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The covariant derivative and Cesàro’s immobility conditions

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1. – In certain investigations, the introduction of a *time-dependent coordinate system* is suggested, and indeed required by the nature of the problem. It is then similar to the introduction of general curvilinear coordinates. For example, one recalls Darboux’s *trièdre mobile* (Cesàro’s little boat), the theory of relative motion, and especially the motion of a mass-point on a deformable surface (or in a deformable space). In the latter case, one can hardly speak of a time-independent system at all.

The introduction of such a coordinate system (we would like to call it *rheonomic*) follows from a time-dependent *kinematical* transformation:

$$\bar{x}^i = \bar{x}^i(x^k, t) \qquad (i, k = 1, \ldots, n). \tag{1}$$

Naturally, “time” is only a conventional name for the parameter t. We would now like to construct an *absolute differential calculus* for the “kinematical” group (1). The juxtaposition of the usual “geometric” group:

$$\bar{x}^i = \bar{x}^i(x^k) \tag{2}$$

and the other one (1) will lead one to distinguish n-tuples that transform by *only* (2) according to the formulas:

$$\bar{X}^i = \frac{\partial \bar{x}^i}{\partial x^k} X^k, \qquad \left(\underline{X}^i = \frac{\partial x^i}{\partial \bar{x}^k} X^k, \text{ resp.}\right) \tag{3}$$

from the ones that also transform by (1). We call the former vectors *weak* and the latter *strong*. The definition for tensors reads analogously. Strong tensors then transform by (1), as if t did not enter into it at all.

In order to explain that, we remark that dx^i / dt is only a weak vector, while $\partial\varphi / \partial x^i$ will be a strong covariant vector only when φ is a strong scalar.

We make the following remark: In a time-independent – i.e., *scleronomic* – system, each point is assigned only n numbers, namely, its coordinates. By contrast, in a rheonomic system, there is also a *velocity* $\partial \bar{x}^i / \partial t$, in addition. Two rheonomic systems can differ even when the same point has the same coordinates in the two systems, namely, when the $\partial \bar{x}^i / \partial t$ in question differ.

The transformation (1) changes the "identity" of the space point to some extent: x^i do not correspond to the same $\bar{x}^i$ for different t. Strictly speaking, one cannot speak of *the same* space point (at least, not in the conventional sense) at different times on the basis of (1).

2. – The usual absolute differential calculus that is based upon (2) replaces the differential of a vector X with the forms:

$$\text{(4)} \qquad \delta X^i = dX^i + \Gamma^i_{jk} X^j dx^k, \qquad dX_i = dX_i - \Gamma^j_{ik} X_j \, dx^k,$$

which will represent a vector subject to (2) only when the Γ^i_{jk} transform according to the formula:

$$\text{(5)} \qquad \bar{\Gamma}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^j} \frac{\partial x^\nu}{\partial \bar{x}^k} \Gamma^\lambda_{\mu\nu} + \frac{\partial \bar{x}^i}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^j \, \partial \bar{x}^k}.$$

However, (4) is only a *weak* vector, so the form (4) is not covariant under the transformation (1). The demand of *strong* covariance implies only that one must extend (4) by one term in dt :

$$\text{(6)} \qquad \begin{aligned} \delta X^i &= dX^i + \Gamma^i_{jk} X^j dx^k + \Lambda^i_j \, X^j \, dt, \\ dX_i &= dX_i - \Gamma^j_{ik} X_j \, dx^k - \Lambda^j_i \, X_j \, dt, \end{aligned}$$

when one establishes that:

$$\text{(7)} \qquad \bar{\Lambda}^i_j = \frac{\partial \bar{x}^i}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^j} \frac{\partial x^\nu}{\partial \bar{x}^k} \Gamma^\lambda_{\mu\nu} + \frac{\partial \bar{x}^i}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^j} \Lambda^\lambda_\mu + \frac{\partial \bar{x}^i}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^j \, \partial \bar{x}^k}.$$

We shall verify that (6) will then transform as a strong vector by (1) according to the laws (3) as a result of equations (7): That is a simple calculation. Therefore, if the Γ^i_{jk} and Λ^i_j are given in *one* well-defined (rheonomic) coordinate system then that will also define *one* "strong" geometry.

The extension of the strong covariant derivative to tensors offers no difficulty: For every tensor, we will have a form:

$$\Gamma^i_{jk} \, T^{\cdots j \cdots} \, dx^k + \Lambda^i_j \, T^{\cdots j \cdots} \, dt, \qquad - \Gamma^j_{ik} \, T_{\cdots j \cdots} \, dx^k - \Lambda^j_i \, T_{\cdots j \cdots} \, dt, \text{ resp.}$$

One easily verifies, moreover, that the strong differential deserves that name, i.e., that the formal laws of differentiation:

$$\delta p = dp \qquad (p\text{, strong scalar}),$$

(8)
$$\delta(\Phi + \Psi) = \delta\Phi + \delta\Psi, \qquad (\Phi, \Psi\text{, strong tensors})$$
$$\delta(\Phi\,\Psi) = \Phi\;\delta\Psi + \Psi\;\delta\Phi$$

will still apply. In order to do that, it is convenient to write (6) in the form:

(9)
$$\delta X^i = dX^i + \omega^i_j\,X^j, \qquad \delta X_i = dX_i - \omega^j_i\,X_j\,,$$

where ω^i_j denotes a differential form.

3. – If there is a coordinate system in which Λ^i_j vanishes (which we can then call scleronomic) then the Λ^i_j can be easily calculated in a different coordinate system. Let the distinguished system be $\bar{x}^i$. The strong differential then has the form:

(10)
$$\delta\bar{X}^i = d\bar{X}^i + \bar{\Gamma}^i_{jk}\,\bar{X}^j\,d\bar{x}^k$$

in it.

Now, due to (3), we have:

$$\delta\bar{X}^i = d\left(\frac{\partial\bar{x}^i}{\partial x^\alpha}X^\alpha\right) + \bar{\Gamma}^i_{jk}\,\frac{\partial\bar{x}^i}{\partial x^\beta}X^\beta\,\frac{\partial\bar{x}^k}{\partial x^\gamma}\,dx^\gamma$$

$$= \frac{\partial\bar{x}^i}{\partial x^\alpha}dX^\alpha + \frac{\partial^2\bar{x}^i}{\partial x^\beta\,\partial x^\gamma}X^\beta dx^\gamma + \frac{\partial^2\bar{x}^i}{\partial x^\beta\,\partial t}X^\beta dt + \bar{\Gamma}^i_{jk}\,\frac{\partial\bar{x}^j}{\partial x^\beta}\,\frac{\partial\bar{x}^k}{\partial x^\gamma}X^\beta\,dx^\gamma$$

$$= \frac{\partial\bar{x}^i}{\partial x^\alpha}\left[dX^\alpha + \left(\frac{\partial x^\alpha}{\partial\bar{x}^h}\frac{\partial^2\bar{x}^h}{\partial x^\beta\,\partial x^\gamma} + \frac{\partial x^\alpha}{\partial\bar{x}^h}\frac{\partial\bar{x}^j}{\partial x^\beta}\frac{\partial\bar{x}^k}{\partial x^\gamma}\right)\bar{\Gamma}^h_{jk}\right]X^\beta dx^\gamma + \frac{\partial x^\alpha}{\partial\bar{x}^h}\frac{\partial^2\bar{x}^h}{\partial x^\beta\,\partial t}X^\beta dt\,,$$

so [cf., (5)]:

(11)
$$\delta\bar{X}^i = \frac{\partial\bar{x}^i}{\partial x^\alpha}\left[dX^\alpha + \Gamma^\alpha_{\beta\gamma}X^\beta dx^\gamma + \frac{\partial x^\alpha}{\partial\bar{x}^h}\frac{\partial^2\bar{x}^h}{\partial x^\beta\,\partial t}X^\beta\,dt\right].$$

One then has:

(12)
$$\Lambda^\alpha_\beta = \frac{\partial x^\alpha}{\partial\bar{x}^h}\frac{\partial^2\bar{x}^h}{\partial x^\beta\,\partial t}\,.$$

Naturally, a system in which the Λ^i_j vanish cannot be constructed in the general case.

4. – We now assume that space is Euclidian and that the system $\bar{x}^i$ is precisely Cartesian. We shall now denote it by y^i, and replace x^i with y^i and $\bar{x}^i$ with x^i in (5):

$$\Gamma^i_{jk} = \frac{\partial x^i}{\partial y^\lambda}\frac{\partial^2 y^\lambda}{\partial \bar{x}^j\,\partial x^k}. \tag{13}$$

In particular, if the transformation:

$$x^i = x^i(y^k, t) \tag{14}$$

is linear and orthogonal in the y^i then we set:

$$\frac{\partial x^i}{\partial y^k} = o^i_k, \qquad \frac{\partial y^k}{\partial x^i} = o^h_k, \tag{15}$$

and will have:

$$o^i_h\, o^i_k = \begin{cases} 1 & h = k, \\ 0 & h \neq k. \end{cases} \tag{16}$$

The distinction between covariant and contravariant vanishes here. From (12), we have:

$$\Lambda^i_j = \Lambda_{ik} = o^i_h\, \dot{o}^i_k, \qquad \Lambda_{ik} = -\Lambda_{ik}. \tag{17}$$

The dot over a symbol means differentiation with respect to t. Now, (17) is nothing but the *angular velocity* of the rectangular coordinate system x^i, which moves according to (14). We denote it by F_{ik} and write (6) in the form:

$$\delta X_i = dX_i + F_{ik}\, X_k\, dt, \tag{18}$$

which is actually common in mechanics, and in particular, the theory of relative motion in vector language, and gives the absolute change in velocity of a vector when referred to a moving system.

If we now choose the moving triad of a space curve C to be the coordinate system x^i and set t equal to the arc-length s then it would be child's play to get F_{ik} from Darboux's formulas ([1]). In fact, the equation that arises from (18):

([1]) Cf., e.g., Kommerell and Kommerell, *Theorie der Raumkurven und krummen Flächen*, Gruyter, 1931, I, pp. 29.

(19) $$\frac{\delta X_i}{ds} = \frac{dX_i}{ds} + F_{ik} X_k$$

is *covariant*. If X_i means the unit tangent $\mathfrak{t}$ to C then $\delta X_i / ds$ will be the curvature vector $\mathfrak{n} / \rho$. The coordinates of those two vectors in the coordinate system x^i are:

$$(1, 0, 0) \qquad \text{and} \qquad (0, \frac{1}{\rho}, 0),$$

resp. If we substitute that in (19) then that will give $F_{21} = 1 / \rho$ for $i = 2$.

Furthermore, if $X_i = \mathfrak{b}$ means the binormal (0, 0, 1) then $\delta\mathfrak{b} / ds$ must yield the torsion $(0, 1 / \tau, 0)$. If we substitute that in (19) then that will once more give $F_{23} = 1 / \tau$ for $i = 2$. However, $F_{12} = 0$ for $i = 1$. Therefore, F_{ik} is given in the form:

(20) $$\begin{Vmatrix} 0 & -1/\rho & 0 \\ 1/\rho & 0 & 1/\tau \\ 0 & -1/\tau & 0 \end{Vmatrix}.$$

Naturally, in three-dimensional space, one can replace F_{ik} with the associated vector $\mathfrak{w}$, which is known to have the components $(1 / \tau, 0, 1 / \rho)$.

5. – We likewise get Cesàro's immobility conditions immediately and compellingly, which we shall write down for the three-dimensional case. x^i keeps its latter meaning here, and we let O be the origin of the immobile coordinate system, while M is the running point along the curve C, and P is a fixed point in space. δ will always mean the *strong* covariant derivative. We have:

(21) $$OP = OM + MP, \qquad \frac{\delta OP}{ds} = \frac{\delta OM}{ds} + \frac{\delta MP}{ds}.$$

Now, $\delta OP / ds = 0$ in the immobile coordinate system, *so that will also be true on any* coordinate system. One then has:

$$\frac{\delta MP}{ds} = -\frac{\delta OM}{ds}$$

or

$$\frac{\delta MP}{ds} = -\mathfrak{t},$$

where $\mathfrak{t}$ means the unit tangent to C at M. Those are actually the stated immobility conditions already. In order to obtain them in their usual form, we go over to coordinates and imagine that the tangent $\mathfrak{t}$ has the components (1, 0, 0) in the system x^i. When we consider formulas (19) and (20), it will follow that:

$$\frac{dx^1}{ds} - \frac{x^2}{\rho} = -1, \qquad \frac{dx^2}{ds} + \frac{x^1}{\rho} + \frac{x^3}{\tau} = 0, \qquad \frac{dx^3}{ds} - \frac{x^2}{\tau} = 0, \tag{22}$$

are precisely the Cesàro equations. [Cf., e.g., Kowalewski, *Allgemeine natürliche Geometrie*, pp. 76, Gruyter, Leipzig, 1931. Equations (8) on pp. 77 are identical to the second equation (21).]

6. – It is just as easy to represent the theory of relative motion. We first get:

$$\frac{\delta OP}{ds} = \frac{\delta OM}{ds} + \frac{\delta MP}{ds}$$

from (21), so:

$$v^i = \underset{0}{v}{}^i + \frac{dx^i}{dt} + F_{ik}\, x^k = \underset{m}{v}{}^i + \underset{r}{v}{}^i . \tag{23}$$

Here, the weak vector dx^i / dt is the relative velocity, while the two remaining summands define the velocity of convection, which agrees with known formulas. If one strongly covariant differentiates (23) then one will get the known formulas for the convective and Coriolis accelerations by an easy calculation.

Nothing prevents us from generalizing the "strong" tensor calculus to transformations:

$$\bar{x}^i = f^i(x^k, t_1, \ldots, t_r) \tag{24}$$

that depend upon several parameters. We would obtain the expression:

$$\delta u^i = du^i + \Gamma^i_{jk}\, u^j\, dx^k + \Lambda^i_{j\alpha}\, u^j dt_\alpha , \tag{25}$$

in which the transformation rules for the $\Lambda^i_{j\alpha}$ are very easy to deduce. If that is to define the differential of a tensor then we will obtain a group of terms like in (25) for each tensor index.

One will arrive at a "strong" displacement in an extrinsic (i.e., Levi-Civita) way when one considers, e.g., a simply-infinite family of spaces $\mathfrak{B}\,(t)$ that are layered upon each other and embedded in a larger space $\mathfrak{A}$. The transformation (1) follows from a change in the mutual arrangement of points on the various $\mathfrak{B}\,(t)$. "Strong" vectors are the ones that lie in $\mathfrak{B}\,(t)$. One obtains the strong differentials by projecting the differentials that exist in $\mathfrak{A}$ onto $\mathfrak{B}(t)$. The direction of projection is arbitrary.

If one looks at things that way then the “strong” geometry will actually appear to be the integrable case of non-holonomic geometry that was developed by various authors in recent times.

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