime \prime}} \subset U_{J^{\prime}}$, hence, the stated conclusion will be true since $G$ coincides with $F^{\prime}$ at $(x, y, t)$. Therefore, for any $J \in N$, one will have $X_{J} \times X_{J} \times[0,1] \subset Y\left(N^{\prime}\right)$, and in particular, $X_{i} \times X_{i} \times[0,1] \subset Y\left(N^{\prime}\right)$ for every $i \in I$. Therefore, $F^{\prime}$ is the function $F$ that one wishes to construct.

## Corollary:

With the same hypotheses as in the lemma, let $f, f^{\prime}$ be two continuous maps of a space $A$ into $E$ such that for any $u \in A$, there will be an $i \in I$ for which $f(u) \in X_{i}$ and $f^{\prime}(u) \in X_{i} . f$ and $f^{\prime}$ will then be homotopic.

Indeed, $F\left(f(u), f^{\prime}(u), t\right)$ is a homotopy that connects $f$ to $f^{\prime}$.

We can now move on to the proof of our theorem. First, let $\mathfrak{U}=\left(U_{i}\right)$ be no particular locallyfinite covering of a normal space $E$ with open subsets $U_{i}$. There will then be a partition of unity $\left(f_{i}\right)$ subordinate to $\mathfrak{U}$. For $p \in E$, set $f(p)=\left(f_{i}(p)\right) . f$ will be a continuous map of $E$ into the nerve $N$ of $\mathfrak{U}$, that will be realized geometrically in conformity with the definition that was recalled at the end of $\S$ 1. If $p \in E$ and $J$ is the set of $i \in I$ such that $p \in U_{i}$ then $f(p)$ will be in the simplex $\Sigma_{J}$ of $N$. Therefore, if $\left(f_{i}^{\prime}\right)$ is another partition of unity that is subordinate to $\mathfrak{U}$ then the line segment that connects $f(p)$ and $f^{\prime}(p)$ is contained in $\Sigma_{J}$, so it will be contained $N$. As a result, the map $p \rightarrow$ $(1-t) f(p)+t f^{\prime}(p)$ is a homotopy that connects $f$ to $f^{\prime}$. The homotopy class of $f$ is then determined completely by the given of $\mathfrak{U}$.

Now suppose that the homotopy groups for all of the $U_{J}=\bigcap_{i \in J} U_{i}$ are null for $J \in N$. In other words, any continuous map of the frontier of a simplex in dimension $m$ into one of the $U_{J}$ can be extended to the entire simplex. For $m=1$, that will amount to saying that $U_{J}$ is path-connected. For any $J \in N$, let $e_{J}$ be the center of gravity of $\Sigma_{J}$. Consider all increasing sequences $J_{0} \subset J_{1} \subset \ldots$ $\subset J_{m}$ of elements of $N$ that are all distinct. For one such sequence, let $\Sigma^{\prime}\left(J_{0}, \ldots, J_{m}\right)$ be the simplex with its vertices at $e_{J_{0}}, \ldots, e_{J_{m}}$. $N$ is the union of all of those simplexes, which will form a barycentric subdivision. One then defines a continuous map $g$ of $N$ into $E$ by recurrence such that $g\left(\Sigma^{\prime}\left(J_{0}, \ldots, J_{m}\right)\right) \subset U_{J_{0}}$ for every sequence $J_{0}, \ldots, J_{m}$. One takes $g\left(e_{J}\right)$ arbitrarily in $U_{J}$ for any $J$ $\in N$. Suppose that $g$ is defined on the simplexes of the barycentric subdivision of $N$ of dimension
$\leq m-1 . g$ will then be defined on the frontier of the simplex $\Sigma^{\prime}\left(J_{0}, \ldots, J_{m}\right)$, which is the union of the simplexes $\Sigma_{\mu}^{\prime}=\Sigma^{\prime}\left(J_{0}, \ldots, J_{\mu-1}, J_{\mu+1}, \ldots, J_{m}\right)$ for $0 \leq \mu \leq m$. From the recurrence hypothesis, one will have $g\left(\Sigma_{0}^{\prime}\right) \subset U_{J_{1}} \subset U_{J_{0}}$, and $g\left(\Sigma_{\mu}^{\prime}\right) \subset U_{J_{0}}$ for $1 \leq \mu \leq m$. Therefore, one can extend $g$ to a map of $\Sigma^{\prime}\left(J_{0}, \ldots, J_{m}\right)$ into $U_{J_{0}}$. Moreover, if $g^{\prime}$ is another map of $N$ into $E$ that satisfies the same condition then one can construct a homotopy that joins $g$ to $g^{\prime}$ by an entirely-analogous recurrence. The homotopy class of $g$ is therefore well-defined by the condition that one has imposed upon it.

Now show that under those conditions, $f \circ g$ will be a map of $N$ into $N$ that is homotopic to the identity map. Indeed, let $F_{i}$ be the union of the images of $g$ under all of the simplexes $\Sigma^{\prime}\left(J_{0}\right.$, $\ldots, J_{m}$ ) for which $i \in J_{0}$. Since there are a finite number of those simplexes, $F_{i}$ will be a compact subset of $U_{i}$. For each $i$, let $U_{i}^{\prime}$ be an open subset of $U_{i}$ that contains $F_{i}$ such that $\bar{U}_{i}^{\prime} \subset U_{i}$, and that the $U_{i}^{\prime}$ again form a covering of $E$. Since the choice of the partition $\left(f_{i}\right)$ subordinate to $\mathfrak{U}$ has no effect upon the homotopy class of $f$, one can suppose that it is chosen in such a way that $f_{i}>0$ on $F_{i}$ and $f_{i}=0$ outside of $U_{i}^{\prime}$ for every $i \in I$. Let $x=\left(x_{i}\right) \in N$ and set $\varphi_{i}(x)=\min \left[x_{i}, f_{i}(g(x))\right]$. The $\varphi_{i}$ are continuous functions $\geq 0$ on $N$, and one will have $\varphi_{i}(x)=0$ if $x_{i}=0$, i.e., if $x$ does not belong to $S t_{i}$. Moreover, from what was just shown, for every $x \in N$, there will be an $i$ such that $\varphi_{i}(x)>0$. It will then result that $\varphi=\sum_{i} \varphi_{i}$ is a continuous function that is $>0$ everywhere on $N$, and as a result that the $h_{i}=\varphi_{i} / \varphi$ will form a partition of unity on $N$ that is subordinate to the covering $\left(S t_{i}\right)$. Therefore, if one sets $h(x)=\left(h_{i}(x)\right)$ then $h$ will be a map of $N$ into $N$. If $J$ is the set of $i \in I$ such that $h_{i}(x)>0$ when $x \in N$ then one will have $x_{i}>0$ and $f_{i}(g(x))>0$ for all $i \in J$. Therefore $h(x)$ will be in $\Sigma_{J}$, and $x$ and $f\left(g(x)\right.$ will both be in $S t_{J}$ in such a way that the line segments that connect $h(x)$ to $x$, on the one hand, and to $f(g(x))$, on the other, will be contained in $N$. As before, one can conclude from this that $h$ is homotopic to the identity map, on the one hand, and to $f \circ g$, on the other.

Finally, let $p \in E$, and let $J$ be the set of $i \in I$ such that $f_{i}(p)>0$. One will then have $p \in U_{i}^{\prime}$ for every $i \in J$. One will have $f(p) \in \Sigma_{J}$, so $f(p)$ will belong to one of the simplexes of the barycentric subdivision of $\Sigma_{J}$. However, they will be the simplexes $\Sigma^{\prime}\left(J_{0}, \ldots, J_{m}\right)$ with $J_{m} \subset J$, with the notations that were employed above. Therefore, if one can take $i \in J_{0}$ then one will have $g(f$ $(p)) \in F_{i}$. Hence, $p$ and $g(f(p))$ are both in $U_{i}^{\prime}$. Set $X_{i}=\bar{U}_{i}^{\prime}$. Let $N^{\prime}$ be the nerve of the family $\left(X_{i}\right)$. One will have $N^{\prime} \subset N$. If one further supposes now that the $U_{J}$ have the extension property, i.e., that U is topologically simple, then one will see that the families $\left(X_{i}\right)_{i \in I}$ and $\left(U_{J}\right)_{J \in N^{\prime}}$ will satisfy all of the conditions of the lemma that was just proved. From the corollary to that lemma, one can then assert that $g \circ f$ is homotopic to the identity map on $E$ provided that $E \times E \times[0,1]$ is normal. The stated theorem is therefore proved completely.

Suppose, in particular, that one of the $U_{i}$ is covered by the union of the others, so one will have $U_{i}=\bigcup_{j \neq i} U_{i j}$, and that $U_{i} \times U_{i} \times[0,1]$ is normal. The non-vacuous $U_{i j}$ will then define a
topologically-simple covering of $U_{i}$ whose nerve has the same homotopy type as $U_{i}$. That type is trivial since $U_{i}$ has the extension property and is therefore contractible. If one omits $U_{i}$ from the covering $\mathfrak{U}$ then what remains will again be a covering $\mathfrak{U}^{\prime}$ of $E$ by virtue of the hypothesis. The nerve $\mathfrak{U}^{\prime}$ of $\mathfrak{U}$ is deduced from $N$ by subtracting $S t_{i}$, and the frontier of $S t_{i}$ will be nothing but the nerve of the covering $\left(U_{i j}\right)_{j \neq i}$ of $U_{i}$, so it will be a finite complex that is homotopically trivial (i.e., it is contractible). As one can easily see, that is equivalent to saying that there exists a retraction of the adherence $\overline{S t_{i}}$ of $S t_{i}$ onto its frontier $\overline{S t_{i}} \cap N^{\prime}$, so there will be a retraction of $N$ onto $N^{\prime}$, and similarly there will exist such a retraction that depends continuously on one parameter, i.e., a continuous map $F(x, t)$ of $N \times[0,1]$ into $N$ such that $F(x, 0)=x$ and $F(x, 1) \in N^{\prime}$ for every $x \in$ $N, F(x, t)=x$ for every $t$ and every $x \in N^{\prime}$, and $F(x, t) \in \overline{S t_{i}}$ for every $t$ and every $x \in \overline{S t_{i}}$. In particular, de Rham has shown (loc. cit., Note 6) that if one confines oneself to considering the family of simple coverings that he called "convex" of a compact differentiable manifold, one can always pass from one of those coverings to the other by successive insertions and omissions of superfluous sets. The result that we just proved in a manner that is a bit more precise that of de Rham is the effect of those operations in the nerves of the corresponding coverings.
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[^0]:    ( ${ }^{1}$ ) Harvard Course, 1948. Seminar at l'E.N.S., Paris 1948-1949 and 1950-1951.

[^1]:    $\left({ }^{3}\right)$ I must thank N. Hamilton for the operator $K$. My original proof appealed to Whitney's prolongation theorem, instead of $K$.

[^2]:    $\left({ }^{4}\right)$ S. Eilenberg, "Singular homology in differentiable manifolds," Ann. Math. 48 (1947), pp. 670.

[^3]:    $\left({ }^{6}\right)$ Loc. cit., note 2.

[^4]:    $\left({ }^{7}\right)$ In the previously-cited work (Note 2), de Rham reproduced part of the proof that follows, but reduced to something that would suffice for the special case that he had in mind. A result that relates to ours was published by K. Borsuk for finite-dimensional spaces ["On the imbedding of systems of compacta in simplicial complexes," Fund. Math. 35 (1948), pp. 217]. It would seem that the proofs have nothing in common.

