"Über Integralinvarianten der continuierlichen Transformationsgruppen," Bull. int. de l'Acad. des Sci. de Cracovie (1889-1900), 127-130.

On integral invariants of continuous transformation groups

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The focal point for this treatise will be the definition of integral invariants for continuous transformation groups.

Namely, if a finite or infinite transformation group $T^{(p)}$ in the variables $x_1, \ldots, x_n$; $y_1, \ldots, y_n$; $z_1, \ldots, z_n$, where the y_i denote functions of the x_i, and z_i are differential quotients of those functions up to certain order p, is arranged such that every transformation of that group leaves the form of the element under the integral sign in the integral:

$$\int \Omega(x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n)\, dx_1\, dx_2 \cdots dx_n$$

invariant then that integral will be called an *integral invariant of the first kind* of that group.

Moreover, if one denotes the most general infinitesimal transformation of the group $T^{(p)}$ by:

$$T^{(p)} f = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \eta_k \frac{\partial f}{\partial y_k} + \sum_{l=1}^{N} \zeta_l \frac{\partial f}{\partial z_l},$$

in which ξ_i and η_k are functions of the x_i and y_k, but ζ_l are certain functions of x_i, y_k, z_l, then the function Ω in the integral invariant of the first kind of that group $T^{(p)}$ will be a solution of the system of differential equations that follows from the differential equation:

$$T^{(p)} \Omega + \Omega \sum_{i=1}^{n} \left(\frac{\partial \xi_i}{\partial x_i} \right) = 0$$

by means of the arbitrary quantities that are present in $T^{(p)} f$. The parentheses in that equation refer to the fact that the differentiations must be performed with respect to the x_i that occur explicitly and implicitly. Conversely, every solution of the aforementioned system will produce an integral invariant of the first kind. Now, on the basis of that fact, we will now derive the fact that as long as integral invariants of the first kind exist, the most-general such integral invariant will have the form:

$$\int \Phi(I_1, I_2, \ldots, I_p)\, H\, dx_1\, dx_2 \cdots dx_n ,$$

in which Φ is an arbitrary function, $I_1, I_2, \ldots, I_p$ are all mutually-independent invariants of the group $T^{(p)}$, and H denotes any non-zero solution of the system in question, so it will not be an invariant of the group $T^{(p)}$, in general.

Those general considerations will be illustrated by some examples, and in particular, we should mention the integral invariants that **Poincaré** derived for use in the theory of linear substitutions (Acta math. Bd., pp. 6-8.)

In order to do that, we must consider integral invariants of the second kind for finite continuous transformation groups. Namely, as soon as one performs any arbitrary transformation of the r-parameter group T on the variables x_i in the integral:

$$\int \Omega(x_1, \ldots, x_n; l_1, \ldots, l_m)\, dx_1\, dx_2 \cdots dx_n$$

and the performs the corresponding transformation from a group L that isomorphic to T on the variables l_μ such that the form of the element under the integral sign remains invariant, the author calls that integral an *integral invariant of the second kind* of the group T relative to the group L.

Now, if one also has that the infinitesimal transformations:

$$X_1 f = \sum_{i=1}^{n} \xi_{ki}(x_1, x_2, \ldots, x_n) \frac{\partial f}{\partial x_i} \qquad (k = 1, 2, \ldots, r)$$

of the group T correspond to the infinitesimal transformations:

$$L_k f = \sum_{\mu=1}^{m} \lambda_{k|\mu}(l_1, l_2, \ldots, l_m) \frac{\partial f}{\partial l_\mu} \qquad (k = 1, 2, \ldots, r)$$

of the group L that isomorphic to T then one will find the most general integral invariant of the second kind by integrating the system, namely:

$$\int \Phi(\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_p)\, \bar{H}\, dx_1\, dx_2 \cdots dx_n,$$

in which Φ is, in turn, an arbitrary function, $\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_p$ are all mutually-independent invariants of the total group T and L, and $\bar{H}$ refers to any non-zero solution to the system in question, so it will not be an invariant of the total group, in general. As long as the group T possesses integral invariants of the first kind, $\bar{H}$ can be set equal to the corresponding function H, so to a function that depends upon only the variables x_i .

If L is the parameter group of the group T, and the finite equations of that group T are given then one can calculate the integral invariants of the second kind more simply insofar as the invariants $\bar{I}_i$, which are n in number in this case, are known to be much easier to exhibit.

Finally, the integral invariants of the one-parameter groups will be considered more thoroughly. Namely, all integral invariants of the first kind and all integral invariants of the second kind of the one-parameter group:

$$\delta x_i = \xi_i\,(x_1, x_2, \ldots, x_n)\;\delta t \qquad (i = 1, 2, \ldots, n)$$

will be determined precisely in the context of the parameter group of the parameter t. Conversely, all one-parameter groups that admit a given integral of the first kind (a given integral of the second kind that depends upon t, respectively) will be exhibited. The author will conclude with the remark that the latter problem coincides with the following problem: For a stationary moving fluid, determine all possible distributions of the density of that fluid when the velocity components of its particles are given, and conversely, exhibit all possible velocity components when the distribution of the density is given.