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On infinitesimal geometry: *p*-dimensional surfaces in an *n*-dimensional space

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The first problem in infinitesimal geometry is to consider an isolated continuous *manifold by itself.* The next thing to address is the study of a p-dimensional *manifold* ("surface") *that is embedded in a higher-dimensional manifold* (the n-dimensional "space"). That problem, of which the theory of curves and surfaces in three-dimensional Euclidian space is a special case, shall be discussed here. I would like to show how the basic concepts and formulas of that theory can be obtained from the infinitesimal geometry of the isolated manifold in a unified, intuitive way with no new calculations.

Let $x_1, x_2, \ldots, x_n$ be coordinates in n-dimensional space, and let $y_1, y_1, \ldots, y_p$ be coordinates on the p-dimensional surface. The equations of embedding, which give the spatial location x_J where an arbitrary surface point $P = (y_\alpha)$ is found, might read:

$$x_J = x_J\,(y_1, y_1, \ldots, y_p) \qquad (J = 1, 2, \ldots, n)\ .$$

Inside of the n-dimensional vector space that belongs to P is established the p-dimensional *tangent space* $\mathfrak{T} = \mathfrak{T}_P$, which is spanned by the vectors:

$$\mathbf{e}_\alpha = \left(e_\alpha^1, e_\alpha^2, \ldots, e_\alpha^n\right) = \left(\frac{\partial x_1}{\partial y_\alpha}, \frac{\partial x_2}{\partial y_\alpha}, \ldots, \frac{\partial x_n}{\partial y_\alpha}\right) \qquad (\alpha = 1, 2, \ldots, p)\ .$$

I. – We next assume that *space carries an affine connection.* In order to be able to define it on the surface, we must assume that the arbitrary surface point P is associated with not only the p-dimensional tangent space, but also a $q = (n - m)$-dimensional *normal space* $\mathfrak{N} = \mathfrak{N}_P$. It likewise consists of a linear family of vectors in P. $\mathfrak{T}$ and $\mathfrak{N}$ can have no vector besides 0 in common, such that each vector in P will be additively composed of a tangent vector and a normal vector in one and only way. I would like to refer to a *complemented* surface in space under those circumstances in a way that should be easy to understand. Let $\mathbf{e}_i$ ($i = p + 1, \ldots, n$) be q independent vectors that span the normal space. If I apply the splitting of the vector space into $\mathfrak{T} + \mathfrak{N}$ to the parallel displacement of an arbitrary vector in P to the infinitely-close surface point P' then I will get the following:

1. A tangent vector $\mathbf{t}$ at P will give rise to a vector $\mathbf{t}' + d\mathbf{n}$ at P' ($\mathbf{t}'$ tangent, $\mathbf{n}$ normal). I shall refer to the rule $\mathbf{t} \to \mathbf{t}'$, by which a surface vector at P will go to a surface vector P', as the *affine connection on the surface.* In ordinary surface theory, we refer to the law $\mathbf{t} \to d\mathbf{n}$, by which a surface vector at P and an infinitesimal displacement in the surface gives rise to a normal vector $d\mathbf{n}$ at P, as *curvature*. The curvature measures the extent to which the tangent space turns into the normal space as one proceeds across the surface by parallel displacement.

2. A normal vector $\mathbf{n}$ at P gives rise to a vector $\mathbf{n}' + d\mathbf{t}$ ($\mathbf{n}'$ normal, $d\mathbf{t}$ tangent). The infinitesimal linear map $\mathbf{n} \to \mathbf{n}'$ from $\mathfrak{N}_P$ to $\mathfrak{N}_{P'}$ is the *torsion*. One can call the law $\mathbf{n} \to d\mathbf{t}$, which makes a normal vector and an infinitesimal displacement in the surface give rise to an infinitesimal tangent vector, and which shows how the normal space turns into the tangent space, the *transverse curvature*, to distinguish it from the "longitudinal" one that was mentioned in 1.

If:

$$dv^{\mathfrak{i}} = -dv^{\mathfrak{i}}_{\mathfrak{k}} \cdot v^{\mathfrak{k}}, \qquad dv^{\mathfrak{i}}_{\mathfrak{t}} = \Gamma^{\mathfrak{i}}_{\mathfrak{t}\rho}(dy)^{\rho} \qquad (\Gamma^{\mathfrak{i}}_{\alpha\beta} = \Gamma^{\mathfrak{i}}_{\beta\alpha})$$

	$1, \ldots, p$	$p+1, \ldots, n$
$1, \ldots, p$	affine connection	trans-verse curv-ature
$p+1, \ldots, n$	longitudinal curvature	torsion

is the formula for the infinitesimal parallel displacement of an arbitrary vector $v^{\mathfrak{i}}\,\mathbf{e}_{\mathfrak{i}}$ (the German indices run through all values from 1 to n, the Greek ones run through only 1 to p, and $(dy)^{\alpha}$ stands in place of dy_{α}, due to its contravariant nature) that is carried out on the surface then the splitting will correspond to four components of the decomposition of the square matrix of the coefficients $dv^{\mathfrak{i}}_{\mathfrak{k}}$ ($\mathfrak{i}$ is the row index and $\mathfrak{k}$ is the column index) that is suggested to the left. We will employ Latin indices for the normal components under the splitting and Greek indices for the tangential ones, set $\mathbf{e}_{p+i} = \overline{\mathbf{e}}_i$, and denote an arbitrary tangent vector by $v^{\alpha}\,\mathbf{e}_{\alpha}$ and an arbitrary normal vector by $v^i\,\overline{\mathbf{e}}_i$. Then:

tt) $\Gamma^{\alpha}_{\rho\sigma} = \Gamma^{\alpha}_{\sigma\rho}$ are the components of the affine connection of the surface. The rule $\mathbf{t} \to \mathbf{t}'$ reads: $dv^{\alpha} = -\Gamma^{\alpha}_{\rho\sigma} v^{\rho}(dy)^{\sigma}$, as a formula.

nt) $G^i_{\alpha\beta} = \Gamma^{p+i}_{\alpha\beta}$ are the components of the curvature: The two infinitesimal displacements d and δ on the surface belong to the normal vectors with the components $-G^i_{\alpha\beta}\,(dy)^{\alpha}\,(dy)^{\beta}$ ($i = 1, 2, \ldots,$

q). As in ordinary surface theory, one also has the symmetry law $G^i_{\alpha\beta} = G^i_{\beta\alpha}$ for the "second fundamental form" here, as well.

tn) $\bar{G}^{\alpha}_{i\beta} = \Gamma^{\alpha}_{p+i,\beta}$ are the components of the transverse curvature, which allows a tangential dy to give rise to another one dy by way of a normal vector $\bar{v}$: $(\delta y)^\alpha = -G^i_{\alpha\beta}\,\bar{v}^i\,(dy)^\beta$.

nn) $\mathsf{T}^i_{k\alpha} = \Gamma^{p+i}_{p+k,\alpha}$ torsion: $d\bar{v}^i = -\mathsf{T}^i_{k\alpha}\,\bar{v}^k\,(dy)^\alpha$.

If a vector with the components u^J in the coordinate system of the x_J changes into the vector $u^J + du^J$ under the displacement $(dx)^J$ of the origin in space then the vector with components $du^J + d\beta^J_K \cdot u^K$ will measure its invariant change; B are the components of the spatial affine connection in $d\beta^J_K = \mathsf{B}^J_{KN}(dx)^N$. If we apply that to the "unit vectors" $\mathbf{u} = \mathbf{e}_{\mathfrak{i}} = \left(e^1_{\mathfrak{i}}, e^2_{\mathfrak{i}}, \ldots, e^n_{\mathfrak{i}}\right)$ that belong to the surface for a displacement *in the surface* and likewise express the invariant change in the $\mathbf{e}_{\mathfrak{i}}$ in their own coordinate system then we will get the *fundamental formulas of surface theory*, which one cares to couple with the name of **Frenet** in the case of $p = 1$:

$$\frac{\partial e^J_{\mathfrak{i}}}{\partial y_\alpha} + \mathsf{B}^J_{MN}\,e^M_{\mathfrak{i}}\,e^N_\alpha = \Gamma^{\mathfrak{j}}_{\mathfrak{i}\alpha}\,e^J_{\mathfrak{j}}. \tag{1}$$

If space is *planar* and the spatial coordinate system that is employed is a linear one (so $\mathsf{B} = 0$) then those equations will read:

$$\frac{\partial \mathbf{e}_{\mathfrak{i}}}{\partial y_\alpha} = \Gamma^{\mathfrak{k}}_{\mathfrak{i}\alpha}\,\mathbf{e}_{\mathfrak{k}}, \tag{2}$$

in particular, or when decomposed:

$$(3)\qquad t)\quad \frac{\partial \mathbf{e}_\beta}{\partial y_\alpha} = \Gamma^{\rho}_{\alpha\beta}\,\mathbf{e}_\rho + G^r_{\alpha\beta}\,\bar{\mathbf{e}}_r, \qquad n)\quad \frac{\partial \bar{\mathbf{e}}_i}{\partial y_\alpha} = \bar{G}^{\rho}_{i\alpha}\,\mathbf{e}_\rho + \mathsf{T}^r_{i\alpha}\,\bar{\mathbf{e}}_r.$$

II. – From the affine connection, we now come to the *metric.* If the space is a metric **Riemannian** space then its metric will carry over with no further discussion to every surface that is *embedded* in it. In order for the fundamental metric form on the surface to never be a degenerate one, the fundamental metric form of space must be definite, and we would, in fact, like to assume that in what follows. The surface will be *complemented in space in the natural way* – i.e., the normal space $\mathfrak{N}$ will consist of all vectors that are perpendicular to $\mathfrak{T}$. We choose q vectors of length 1 that are mutually perpendicular to be the $\bar{\mathbf{e}}_i = \mathbf{e}_{p+i}$. The components $g_{\mathfrak{i}\mathfrak{k}}$ of the metric field in space at the surface point P, which are the scalar products $(\mathbf{e}_\mathfrak{i} \cdot \mathbf{e}_\mathfrak{k})$, then define the accompanying matrix. In it:

$$ds^2 = g_{\alpha\beta}\,(dy)^\alpha\,(dy)^\beta$$

us the *fundamental metric form on the surface.*

$$\left[\begin{array}{ccc|c} g_{11} & \cdots & g_{1p} & 0\cdots 0 \\ \cdots & \cdots & \cdots & \cdots\cdots \\ g_{p1} & \cdots & g_{pp} & 0\cdots 0 \\ \hline 0 & \cdots & 0 & 1\cdots 0 \\ \cdots & \cdots & \cdots & \cdots\cdots \\ 0 & \cdots & 0 & 0\cdots 1 \end{array}\right]$$

The metric space is provided with an affine connection that is characterized uniquely by the fact that the length of a vector remains unchanged of a vector parallel translation. Under the splitting $\mathfrak{T} + \mathfrak{N}$, that will imply the following statements:

1. The map $\mathbf{t} \to \mathbf{t}'$ from $\mathfrak{T}_P$ to $\mathfrak{T}_{P'}$ leaves the length of the vector $\mathbf{t}$ unchanged, or: *The affine connection on the surface is the one that belongs to the metric that is in effect on the surface.*

2. For the maps $\mathbf{t} \to d\mathbf{n}$ and $\mathbf{n} \to d\mathbf{t}$, one has $(\mathbf{t} \cdot d\mathbf{t}) + (\mathbf{n} \cdot d\mathbf{n}) = 0$. *The transverse curvature will lead back to the longitudinal curvature* in that way.

3. The map $\mathbf{n} \to \mathbf{n}'$ (viz., the *torsion*) is also a *congruence*; i.e., an infinitesimal "rotation." One can express that in formulas by saying that the equation:

$$dg_{\mathfrak{i}\mathfrak{k}} = g_{\mathfrak{i}\mathfrak{r}}\,d\gamma^{\mathfrak{r}}_{\mathfrak{k}} + g_{\mathfrak{k}\mathfrak{r}}\,d\gamma^{\mathfrak{r}}_{\mathfrak{i}} \tag{4}$$

splits into:

$$\begin{cases} \qquad\qquad tt) \quad \dfrac{\partial g_{\mu\nu}}{\partial y_\alpha} = g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} + g_{\nu\rho}\Gamma^{\rho}_{\mu\alpha}, \\ nt) \quad \text{or} \quad tn) \quad 0 = G^{i}_{\alpha\beta} + g_{\alpha\rho}\bar{G}^{\rho}_{i\beta}, \\ \qquad\qquad nn) \quad 0 = \mathsf{T}^{i}_{k\alpha} + \mathsf{T}^{k}_{i\alpha}. \end{cases} \tag{5}$$

III. – If the vector $v^\mathfrak{i}\,\mathbf{e}_\mathfrak{i}$ goes around an infinitely small two-dimensional element on the surface with the components:

$$(\Delta y)^{\alpha\beta} = (dy)^\alpha\,(\delta y)^\beta - (dy)^\beta\,(\delta y)^\alpha$$

then it will suffer a change of:

$$\Delta v^\mathfrak{i} = \Delta r^{\mathfrak{i}}_{\mathfrak{k}} \cdot v^\mathfrak{k} = \tfrac{1}{2} R^{\mathfrak{i}}_{\mathfrak{k}\alpha\beta}\, v^\mathfrak{k}\,(\Delta y)^{\alpha\beta},$$

where

$$R^{\mathfrak{i}}_{\mathfrak{k}\alpha\beta} = \left(\frac{\partial \Gamma^{\mathfrak{i}}_{\mathfrak{k}\beta}}{\partial y_\alpha} - \frac{\partial \Gamma^{\mathfrak{i}}_{\mathfrak{k}\alpha}}{\partial y_\beta}\right) + \left(\Gamma^{\mathfrak{i}}_{\mathfrak{r}\alpha}\,\Gamma^{\mathfrak{r}}_{\mathfrak{k}\beta} - \Gamma^{\mathfrak{i}}_{\mathfrak{r}\beta}\,\Gamma^{\mathfrak{r}}_{\mathfrak{k}\alpha}\right).$$

In order to avoid confusion, I shall use the term *vorticity* for this, instead of the **Riemannian** term "curvature," which generally seems appropriate to the change in a quantity under a circuit of a surface element. In metric space, $\Delta r_{\mathrm{i}\mathfrak{k}} = g_{\mathrm{ij}}\,\Delta r^{\mathrm{j}}_{\mathfrak{k}}$ is a skew-symmetric matrix, because the vector does not change in length under a circuit. The square matrix of the $\Delta r^{\mathrm{j}}_{\mathfrak{k}}$, in turn, decomposes into the four components *tt*, *tn*, *nn* (tangential and normal components of the changes in a tangential and normal vector, resp.). I shall write down the explicit expressions for the case of a metric space.

$$tt)\qquad R_{\gamma\delta;\alpha\beta} = S_{\gamma\delta;\alpha\beta} + (G^r_{\gamma\alpha}\,G^r_{\delta\beta} - G^r_{\gamma\beta}G^r_{\delta\alpha}),$$

$$nn)\qquad R_{p+i,\,p+k\,;\alpha\beta} = U_{ik\,;\alpha\beta} + g^{\rho\sigma}(G^i_{\alpha\vartheta}\,G^k_{\beta\sigma} - G^i_{\beta\rho}G^k_{\alpha\sigma}).$$

$S_{\gamma\delta;\alpha\beta}$ is the *longitudinal surface vorticity* (or the "**Riemannian** curvature" of the surface): Namely, it is the change in a tangential vector that goes around a two-dimensional element that lies in the surface according to the displacement law $\mathbf{t} \to \mathbf{t}'$; it depends upon only the affine connection on the surface (or its fundamental metric form). However, the *transverse surface vorticity* $U_{ik\,;\alpha\beta}$ is the change in a normal vector that goes around the surface element according to the displacement law $\mathbf{n} \to \mathbf{n}'$; it belongs to the torsion, just as the longitudinal vorticity belongs to the affine connection on the surface. I shall refer to the component:

$$nt)\qquad C^i_{\gamma,\alpha\beta} = \left(\frac{\partial G^i_{\gamma\beta}}{\partial y_\alpha} - \frac{\partial G^i_{\gamma\alpha}}{\partial y_\beta}\right) + \left(\Gamma^\rho_{\gamma\alpha}G^i_{\alpha\rho} - \Gamma^\rho_{\gamma\beta}\,G^i_{\beta\rho}\right) + \left(G^r_{\gamma\beta}\,\mathrm{T}^i_{r\alpha} - G^i_{r\alpha}\,\mathrm{T}^i_{r\beta}\right)$$

as the *Codazzi tensor;* due to the skew-symmetry of $\Delta r_{\mathrm{i}\mathfrak{k}}$, the *tn*) component $\bar{C}^\gamma_{i,\alpha\beta}$ is essentially identical to it: $C^i_{\gamma,\alpha\beta} + g_{\gamma\rho}\,\bar{C}^\rho_{i,\alpha\beta} = 0$.

If the space is planar then *the change in a vector under a circuit of a surface element* will be equal to zero:

$$(7)\qquad \left\{\begin{array}{rl} tt) & \text{longitudinal surface vorticity} = \sum_r (G^r_{\alpha\delta}G^r_{\beta\gamma} - G^r_{\beta\delta}G^r_{\alpha\gamma}), \\ nn) & \text{transversal surface vorticity} = \sum_r g^{\rho\sigma}(G^k_{\alpha\sigma}G^i_{\beta\rho} - G^k_{\beta\sigma}G^i_{\alpha\rho}), \\ nt)\ \text{ or }\ tn) & \text{the } \textbf{Codazzi} \text{ tensor} = 0. \end{array}\right.$$

These are the *integrability conditions* for the "fundamental formulas" (2) [(3), resp.]. If they are fulfilled then those equations will have one and only one solution with arbitrarily-given initial values, when they are considered to be differential equations for the unknowns $\mathbf{e}_{\mathrm{i}}$. If $\mathbf{x}$ means the vector that leads from the origin to the surface point and has the components x_J then one can determine $\mathbf{x}$ from $\partial\mathbf{x} / \partial y_\alpha = \mathbf{e}_\alpha$, since from (3*t*), $\dfrac{\partial \mathbf{e}_\alpha}{\partial y_\beta} = \dfrac{\partial \mathbf{e}_\beta}{\partial y_\alpha}$. In the metric case, the solutions $\mathbf{e}_{\mathrm{i}}$

will fulfill the equations $(\mathbf{e}_{\mathfrak{i}} \cdot \mathbf{e}_{\mathfrak{k}}) = g_{\mathfrak{ik}}$ on the surface when they are true for the initial values, since from (2), the quantities $g^*_{\mathfrak{ik}} = (\mathbf{e}_{\mathfrak{i}} \cdot \mathbf{e}_{\mathfrak{k}})$ satisfy the relations:

(4*; cf., [**4**])
$$dg^*_{\mathfrak{ik}} = g^*_{\mathfrak{ir}}\, d\gamma^{\mathfrak{r}}_{\mathfrak{k}} + g^*_{\mathfrak{kr}}\, d\gamma^{\mathfrak{r}}_{\mathfrak{i}} .$$

For a given fundamental metric form, curvature, and torsion there always exists one and only one (in the sense of congruence) *surface in Euclidian space, assuming that the given quantities satisfy the conditions* (6) [(7), *resp.*] (*fundamental theorem of surface theory*).

The plane is not the only metrically-homogeneous space. The "sphere" of (positive or negative) constant curvature λ, whose fundamental metric form reads:

$$(dx_1^2 + dx_2^2 + \cdots + dx_n^2) + \frac{\lambda\,(x_1\, dx_1 + \cdots + x_n\, dx_n)^2}{1 - \lambda\,(x_1^2 + x_2^2 + \cdots + x_n^2)},$$

is also of that type. Therefore, one can also pose the problem of determining a surface from its metric, curvature, and torsion in such a *spherical space*. Its solution is achieved in the same way. The "fundamental formulas" and "integrability conditions" can be simply adapted with the following two modifications: One adds the terms $-\lambda\, g_{\alpha\beta} \cdot \mathbf{x}$ to the right-hand side of (3_t) and the terms $-\lambda\,(g_{\alpha\delta}\, g_{\beta\gamma} - g_{\alpha\gamma}\, g_{\beta\delta})$ to the right-hand side of (7_{tt}).

In an arbitrary space with an affine connection whose vorticity has the components R^J_{KAB} in the coordinate system of the x_J, the equations:

$$R^{\mathfrak{i}}_{\mathfrak{k}\alpha\beta}\, e^J_{\mathfrak{i}} = \mathrm{R}^J_{KAB}\, e^K_{\mathfrak{k}}\, e^A_\alpha e^B_\beta$$

will appear in place of (6).

All of that is indeed quite trivial; however, it must still be said. It would be very desirable to incorporate the concepts of infinitesimal parallel translation, the conception of the **Riemannian** curvature as a vector vorticity, and the idea that the affine connection on the surface and its curvature and torsion define a natural totality into ordinary surface theory; the gain in intuitiveness and clarity would be significant. Furthermore, it would seem expedient to adapt the theory of curves to the representation that was given here insofar as the axis-cross in the normal plane no longer employs the principal normal and binormal, since the theory suffers from the inconvenience that they will be indeterminate when the curvature vanishes.

Bibliography

1. **A. Voss**, "Zur Theorie der…Krümmung höherer Mannigfaltigkeiten," Math. Ann. **16** (1880), 129-178.

2. **T. Levi-Civita**, "Nozione di parallelismo in una varietà qualunque…," Rend. del Circ. Math. di Palermo **42** (1917).

3. **H. Weyl**, *Raum, Zeit, Materie*, 4th ed., Springer, 1921, Chap. II.

4. **W. Blaschke**, "Frenets Formeln für den Raum von Riemann," Math. Zeit. **6** (1920), 94-99.

5. **G. Juvet**, "Les formules de Frenet pour un espace de M. Weyl," C. R. Acad. Sci. Paris **172** (27 June 1921), pp. 1647.