"Anwendungen der Variationsrechnung auf partielle Differentialgleichungen mit zwei unabhängigen Variabeln," Math. Ann. **57** (1903), 185-194.

Applications of the calculus of variations to partial differential equations with two independent variables

By

Yoshiye in Tokyo

Translated by D. H. Delphenich

In his lectures on partial differential equations during the Winter semester 1900-01, Herr Prof. Hilbert showed how one could apply the methods of the calculus of variations to partial differential equations. Using his ideas, I have carried out the calculations, and in that way arrived at the known equations for characteristics and the conditions for a system to be in involution.

First-order partial differential equation with two independent variables.

We write the given partial differential equation in the form:

$$F(x, y, z, p, q) = 0$$

p and q are abbreviations for $\partial z / \partial x$ and $\partial z / \partial y$, resp., in it.

We shall also use some other known notations here: For example, F_x , F_y , ... stand for $\frac{\partial F}{\partial x}$,

 $\frac{\partial F}{\partial y}$, ..., resp., while x', y', ... mean the differential quotients of x, y, ..., resp., with respect to

an unknown variable u. (From symmetry, we imagine that x, y, ... are functions of u.) We let ξ denote an arbitrary differentiable function of u and let u_0, u_1 denote two well-defined values of u.

Having done that, we define the integral:

$$\int_{u_0}^{u_1} \xi(p \, x' + q \, y' - z') \, du \, .$$

If z is function of only x and y then obviously that integral will be independent of the path of integration from u_0 to u_1 , and indeed it will have the value zero, independently of the choice of the function. Conversely: If that integral is independent of the path of integration and always has the

value zero then we will know that due to the arbitrariness of the function x, we must have z' = px' + qy', assuming that px' + qy' - z' remains continuous. Hence, z will be a function of only x and y.

That shows us that when a solution z = z (x, y) to the partial differential equation F (x, y, z, p, q) = 0 exists, the integral above for that z will vanish identically, and that z satisfies the equation F = 0. Conversely: When the five functions x, y, p, q, z satisfy the equation F = 0, and the integral above is independent of the path of integration, and it has the value zero, then the function z = z

z(x, y) will exist that satisfies the partial differential equation $F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$.

Thus, the problem of integrating the differential equation F(x, y, z, p, q) = 0 is equivalent to the problem of making the integral $\int_{u_0}^{u_1} \xi(p x' + q y' - z') du$ independent of the path of integration under the condition that F(x, y, z, p, q) = 0, i.e., from the rule in the calculus of variations:

$$\delta \int_{u_0}^{u_1} \left[\xi \left(p \, x' + q \, y' - z' \right) + \lambda \, F \right] \, du = 0,$$

where λ is the Lagrange factor.

That line of reasoning has a geometric interpretation: Namely, we consider x, y, z, p, q to be a surface element, i.e., a point x, y, z and a plane p(X-x) + q(Y-y) - (Z-z) = 0 that includes the point x, y, z. When those surface elements fulfill the conditions above, they will define a union of surface elements whose elements are included in the family that is represented by the equation F(x, y, z, p, q) = 0. The union is one-dimensional here, like a strip that lies in the integral surface, because we consider x, y, z, p, q to be functions of one independent variable u.

One can replace the vanishing of the variation of the integral with the Lagrange equations. After a slight modification, they will read:

$$0 = \lambda (F_x + p F_z) - \xi p',$$

$$0 = \lambda (F_y + q F_z) - \xi q',$$

$$0 = \lambda F_z + \xi',$$

$$0 = \lambda F_p + \xi x',$$

$$0 = \lambda F_a + \xi y'.$$

Those equations, along with the condition equation F = 0, serve to determine the functions *x*, *y*, *z*, *p*, *q*, λ . When those equations are fulfilled, *z* will satisfy the partial differential equation:

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$
.

In fact, one will get the following relation from them:

$$0 = \lambda F' - \xi' (p x' + q y' - z') ,$$

i.e., the five functions x, y, z, p, q satisfy the relation F = 0, along with the other two $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y}$

= q, which was to be proved.

In order to eliminate ξ' , we choose ξ such that the equation $\lambda F_z + \xi' = 0$ will be fulfilled identically. We will then have four homogeneous equations in λ and ξ here. If we eliminate λ and ξ from them then we will get three equations:

$$\begin{vmatrix} F_{x} + p F_{z} & -p' \\ F_{p} & x' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_{y} + q F_{z} & -q' \\ F_{q} & y' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_{p} & x' \\ F_{q} & y' \end{vmatrix} = 0.$$

If one adds the first two equations and compares them with the equation F(x, y, z, p, q) = 0 then one will find that p x' + q y' - z' = 0, assuming that $F_z \neq 0$.

One can put those three equations into the form:

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{-dp}{F_x + p F_z} = \frac{-dq}{F_y + q F_z}.$$

Those three equations, along with F = 0, determine four functions y, z, p, q as functions of x. From the equation:

$$0 = \lambda F' - \xi' (p x' + q y' - z'),$$

one knows that the relation F = 0 can be replaced with the equation:

$$p\,x'+q\,y'-z'=0\,,$$

with the condition that the initial values of x, y, z, p, q fulfill the equation F = 0, because F' = 0 gives only the relation F = const. Hence, one has the equations:

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x + pF_z} = \frac{-dq}{F_y + qF_z}$$

for the determination of *y*, *z*, *p*, *q* as functions of *x*, under the condition *F* (x_0 , y_0 , z_0 , p_0 , q_0) = 0, where x_0 , y_0 , z_0 , p_0 , q_0 mean the initial values.

We will now consider the case in which F_z vanishes identically. We will then get:

$$0 = \lambda F_x - c p',$$

$$0 = \lambda F_y - c q',$$

$$0 = \lambda F_p + c x',$$

$$0 = \lambda F_a + c y',$$

in which c is a constant. Eliminating λ and c from them will give us:

$$\begin{vmatrix} F_x & -p' \\ F_p & x' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_y & -q' \\ F_q & y' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_p & x' \\ F_q & y' \end{vmatrix} = 0.$$

From the first two of them, we will get the relation:

$$F'=0$$
.

We will then have the equations:

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x} = \frac{-dq}{F_y}$$

for determining y, z, p, q, along with the condition $F(x_0, y_0, z_0, p_0, q_0) = 0$.

The ordinary differential equations for determining y, z, p, q define the one-dimensional union of surface elements that belong to the family F(x, y, z, p, q) = 0. (Here, we are excluding the solutions for which F_p , F_q , $F_x + p F_z$, $F_y + q F_z$ vanish simultaneously, i.e., the singular solutions.) One calls those unions of elements *characteristic strips*. All integral surfaces that include the surface element x_0 , y_0 , z_0 , p_0 , q_0 will also include all surface elements of the characteristic strips that start from that element. We have arrived at precisely the differential equations of the characteristic strips by applying the calculus of variations.

The second-order partial differential equations.

We can treat the second-order partial differential equations in precisely the same way that we treated first-order partial differential equations. We denote the given equation by:

$$F(x, y, z, p, q, r, s, t) = 0$$
,

in which *r*, *s*, *t* stand for $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$, resp.

The integral:

$$\int_{u_0}^{u_1} [\xi^{(1)} (p x' + q y' - z') + \xi_1^{(2)} (r x' + s y' - p') + \xi_2^{(2)} (s x' + t y' - q')] du$$

appears in place of $\int \xi (p x' + q y' - z') du$, where all ξ mean arbitrary functions of u.

We now seek all one-dimensional unions of curvature elements that belong to the family F(x, y, z, p, q, r, s, t) = 0, or what amounts to the same thing, we must make the integral above independent of the path of integration (it must even be always zero) under the condition that F = 0. The Lagrange equations will follow from that. If we set:

$$\left(\frac{\partial F}{\partial x}\right) = F_x + p F_z + r F_p + s F_q,$$
$$\left(\frac{\partial F}{\partial y}\right) = F_y + q F_z + s F_p + t F_q$$

then by a small calculation that is similar to the one in the case of first-order differential equations, we will get:

$$\begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & -r' & -s' \\ F_r & x' & 0 \\ F_t & 0 & y' \end{vmatrix} = 0, \qquad \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & -s' & -t' \\ F_r & x' & 0 \\ F_t & 0 & y' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_s & y' & x' \\ F_r & x' & 0 \\ F_t & 0 & y' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_s & y' & x' \\ F_r & x' & 0 \\ F_t & 0 & y' \end{vmatrix} = 0, \qquad \begin{vmatrix} F_s & y' & x' \\ F_r & x' & 0 \\ F_t & 0 & y' \end{vmatrix} = 0,$$

In fact, one will get only those six equations for the determination of seven quantities y, z, p, q, r, s, t as functions of x when one again considers x to be the independent variable. The equation:

$$p x' + q y' - z' = 0$$

is not counted among them, since one can derive it from the six equations above. If one multiplies the first rows of the determinants by x', y', s', and adds them, then multiplies the second and third rows of the resulting determinants and adds them to all of the first rows then since:

$$r x' + s y' - p' = 0$$
, $s x' + t y' - q' = 0$,

one will get the equation:

$$\begin{vmatrix} F' + (p x' + q y' - z')F_z & 0 & 0 \\ F_r & x' & 0 \\ F_t & 0 & y' \end{vmatrix} = 0,$$

i.e.:

$$F' + (p x' + q y' - z') F_z = 0.$$

That equation says that one can replace the equation F = 0 with the relation rx' + sy' - p' = 0 and $F(x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0) = 0$, where x_0, y_0, \dots mean the initial values of x, y, \dots

One also calls the one-dimensional unions of elements "characteristics." I will not go into more details about characteristics.

One can treat the general partial differential equations of order n in precisely the same way as one did here. I shall write down only the results.

If one writes:

$$p_1^{(n)}, p_2^{(n)}, ..., p_n^{(n)}, p_{n+1}^{(n)}$$

for the quantities:

$$\frac{\partial^n z}{\partial x^n}, \frac{\partial^n z}{\partial x^{n-1} \partial y}, \dots, \frac{\partial^n z}{\partial x \partial y^{n-1}}, \frac{\partial^n z}{\partial y^n},$$

resp., and one sets:

$$\left(\frac{\partial F}{\partial x}\right) = F_x + p F_z + r F_p + s F_q + \dots + p_1^{(n)} F_{p_1^{(n-1)}} + \dots + p_n^{(n)} F_{p_n^{(n-1)}},$$
$$\left(\frac{\partial F}{\partial y}\right) = F_y + q F_z + s F_p + t F_q + \dots + p_2^{(n)} F_{p_1^{(n-1)}} + \dots + p_{n+1}^{(n)} F_{p_n^{(n-1)}},$$

then one will get the equations:

$$0 = \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & -\left(p_{1}^{(n)}\right)' & -\left(p_{2}^{(n)}\right)' & \cdots & -\left(p_{n}^{(n)}\right)' \\ F_{p_{1}^{(n)}} & x' & 0 & \cdots & 0 \\ F_{p_{2}^{(n)}} & y' & x' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{p_{n}^{(n)}} & 0 & \cdots & \cdots & x' \end{vmatrix} \right|,$$
$$0 = \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & -\left(p_{2}^{(n)}\right)' & -\left(p_{3}^{(n)}\right)' & \cdots & -\left(p_{n+1}^{(n)}\right)' \\ F_{p_{1}^{(n)}} & x' & 0 & \cdots & 0 \\ F_{p_{2}^{(n)}} & y' & x' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{p_{n}^{(n)}} & 0 & \cdots & \cdots & x' \end{vmatrix} \right|,$$

$$0 = \begin{vmatrix} F_{p_1^{(n)}} & x' & 0 & \cdots & \cdots & 0 \\ F_{p_2^{(n)}} & y' & x' & \cdots & \cdots & 0 \\ F_{p_3^{(n)}} & 0 & y' & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{p_n^{(n)}} & 0 & 0 & \cdots & y' & x' \\ F_{p_{n+1}^{(n)}} & 0 & 0 & \cdots & 0 & y' \end{vmatrix},$$

and

$$p x' + q y' - z' = 0,$$

$$r x' + s y' - p' = 0,$$

$$s x' + t y' - q' = 0,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$p_1^{(n)} x' + p_2^{(n)} y' - (p_1^{(n-1)})' = 0,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$p_n^{(n)} x' + p_{n+1}^{(n)} y' - (p_n^{(n-1)})' = 0$$

for the determination of the functions y, z, p, q, ..., $p_{n+1}^{(n)}$, under the condition that F = 0 for the initial values of those functions. We then have $\frac{1}{2}n(n+1)+3$ equations for determining $\frac{1}{2}(n+1)(n+2) + 1$ functions.

That system of first-order ordinary differential equations determines the characteristics of the partial differential equation F = 0.

The conditions for two partial differential equations to have a common solution.

We can also derive the conditions for a system to be in involution by our method. We will first address first-order partial differential equations, which read:

$$F(x, y, z, p, q) = 0, G(x, y, z, p, q) = 0.$$

In order for those two equations to a have a common solution, on the same grounds as before, it is necessary that the first variation of the variation:

$$J = \int_{u_0}^{u_1} [\xi (p \, x' + q \, y' - z') + \lambda F + \mu G] du$$

must vanish. (Here, μ , like λ , is an undetermined Lagrange factor.)

The calculus of variations gives us a system of ordinary differential equations that read:

$$\begin{split} \lambda \left(\frac{\partial F}{\partial x} \right) &+ \mu \left(\frac{\partial G}{\partial x} \right) - \xi p' = 0 , \\ \lambda \left(\frac{\partial F}{\partial y} \right) &+ \mu \left(\frac{\partial G}{\partial y} \right) - \xi q' = 0 , \\ \lambda F_p &+ \mu G_p &+ \xi x' = 0 , \\ \lambda F_q &+ \mu G_q &+ \xi y' = 0 , \end{split}$$

in which $\left(\frac{\partial F}{\partial x}\right)$ stands for the expression $\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z}$. (As before, ξ is determined in such a way that $0 = \lambda F_z + \mu G_z + \xi'$ is fulfilled identically. Hence, ξ will be determined up to a constant.)

We are now in a position to derive the relations that are free of ξ , λ , μ : Those relations are the desired conditions.

To that end, we solve the first and third equation for λ and μ :

$$\lambda \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_p & G_p \end{vmatrix} = \xi \begin{vmatrix} p' & \left(\frac{\partial G}{\partial x}\right) \\ -x' & G_p \end{vmatrix}, \qquad \mu \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_p & G_p \end{vmatrix} = \xi \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & p' \\ F_p & -x' \end{vmatrix}.$$

The second and fourth equation also give us:

$$\lambda \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_q & G_q \end{vmatrix} = \xi \begin{vmatrix} q' & \left(\frac{\partial G}{\partial y}\right) \\ -y' & G_q \end{vmatrix}, \qquad \mu \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_q & G_q \end{vmatrix} = \xi \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & q' \\ F_q & -y' \end{vmatrix}.$$

Due to the relations G = 0 and p x' + q y' - z' = 0, the sums of the two determinants:

$$\begin{vmatrix} p' & \left(\frac{\partial G}{\partial x}\right) \\ -x' & G_p \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} q' & \left(\frac{\partial G}{\partial y}\right) \\ -y' & G_q \end{vmatrix}$$

will vanish. That gives us the desired condition:

$$\begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_p & G_p \end{vmatrix} + \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_q & G_q \end{vmatrix} = 0.$$

Here, we have used only the first two equations. The other two equations give us precisely the same condition as that one. That condition for the system to be in involution agrees with the known condition: When z does not enter into F or G, the left-hand side will go to the known bracket expression.

Things are different for the second-order partial differential equations. We must then deal with the integral:

$$J = \int_{u_0}^{u_1} [\xi(p x' + q y' - z') + \eta(r x' + s y' - p') + \zeta(s x' + t y' - q') + \lambda F + \mu G] du.$$

(Previously, we used $\xi^{(1)}$, $\xi_1^{(2)}$, $\xi_2^{(2)}$ in place of ξ , η , ζ .)

The condition $\delta J = 0$ gives us the relations:

$$\begin{split} \lambda \left(\frac{\partial F}{\partial x} \right) &+ \mu \left(\frac{\partial G}{\partial x} \right) - \eta \, r' - \zeta \, s' = 0 , \\ \lambda \left(\frac{\partial F}{\partial y} \right) &+ \mu \left(\frac{\partial G}{\partial y} \right) - \eta \, s' - \zeta \, t' = 0 , \\ \lambda \, F_r &+ \mu \, G_r + \eta \, x' = 0 , \\ \lambda \, F_s &+ \mu \, G_s + \eta \, y' + \zeta \, x' = 0 , \\ \lambda \, F_t &+ \mu \, G_t + \zeta \, y' = 0 , \end{split}$$

 $\left(\frac{\partial F}{\partial x}\right)$ shall mean the expression $F_x + p F_z + r F_p + s F_q$, and likewise $\left(\frac{\partial F}{\partial y}\right)$ means the expression $F_y + q F_z + s F_p + t F_q$. The functions ξ , η , ζ are determined, up to constants, by the differential equations:

$$0 = \lambda F_z + \mu G_z + \xi', \qquad 0 = \lambda F_p + \mu G_p + \xi x' + \eta', \qquad 0 = \lambda F_q + \mu G_q + \zeta'.$$

We get from the first and third equation that:

$$\lambda \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_r & G_r \end{vmatrix} = \begin{vmatrix} \eta r' + \zeta s' & \left(\frac{\partial G}{\partial x}\right) \\ -\eta x' & G_r \end{vmatrix}, \qquad \mu \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_r & G_r \end{vmatrix} = \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \eta r' + \zeta s' \\ F_r & -\eta x' \end{vmatrix}.$$

The second and fifth equation give us:

$$\lambda \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_t & G_t \end{vmatrix} = \begin{vmatrix} \eta s' + \zeta t' & \left(\frac{\partial G}{\partial y}\right) \\ -\zeta y' & G_t \end{vmatrix}, \qquad \mu \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_t & G_t \end{vmatrix} = \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \eta s' + \zeta t' \\ F_t & -\zeta y' \end{vmatrix}.$$

One can easily find that the fourth equation is fulfilled when the last four equations are fulfilled, which must naturally be the case because if the equations:

$$F = 0, \qquad G = 0, \qquad p x' + q y' - z' = 0, \qquad r x' + s y' - p' = 0, \qquad s x' + t y' - q' = 0$$

are fulfilled then one of the five equations above will be a consequence of the other four. Hence, it is not necessary to use the fourth equation in the five equations above.

We get two equations from the last four equations:

$$\begin{split} \lambda \zeta \begin{vmatrix} \left(\frac{\partial F}{\partial x} \right) & \left(\frac{\partial G}{\partial x} \right) \\ F_r & G_r \end{vmatrix} + \lambda \eta \begin{vmatrix} \left(\frac{\partial F}{\partial y} \right) & \left(\frac{\partial G}{\partial y} \right) \\ F_t & G_t \end{vmatrix} = s'(\zeta^2 G_r - \eta \zeta G_s + \eta^2 G_t) , \\ \mu \zeta \begin{vmatrix} \left(\frac{\partial F}{\partial x} \right) & \left(\frac{\partial G}{\partial x} \right) \\ F_r & G_r \end{vmatrix} + \mu \eta \begin{vmatrix} \left(\frac{\partial F}{\partial y} \right) & \left(\frac{\partial G}{\partial y} \right) \\ F_t & G_t \end{vmatrix} = -s'(\zeta^2 F_r - \eta \zeta F_s + \eta^2 F_t) . \end{split}$$

The two left-hand sides have a common factor:

$$\zeta \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_r & G_r \end{vmatrix} + \eta \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_t & G_t \end{vmatrix}$$

that is linear in η and ζ .

The two quadratic factors in ζ and η must also have a common linear factor. That common solution for ζ / η corresponds to the common factor in the left-hand sides. That linear factor will then vanish for those values of ζ / η . As a result, if:

$$F(x, y, z, p, q, r, s, t) = 0$$
, $G(x, y, z, p, q, r, s, t) = 0$

have a solution in common then one must have:

First of all:

$$m^{2} G_{r} - m G_{s} + G_{t} = 0,$$

$$m^{2} F_{r} - m F_{s} + F_{t} = 0$$
have a common solution for *m*,

Secondly:

$$m \begin{vmatrix} \left(\frac{\partial F}{\partial x}\right) & \left(\frac{\partial G}{\partial x}\right) \\ F_r & G_r \end{vmatrix} + \begin{vmatrix} \left(\frac{\partial F}{\partial y}\right) & \left(\frac{\partial G}{\partial y}\right) \\ F_t & G_t \end{vmatrix} = 0.$$

From the first condition, namely, that the two equations:

$$m^2 G_r - mG_s + G_t = 0,$$

$$m^2 F_r - m F_s + F_t = 0$$

must have a common solution *m*, one will get the relation:

$$\begin{vmatrix} F_r & F_s & F_t & 0 \\ 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 \\ 0 & G_r & G_s & G_t \end{vmatrix} = 0 .$$

The conditions that were obtained here agree precisely with the conditions that were given in Goursat's textbook, t. II.